Conformal blocks from AdS

Per Kraus (UCLA)

Based on: Hijano, PK, Snively 1501.02260
Hijano, PK, Perlmutter, Snively 1508.00501, 1508.04987

Introduction

- Goal in this talk is to further develop understanding of structure of AdS/CFT correlation functions
- Focus on conformal block expansion of CFT correlators. How does this work in AdS/CFT?
- Mature subject with many results

e.g. D'Hoker, Freedman, Mathur, Rastelli
Heemskerk, Penedones, Polchinski, Sully
Fitzpatrick, Kaplan, Walters

Conformal Blocks

- Conformal block expansion builds up correlators of local operators out of basic CFT data: spectrum of primaries and their OPE coefficients
- Mostly focus on d-dimensional Euclidean CFT, with conformal group SO(d+1,1)
- For d=2 have enhancement to Vir x Vir or larger (e.g. W-algebras)
- Want to isolate all the structure of correlators fixed by symmetry

- local operators/states fall into representations of conformal algebra:

$$\text{primary}: \quad O_{\Delta,l}(x)$$
$$\text{descendants}: \quad \partial_\mu O_{\Delta,l}(x)\ ,\ldots$$

OPE: $$O_i(x)O_j(y) = \sum_k C_{ijk}\frac{O_k(y)}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}} + \text{descendants}$$
$$= \sum_k C_{ijk} D(\Delta_{i,j,k}; x-y; \partial_y) O_k(y)$$

complicated, but fixed by conformal symmetry

- Repeated use of OPE reduces any correlator to two-point functions
- First nontrivial case is 4-point function. Consider correlator of scalar operators for simplicity

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle = \sum_{\Delta,l}\langle O_1(x_1)O_2(x_2)P_{\Delta,l}O_3(x_3)O_4(x_4)\rangle$$

$$P_{\Delta,l} = |O_{\Delta,l}\rangle\langle O_{\Delta,l}| + \text{descendants}$$ projection operator

For external scalars, primaries that appear in OPE are symmetric, traceless tensors

- Each term is a "conformal partial wave":

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle = \sum_{\Delta,l} C_{12i}C_{34i}W_{\Delta_i,l_i}(x_i)$$

$$W_{\Delta_i,l_i}(x_i) = \frac{1}{C_{12i}C_{34i}}\langle O_1(x_1)O_2(x_2)P_{\Delta_i,l_i}O_3(x_3)O_4(x_4)\rangle$$

  - completely fixed by conformal symmetry

- Use conformal symmetry to write in terms of cross ratios:

$$W_{\Delta_i,l_i}(x_i) = \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{1}{2}\Delta_{12}}\left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{1}{2}\Delta_{34}}\frac{G_{\Delta_i,l_i}(u,v)}{(x_{12}^2)^{\frac{1}{2}(\Delta_1+\Delta_2)}(x_{34}^2)^{\frac{1}{2}(\Delta_3+\Delta_4)}}$$

$$u = z\bar{z}\ , \quad v = (1-z)(1-\bar{z})$$

- d=2: conformal algebra enhanced to Vir x Vir. Conformal partial waves/blocks factorize

$$G_{\Delta_i,l_i}(z,\overline{z}) = G_{h_i}(z)G_{\overline{h}_i}(\overline{z})$$

- Since Virasoro reps contain an infinite number of global reps, the Vir blocks are much richer and more complicated. Depend on the central charge

What's known

- CPWs appearing in scalar correlators were obtained by Ferrara et. al. in the 70s. E.g.:

$$G_{\Delta,0}(u,v) \quad \propto \quad u^{\Delta/2} \int_0^1 d\sigma\, \sigma^{\frac{\Delta+\Delta_{34}-2}{2}} (1-\sigma)^{\frac{\Delta-\Delta_{34}-2}{2}} (1-(1-v)\sigma)^{\frac{-\Delta+\Delta_{12}}{2}}$$
$$\times {}_2F_1\left(\frac{\Delta+\Delta_{12}}{2}, \frac{\Delta-\Delta_{12}}{2}, \Delta - \frac{d-2}{2}, \frac{u\sigma(1-\sigma)}{1-(1-v)\sigma}\right)$$

