Conformal blocks from AdS

Per Kraus (UCLA)

Based on: Hijano, PK, Snively 1501.02260
Hijano, PK, Perlmutter, Snively 1508.00501, 1508.04987
Goal in this talk is to further develop understanding of structure of AdS/CFT correlation functions.

Focus on conformal block expansion of CFT correlators. How does this work in AdS/CFT?

Mature subject with many results

e.g. D’Hoker, Freedman, Mathur, Rastelli, Heemskerk, Penedones, Polchinski, Sully, Fitzpatrick, Kaplan, Walters
Conformal Blocks

Conformal block expansion builds up correlators of local operators out of basic CFT data: spectrum of primaries and their OPE coefficients.

Mostly focus on d-dimensional Euclidean CFT, with conformal group SO(d+1,1).

For d=2 have enhancement to Vir x Vir or larger (e.g. W-algebras).

Want to isolate all the structure of correlators fixed by symmetry.
local operators/states fall into representations of conformal algebra:

primary: \( O_{\Delta,l}(x) \)

descendants: \( \partial_{\mu} O_{\Delta,l}(x) , \ldots \)

OPE:

\[
O_i(x)O_j(y) = \sum_k C_{ijk} \frac{O_k(y)}{|x-y|^{\Delta_i + \Delta_j - \Delta_k}} + \text{descendants} \\
= \sum_k C_{ijk} D(\Delta_{i,j,k}; x - y; \partial_y) O_k(y)
\]

complicated, but fixed by conformal symmetry

Repeated use of OPE reduces any correlator to two-point functions

First nontrivial case is 4-point function. Consider correlator of scalar operators for simplicity
\[ \langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4) \rangle = \sum_{\Delta, l} \langle O_1(x_1)O_2(x_2)P_{\Delta, l}O_3(x_3)O_4(x_4) \rangle \]

\[ P_{\Delta, l} = |O_{\Delta, l}\rangle \langle O_{\Delta, l}| + \text{descendants} \]

For external scalars, primaries that appear in OPE are symmetric, traceless tensors

Each term is a “conformal partial wave”:

\[ \langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4) \rangle = \sum_{\Delta, l} C_{12i}C_{34i}W_{\Delta_i, l_i}(x_i) \]

\[ W_{\Delta_i, l_i}(x_i) = \frac{1}{C_{12i}C_{34i}} \langle O_1(x_1)O_2(x_2)P_{\Delta_i, l_i}O_3(x_3)O_4(x_4) \rangle \]

• completely fixed by conformal symmetry

Use conformal symmetry to write in terms of cross ratios:

\[ W_{\Delta_i, l_i}(x_i) = \left( \frac{x_{24}^2}{x_{14}^2} \right)^{\frac{1}{2} \Delta_{12}} \left( \frac{x_{14}^2}{x_{13}^2} \right)^{\frac{1}{2} \Delta_{34}} \frac{G_{\Delta_i, l_i}(u,v)}{(x_{12}^2)^{\frac{1}{2}(\Delta_1+\Delta_2)}(x_{34}^2)^{\frac{1}{2}(\Delta_3+\Delta_4)}} \]

\[ u = z\bar{z}, \quad v = (1-z)(1-\bar{z}) \]
d=2: conformal algebra enhanced to Vir x Vir. Conformal partial waves/blocks factorize

\[ G_{\Delta_i, l_i}(z, \bar{z}) = G_{h_i}(z)G_{\bar{h}_i}(\bar{z}) \]

Since Virasoro reps contain an infinite number of global reps, the Vir blocks are much richer and more complicated. Depend on the central charge
What’s known

CPWs appearing in scalar correlators were obtained by Ferrara et. al. in the 70s. E.g.:

\[ G_{\Delta,0}(u,v) \propto u^{\Delta/2} \int_0^1 d\sigma \, \sigma^{\frac{\Delta+\Delta_{12}}{2}-2} (1-\sigma)^{\frac{\Delta-\Delta_{12}}{2}-2} (1-(1-v)\sigma)^{-\Delta+\Delta_{12}} \times {}_2F_1 \left( \frac{\Delta + \Delta_{12}}{2}, \frac{\Delta - \Delta_{12}}{2}, \Delta - \frac{d-2}{2}, \frac{u\sigma(1-\sigma)}{1-(1-v)\sigma} \right) \]