- In d=2,4,6, there are closed form expressions in terms of hypergeometric functions (Dolan, Osborn)

- For d=2 Virasoro blocks, no closed form expressions available in general. Simplifications occur in the limit of large c

Dolan/Osborn:

2, 4, 6. For example, in $d = 2$ we have

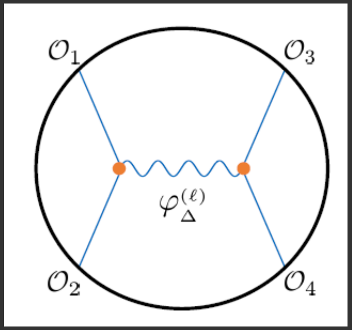
$$G_{\Delta,\ell}(z,\overline{z}) = |z|^{\Delta-\ell} \times \Big[z^{\ell}\,{}_2F_1\left(\frac{\Delta-\Delta_{12}+\ell}{2}, \frac{\Delta+\Delta_{34}+\ell}{2}, \Delta+\ell; z\right) \times {}_2F_1\left(\frac{\Delta-\Delta_{12}-\ell}{2}, \frac{\Delta+\Delta_{34}-\ell}{2}, \Delta-\ell; \overline{z}\right) + (z \leftrightarrow \overline{z})\Big]$$

and in $d = 4$ we have

$$G_{\Delta,\ell}(z,\overline{z}) = |z|^{\Delta-\ell} \frac{1}{z-\overline{z}} \times \Big[z^{\ell+1}\,{}_2F_1\left(\frac{\Delta-\Delta_{12}+\ell}{2}, \frac{\Delta+\Delta_{34}+\ell}{2}; \Delta+\ell; z\right) \times {}_2F_1\left(\frac{\Delta-\Delta_{12}-\ell}{2}-1, \frac{\Delta+\Delta_{34}-\ell}{2}-1; \Delta-\ell-2; \overline{z}\right) - (z \leftrightarrow \overline{z})\Big]$$

AdS: Witten diagrams

- basic exchange diagram



$$\int_y \int_{y'} G_{b\partial}(y,x_1) G_{b\partial}(y,x_2) \times G_{bb}(y,y';\Delta,\ell) \times G_{b\partial}(y',x_3) G_{b\partial}(y',x_4)$$

propagators in Poincare coords: $ds^2 = \frac{du^2 + dx^i dx^i}{u^2}$

$$G_{b\partial}(y,x_i) = \left(\frac{u}{u^2+|x-x_i|^2}\right)^{\Delta}$$

$$G_{bb}(y,y';\Delta,\ell=0) = e^{-\Delta\sigma(y,y')}{}_2F_1\left(\Delta,\tfrac{d}{2};\Delta+1-\tfrac{d}{2};e^{-2\sigma(y,y')}\right)$$

$$\sigma(y,y') = \log\left(\frac{1+\sqrt{1-\xi^2}}{\xi}\right) \quad , \quad \xi = \frac{2uu'}{u^2+u'^2+|x-x'|^2}$$

- integrals are very challenging (D'Hoker, Freedman, Mathur, Rastelli)
- Mellin space helps (Penedones, …)

- Brute force approach to conformal block decomposition involves evaluating integrals and then extracting block coefficients. Messy.
- Mellin space allows further progress
- But would be nice to have an efficient procedure that operates in position space
- We offer one here that requires no explicit integration
- Along the way we answer the question: what is the bulk representation of a conformal block?

Expected large N decomposition

- Assuming a semiclassical bulk, the CPW decomposition admits a 1/N expansion

$$\text{single trace operators}: \ O_i$$

$$\text{double trace operators}: \ [O_i O_j]_{n,\ell} \approx O_i \partial^{2n} \partial_{\mu_1} \ldots \partial_{\mu_\ell} O_j$$

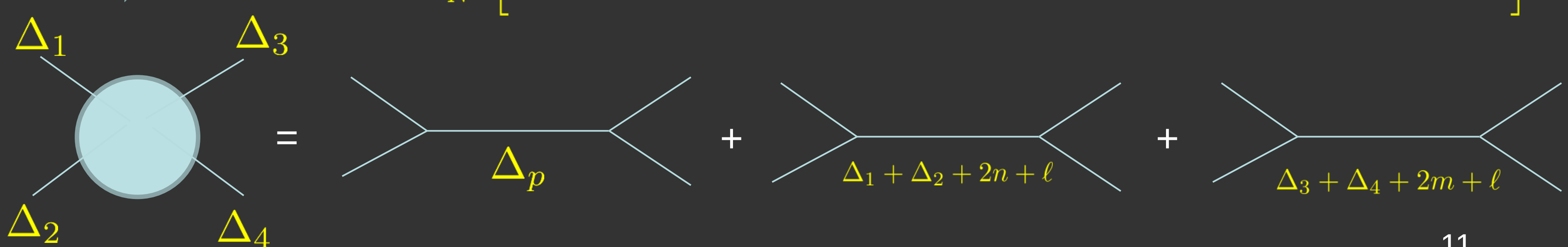
$$\Delta^{(ij)}(n,\ell) = \Delta_i + \Delta_j + 2n + \ell + O(\tfrac{1}{N})$$

- Decomposition of $\langle O_1 O_2 O_3 O_4 \rangle$:

$$O_1 O_2 = [O_1 O_2]_{n,\ell} + \tfrac{1}{N} C_{12p} O_p + \tfrac{1}{N^2} [O_3 O_4]_{n,\ell} + \ldots$$

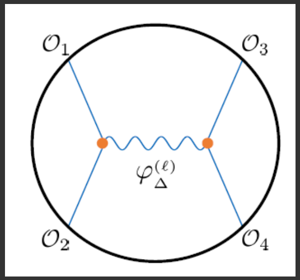
$$O_3 O_4 = [O_3 O_4]_{n,\ell} + \tfrac{1}{N} C_{34p} O_p + \tfrac{1}{N^2} [O_1 O_2]_{n,\ell} + \ldots$$

$$\Rightarrow \quad \langle O_1 O_2 O_3 O_4 \rangle = \tfrac{1}{N^2} \Big[C_{12p} C_{34p} \langle O_p O_p \rangle + \langle [O_1 O_2]_{n,\ell} [O_1 O_2]_{n,\ell} \rangle + \langle [O_3 O_4]_{n,\ell} [O_3 O_4]_{n,\ell} \rangle \Big]$$

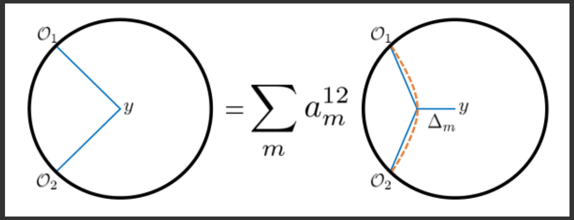


Witten diagram decomposition

$\ell = 0$



- decompose this diagram by rewriting the product of two bulk-boundary propagators



$$G_{b\partial}(y, x_1) G_{b\partial}(y, x_2) = \sum_{m=0}^{\infty} a_m^{12}\, \varphi_{\Delta_m}^{12}(y)$$

$$\Delta_m = \Delta_1 + \Delta_2 + 2m$$

$$\varphi_{\Delta}^{12}(y) \equiv \int_{\gamma_{12}} G_{b\partial}(y(\lambda), x_1) G_{b\partial}(y(\lambda), x_2) G_{bb}(y(\lambda), y; \Delta)$$

- in words: product of two bulk-boundary propagators is equal to a sum over fields sourced on the geodesic connecting the two boundary points

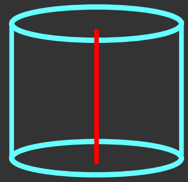
- identity is easy to derive by mapping to global AdS, with boundary points mapped to $t_{1,2} = \pm\infty$

$$ds^2 = \frac{1}{\cos^2\rho}(d\rho^2 + dt^2 + \sin^2\rho\, d\Omega_{d-1}^2)$$

- Product of bulk-boundary propagators:

$$G_{b\partial}(\rho, t; t_1) G_{b\partial}(\rho, t; t_2) \propto (\cos\rho)^{\Delta_1+\Delta_2} e^{-\Delta_{12} t}$$