In $d=2,4,6$, there are closed form expressions in terms of hypergeometric functions (Dolan, Osborn)

For $d=2$ Virasoro blocks, no closed form expressions available in general. Simplifications occur in the limit of large $c$
2, 4, 6. For example, in $d = 2$ we have

$$G_{\Delta, \ell}(z, \bar{z}) = |z|^{\Delta - \ell} \times$$

$$\left[ z^{\ell} \frac{\Gamma_{\ell} F_{1} \left( \frac{\Delta - \Delta_{12} + \ell}{2}, \frac{\Delta + \Delta_{34} + \ell}{2}, \Delta + \ell; z \right)}{\Gamma_{\ell} F_{1} \left( \frac{\Delta - \Delta_{12} - \ell}{2}, \frac{\Delta + \Delta_{34} - \ell}{2}, \Delta - \ell; \bar{z} \right)} \right]$$

and in $d = 4$ we have

$$G_{\Delta, \ell}(z, \bar{z}) = |z|^{\Delta - \ell} \frac{1}{z - \bar{z}} \times$$

$$\left[ z^{\ell+1} \frac{\Gamma_{\ell+1} F_{1} \left( \frac{\Delta - \Delta_{12} + \ell}{2}, \frac{\Delta + \Delta_{34} + \ell}{2}; \Delta + \ell; z \right)}{\Gamma_{\ell+1} F_{1} \left( \frac{\Delta - \Delta_{12} - \ell}{2} - 1, \frac{\Delta + \Delta_{34} - \ell}{2} - 1; \Delta - \ell - 2; \bar{z} \right)} \right] - (z \leftrightarrow \bar{z})$$
AdS: Witten diagrams

- basic exchange diagram

\[ \int_{y'} \int_y G_{b\delta}(y, x_1) G_{b\delta}(y, x_2) \times G_{bb}(y, y'; \Delta, \ell) \times G_{b\delta}(y', x_3) G_{b\delta}(y', x_4) \]

propagators in Poincare coords: 

\[ ds^2 = \frac{du^2 + dx^i dx^i}{u^2} \]

\[ G_{b\delta}(y, x_i) = \left( \frac{u}{u^2 + |x - x_i|^2} \right)^\Delta \]

\[ G_{bb}(y, y'; \Delta, \ell = 0) = e^{-\Delta \sigma(y,y')} {}_2F_1 \left( \Delta, \frac{d}{2}; \Delta + 1 - \frac{d}{2}; e^{-2\sigma(y,y')} \right) \]

\[ \sigma(y, y') = \log \left( \frac{1 + \sqrt{1 - \xi^2}}{\xi} \right), \quad \xi = \frac{2uu'}{u^2 + u'^2 + |x - x'|^2} \]

- integrals are very challenging (D'Hoker, Freedman, Mathur, Rastelli)

- Mellin space helps (Penedones, ...)

Brute force approach to conformal block decomposition involves evaluating integrals and then extracting block coefficients. Messy.

Mellin space allows further progress

But would be nice to have an efficient procedure that operates in position space

We offer one here that requires no explicit integration

Along the way we answer the question: what is the bulk representation of a conformal block?
Expected large N decomposition

Assuming a semiclassical bulk, the CPW decomposition admits a 1/N expansion

single trace operators: \( O_i \)

double trace operators: \([O_i O_j]_{n,\ell} \approx O_i \partial^{2n} \partial_{\mu_1} \cdots \partial_{\mu_\ell} O_j\)

\( \Delta^{(ij)}(n, \ell) = \Delta_i + \Delta_j + 2n + \ell + O\left(\frac{1}{N}\right)\)

Decomposition of \( \langle O_1 O_2 O_3 O_4 \rangle \):

\[
O_1 O_2 = [O_1 O_2]_{n,\ell} + \frac{1}{N} C_{12p} O_p + \frac{1}{N^2} [O_3 O_4]_{n,\ell} + \ldots \\
O_3 O_4 = [O_3 O_4]_{n,\ell} + \frac{1}{N} C_{34p} O_p + \frac{1}{N^2} [O_1 O_2]_{n,\ell} + \ldots
\]

\[
\langle O_1 O_2 O_3 O_4 \rangle = \frac{1}{N^2} \left[ C_{12p} C_{34p} \langle O_p O_p \rangle + \langle [O_1 O_2]_{n,\ell} [O_1 O_2]_{n,\ell} \rangle + \langle [O_3 O_4]_{n,\ell} [O_3 O_4]_{n,\ell} \rangle \right]
\]
Witten diagram decomposition