- Geodesic maps to a line at origin of global AdS

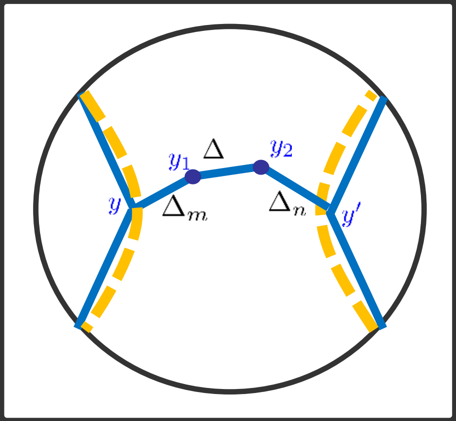


Look for normalizable solution for field of dimension Δ with above time dependence:

$$\varphi_\Delta^{12}(\rho, t) \propto {}_2F_1\left(\frac{\Delta+\Delta_{12}}{2}, \frac{\Delta-\Delta_{12}}{2}; \Delta - \frac{d-2}{2}; \cos^2\rho\right) \cos^\Delta\rho\; e^{-\Delta_{12} t}$$

- Comparing: $G_{b\partial}(y, x_1) G_{b\partial}(y, x_2) = \sum_{m=0}^\infty a_m^{12}\, \varphi_{\Delta_m}^{12}(y)$
- Coefficients a_m^{12} are easily computed

- applying our propagator identity at both vertices, we get a sum of diagrams of type

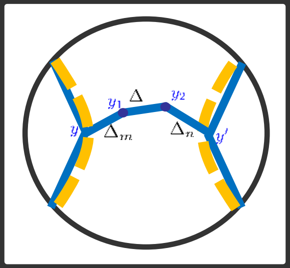


y and y' integrated over geodesics

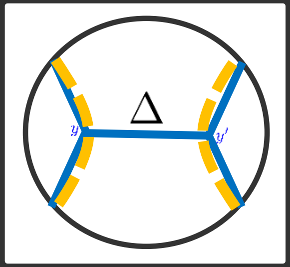
y_1 and y_2 integrated over all of AdS

$$G_{bb}(y,y';\Delta) = \left\langle y \left| \frac{1}{\nabla^2 - m^2} \right| y' \right\rangle \Longrightarrow \int_{y_1}\int_{y_2} G_{bb}(y,y_1;\Delta_m) G_{bb}(y_1,y_2;\Delta) G_{bb}(y_1,y';\Delta_n)$$

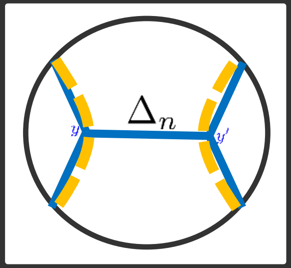
$$= \frac{G_{bb}(y,y';\Delta_m)}{(m_m^2-m_\Delta^2)(m_m^2-m_n^2)} + \frac{G_{bb}(y,y';\Delta)}{(m_\Delta^2-m_m^2)(m_\Delta^2-m_n^2)} + \frac{G_{bb}(y,y';\Delta_n)}{(m_n^2-m_m^2)(m_n^2-m_\Delta^2)}$$