Decompose this diagram by rewriting the product of two bulk-boundary propagators:

\[
G_{b\partial}(y, x_1)G_{b\partial}(y, x_2) = \sum_{m=0}^{\infty} a_m^{12} \varphi_{\Delta_m}^{12}(y)
\]

where

\[
\Delta_m = \Delta_1 + \Delta_2 + 2m
\]

In words: product of two bulk-boundary propagators is equal to a sum over fields sourced on the geodesic connecting the two boundary points.
identity is easy to derive by mapping to global AdS, with boundary points mapped to \( t_{1,2} = \pm \infty \)

\[
ds^2 = \frac{1}{\cos^2 \rho} (d\rho^2 + dt^2 + \sin^2 \rho d\Omega^2_{d-1})
\]

Product of bulk-boundary propagators:

\[
G_{b\partial}(\rho, t; t_1)G_{b\partial}(\rho, t; t_2) \propto (\cos \rho)^{\Delta_1 + \Delta_2} e^{-\Delta_{12} t}
\]

Geodesic maps to a line at origin of global AdS

Look for normalizable solution for field of dimension \( \Delta \) with above time dependence:

\[
\varphi_{\Delta}^{12}(\rho, t) \propto \binom{2 \, F_1}{\Delta + \Delta_{12}, \Delta - \Delta_{12}; \Delta - \frac{d-2}{2}; \cos^2 \rho} \cos^\Delta \rho \ e^{-\Delta_{12} t}
\]

Comparing: \( G_{b\partial}(y, x_1)G_{b\partial}(y, x_2) = \sum_{m=0}^{\infty} a_{m}^{12} \varphi_{\Delta m}^{12} (y) \)

Coefficients \( a_{m}^{12} \) are easily computed
applying our propagator identity at both vertices, we get a sum of diagrams of type 

\[ G_{bb}(y, y'; \Delta) = \langle y \left| \nabla^2 \frac{1}{m^2} \right| y' \rangle \rightarrow \int_{y_1} \int_{y_2} G_{bb}(y, y_1; \Delta_m) G_{bb}(y_1, y_2; \Delta) G_{bb}(y_1, y'; \Delta_n) \]

\[ = \frac{G_{bb}(y, y'; \Delta_m)}{(m_m^2 - m_{\Delta}^2)(m_m^2 - m_n^2)} + \frac{G_{bb}(y, y'; \Delta)}{(m_\Delta^2 - m_m^2)(m_\Delta^2 - m_n^2)} + \frac{G_{bb}(y, y'; \Delta_n)}{(m_n^2 - m_m^2)(m_n^2 - m_{\Delta}^2)} \]
Expansion in terms of geodesic Witten diagrams: exactly like ordinary Witten diagram, except that vertices are only integrated over geodesics, not over all of AdS

Spectrum of operators appearing is what we expected from large N CPW expansion

Suggests that:

geodesic Witten diagram = conformal partial wave
relation can be established by direct computation

Recall integral rep. of Ferrara et. al.

\[ G_\Delta,0(u, v) \propto u^{\Delta/2} \int_0^1 d\sigma \sigma^{\frac{\Delta+\Delta_{34}-2}{2}} (1 - \sigma)^{\frac{\Delta-\Delta_{34}-2}{2}} (1 - (1 - v)\sigma)^{-\frac{\Delta+\Delta_{12}}{2}} \]

\[ \times {}_2F_1 \left( \frac{\Delta + \Delta_{12}}{2}, \frac{\Delta - \Delta_{12}}{2}, \Delta - \frac{d-2}{2}, \frac{u\sigma(1-\sigma)}{1-(1-v)\sigma} \right) \]

after a little rewriting, this can be recognized as a geodesic integral:

\[ \int_{\gamma_{12}} d\lambda \, \varphi_\Delta(y(\lambda)) G_{b\partial}(x_1, y(\lambda), \Delta_1) G_{b\partial}(x_2, y(\lambda), \Delta_2) \]

\[ \varphi_\Delta(y) = \text{field sourced by } \gamma_{34} \]