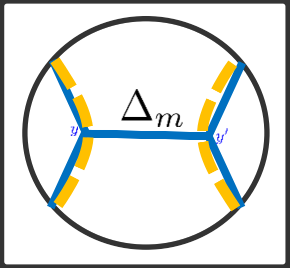
=

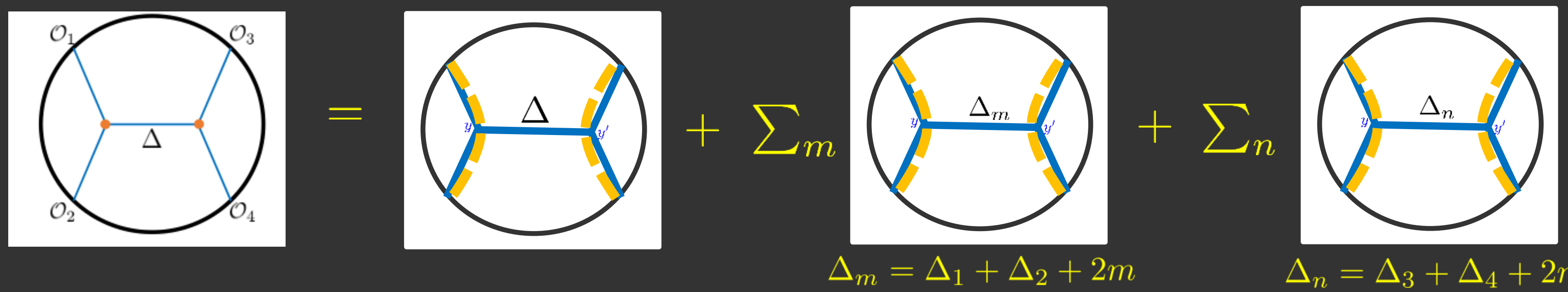


+



+





- Expansion in terms of geodesic Witten diagrams: exactly like ordinary Witten diagram, except that vertices are only integrated over geodesics, not over all of AdS
- Spectrum of operators appearing is what we expected from large N CPW expansion
- Suggests that:

geodesic Witten diagram = conformal partial wave

GWD = CPW

- relation can be established by direct computation
- Recall integral rep. of Ferrara et. al.

$$G_{\Delta,0}(u,v) \quad \propto \quad u^{\Delta/2}\int_0^1 d\sigma\,\sigma^{\frac{\Delta+\Delta_{34}-2}{2}}(1-\sigma)^{\frac{\Delta-\Delta_{34}-2}{2}}(1-(1-v)\sigma)^{\frac{-\Delta+\Delta_{12}}{2}}$$
$$\times {}_2F_1\left(\frac{\Delta+\Delta_{12}}{2},\frac{\Delta-\Delta_{12}}{2},\Delta-\frac{d-2}{2},\frac{u\sigma(1-\sigma)}{1-(1-v)\sigma}\right)$$

after a little rewriting, this can be recognized as a geodesic integral:

$$\int_{\gamma_{12}} d\lambda\ \varphi_\Delta(y(\lambda))G_{b\partial}(x_1,y(\lambda),\Delta_1)G_{b\partial}(x_2,y(\lambda),\Delta_2)$$
$$\varphi_\Delta(y) = \text{ field sourced by } \gamma_{34}$$

- GWD=CPW follows

- another way to establish this uses that CPW is an eigenfunction of the conformal Casimir

$$W_{\Delta_i,l_i}(x_i) = \frac{1}{C_{12i}C_{34i}}\langle O_1(x_1)O_2(x_2)P_{\Delta_i,l_i}O_3(x_3)O_4(x_4)\rangle$$

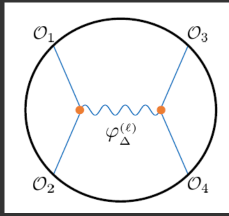
$$(L^1_{AB} + L^2_{AB})^2 W_{\Delta,\ell}(x_i) = C_2(\Delta,\ell)W_{\Delta,\ell}(x_i)$$

$$C_2(\Delta,\ell) = -\Delta(\Delta-d) - \ell(\ell+d-2)$$

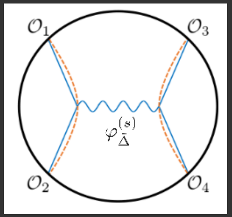
- Can show that GWD obeys this equation for l=0, recalling
  - conformal Casimir = Laplace operator
  - $\nabla^2 G_{bb}(y,y';\Delta) = C_2(\Delta,0)G_{bb}(y,y';\Delta) + \delta(y-y')$
  - integrating vertices over all of AdS, delta function contributes, so ordinary Witten diagrams are not eigenfunctions. But no such contribution for GWD

Comments

- Summary: simple method for computing scalar exchange diagram. No integration needed
- Output are OPE coefficients of double trace operators, in agreement with previous work
- Generalization to spin l exchange diagram with external scalars



decomposes into spin $s \leq l$ GWDs



  - spin-s propagator is pulled back to geodesics

- spinning GWDs reproduce known results for CPWs

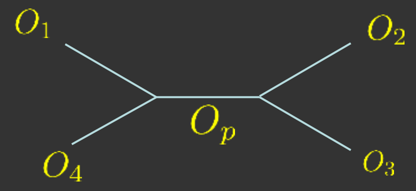
- Easy to decompose exchange diagram into CPWs in crossed channels. Contact diagrams also easy
- Note that geodesics often appear as approximations in the case of $\Delta >> 1$ operators. Here geodesics appear, but there is no approximation being made
- Obvious extensions:
  - adding legs
  - adding loops
  - spinning external operators

need some new propagator identities.

In progress

d=2: Virasoro Blocks

- Virasoro CPWs contain an infinite number of global blocks, and depend on central charge
- Apart from isolated examples, no explicit results
- But:
  - Zamolodchikov recursion relation enables efficient computation in series expansion in small cross ratio
  - simplifications at large c: "semiclassical blocks"



heavy limit: $c, h_i, h_p \to \infty \;\text{ with }\; \frac{h_i}{c}, \frac{h_p}{c} \;\text{ fixed}$

- can apply Zamolodchikov monodromy method

heavy-light limit: $c, h_1, h_2 \to \infty \;\text{ with }\; \frac{h_1}{c}, \frac{h_2}{c}, h_1 - h_2, \; h_3, \; h_4, \; h_p \;\text{ fixed}$

- focus on heavy-light limit

Heavy-light Virasoro Blocks

$$\langle O_{H_1}(\infty,\infty)O_{H_2}(0,0)\,P_p\,O_{L_1}(z,\bar{z})O_{L_2}(1,1)\rangle = \mathcal{F}\,(h_i,h_p,c;z-1)\,\overline{\mathcal{F}}(\bar{h}_i,\bar{h}_p,c;\bar{z}-1)$$

$$c\to\infty \quad \text{with} \quad \frac{h_{H_1}}{c},\ \ \frac{h_{H_2}}{c},\ \ h_{H_1}-h_{H_2},\ \ h_{L_1},\ \ h_{L_2},\ \ h_p \quad \text{fixed}$$

- Fitzpatrick, Kaplan and Walters used a clever conformal transformation to effectively transform away the heavy operators. Virasoro block then related to global block, with result:

$$\mathcal{F}\,(h_i,h_p,c;z-1) = z^{(\alpha-1)h_{L_1}}(1-z^{\alpha})^{h_p-h_{L_1}-h_{L_2}}\,{}_2F_1\left(h_p+h_{12},h_p-\frac{H_{12}}{\alpha},2h_p;1-z^{\alpha}\right)$$

$$\alpha=\sqrt{1-\frac{24h_{H_1}}{c}}$$

- This result has a simple bulk interpretation, now in terms of geodesics in backgrounds dual to the heavy operators

- FKW gave an interpretation in the simple case

$$h_{H_1} = h_{H_2}\ , \quad h_{L_1} = h_{L_2} \gg 1\ , \quad h_p = 0$$ vacuum block

Easiest to understand result by transforming to cylinder: $z = e^{iw} \quad \Rightarrow \quad \mathcal{F}(w) = \left(\sin\frac{\alpha w}{2}\right)^{-2h_L}$

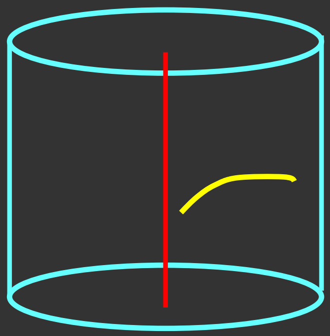
- Consider the conical defect metric

$$ds^2 = \frac{\alpha^2}{\cos^2\rho}\left(\frac{d\rho^2}{\alpha^2} + d\tau^2 + \sin^2\rho d\phi^2\right) \qquad w = \phi + i\tau$$

simple computation: $e^{-mL} \propto \left|\sin\frac{\alpha w}{2}\right|^{-4h_L} = |\mathcal{F}(w)|^2$

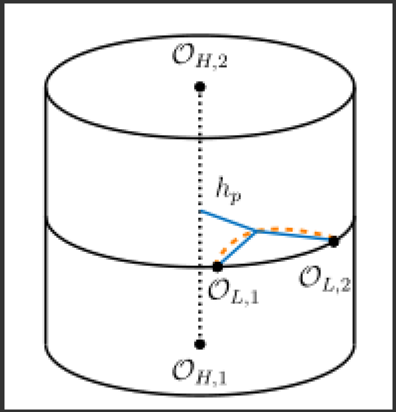
$m^2 = 4h_L(h_L - 1) \approx 4h_L^2$

L = regulated geodesic length



geodesic in conical defect background

- Bulk version of general heavy-light block combines this with our understanding of global case
- Take all operators to be scalars $h = \overline{h}$



background is conical defect dressed with a scalar field

$$\varphi_p(y') = (\cos\rho')^{2h_p} {}_2F_1\left(h_p + \frac{H_{12}}{\alpha}, h_p - \frac{H_{12}}{\alpha}, 2h_p; \cos^2\rho'\right) e^{-2H_{12}\tau'}$$

- We then integrate over geodesic

$$\begin{aligned} W(w,\overline{w}) &= \int_{-\infty}^{\infty} d\lambda'\, \varphi_p(y(\lambda'))\, G_{b\partial}(w_1 = 0, y(\lambda'))\, G_{b\partial}(w_2 = w, y(\lambda')) \\ &= \left(\sin\frac{\alpha w}{2}\right)^{2h_p - 2h_{L_1} - 2h_{L_2}} \int_{-\infty}^{\infty} d\lambda' e^{-2h_{12}\lambda'} (\cosh\lambda')^{-2h_p} \\ &\qquad \times {}_2F_1\left(h_p + \frac{H_{12}}{\alpha}, h_p - \frac{H_{12}}{\alpha}, 2h_p; \frac{\sin^2\frac{\alpha w}{2}}{\cosh^2\lambda'}\right) \end{aligned}$$

- With some effort, integral can be done:

$$W(w,\overline{w}) \propto \left| \sin \frac{\alpha w}{2}^{h_p - h_{L_1} - h_{L_2}} \times {}_2F_1\left(h_p + h_{12}, h_p - \frac{H_{12}}{\alpha}, 2h_p; 1 - e^{i\alpha w}\right) \right|^2$$

  precisely the FKW result in cylinder coordinates
- Since we considered scalar fields, result is the absolute square of the chiral Virasoro block
- Interesting to instead derive just the chiral part. Can be achieved by working with higher spin gauge fields propagating in a conical defect dressed with higher spin fields.

Comments

- heavy-light Virasoro blocks have a simple bulk description. Nontrivial bulk solutions “emerge” from CFT
- semiclassical Virasoro block is the leading term in a 1/c expansion. Subleading correction can be computed in CFT, at least in a series expansion in the cross ratio. These should map to quantum corrections in the bulk, which would be interesting to reproduce. CFT result actually give predictions nonperturbative in c.

Thermality

- Recall the conical defect solution

$$ds^2 = \frac{\alpha^2}{\cos^2\rho}\left(\frac{d\rho^2}{\alpha^2} + d\tau^2 + \sin^2\rho d\phi^2\right) \qquad w = \phi + i\tau$$

$$\alpha = \sqrt{1 - \frac{24h_{H_1}}{c}}$$

- Virasoro blocks are expressed in terms of $e^{i\alpha w}$
- this solution is the bulk dual of heavy operators
- FKW point out that for $h_H > \frac{c}{24}$ parameter α becomes imaginary, and correlators are periodic in imaginary time $\tau \cong \tau + \frac{2\pi}{|\alpha|}$
- an attractive interpretation is that a pure state appears effectively thermal in this regime

Conclusion

- Geodesic Witten diagrams are an efficient method for computing AdS correlators. Will be interesting to see how far this can be pushed: loop diagrams, etc.
- Bulk representation of semiclassical Virasoro blocks in the heavy-light limit. Wealth of data available regarding 1/c corrections