GWD = CPW follows
another way to establish this uses that CPW is an eigenfunction of the conformal Casimir

\[ W_{\Delta_i,l_i}(x_i) = \frac{1}{C_{12i}C_{34i}} \langle O_1(x_1)O_2(x_2)P_{\Delta_i,l_i}O_3(x_3)O_4(x_4) \rangle \]

\[
(L_{AB}^1 + L_{AB}^2)^2 W_{\Delta,\ell}(x_i) = C_2(\Delta, \ell) W_{\Delta,\ell}(x_i)
\]

\[
C_2(\Delta, \ell) = -\Delta(\Delta - d) - \ell(\ell + d - 2)
\]

Can show that GWD obeys this equation for \( l=0 \), recalling

- conformal Casimir = Laplace operator
- \( \nabla^2 G_{bb}(y, y'; \Delta) = C_2(\Delta, 0) G_{bb}(y, y'; \Delta) + \delta(y - y') \)
- integrating vertices over all of AdS, delta function contributes, so ordinary Witten diagrams are not eigenfunctions. But no such contribution for GWD
Summary: simple method for computing scalar exchange diagram. No integration needed

Output are OPE coefficients of double trace operators, in agreement with previous work

Generalization to spin $l$ exchange diagram with external scalars

decomposes into spin $s \leq l$ GWDs

- spin-$s$ propagator is pulled back to geodesics

spinning GWDs reproduce known results for CPWs
Easy to decompose exchange diagram into CPWs in crossed channels. Contact diagrams also easy

Note that geodesics often appear as approximations in the case of $\Delta >> 1$ operators. Here geodesics appear, but there is no approximation being made

Obvious extensions:

- adding legs
- adding loops
- spinning external operators

need some new propagator identities.

In progress
d=2: Virasoro Blocks

Virasoro CPWs contain an infinite number of global blocks, and depend on central charge

Apart from isolated examples, no explicit results

But:

• Zamolodchikov recursion relation enables efficient computation in series expansion in small cross ratio
• simplifications at large c: “semiclassical blocks”

\[
\begin{aligned}
O_1 & \quad O_3 \\
O_2 & \quad O_4 \\
\end{aligned}
\]

heavy limit: \( c, h_i, h_p \to \infty \) with \( \frac{h_i}{c}, \frac{h_p}{c} \) fixed

• can apply Zamolodchikov monodromy method

heavy-light limit: \( c, h_1, h_2 \to \infty \) with \( \frac{h_1}{c}, \frac{h_2}{c}, h_1 - h_2, h_3, h_4, h_p \) fixed

focus on heavy-light limit
Fitzpatrick, Kaplan and Walters used a clever conformal transformation to effectively transform away the heavy operators. Virasoro block then related to global block, with result:

\[
\langle O_{H_1}(\infty, \infty)O_{H_2}(0, 0) P_p O_{L_1}(z, \bar{z})O_{L_2}(1, 1) \rangle = \mathcal{F}(h_i, h_p, c; z - 1) \mathcal{F}(h_i, h_p, c; \bar{z} - 1)
\]

\[
c \to \infty \quad \text{with} \quad \frac{h_{H_1}}{c}, \quad \frac{h_{H_2}}{c}, \quad h_{H_1} - h_{H_2}, \quad h_{L_1}, \quad h_{L_2}, \quad h_p \quad \text{fixed}
\]

This result has a simple bulk interpretation, now in terms of geodesics in backgrounds dual to the heavy operators:

\[
\mathcal{F}(h_i, h_p, c; z - 1) = z^{(\alpha - 1)}h_{L_1}(1 - z^\alpha)^{h_p - h_{L_1} - h_{L_2}}_2 F_1 \left(h_p + h_{12}, h_p - \frac{H_{12}}{\alpha}, 2h_p; 1 - z^\alpha \right)
\]

\[
\alpha = \sqrt{1 - \frac{24h_{H_1}}{c}}
\]
FKW gave an interpretation in the simple case

\[ h_{H1} = h_{H2}, \quad h_{L1} = h_{L2} \gg 1, \quad h_p = 0 \]  vacuum block

Easiest to understand result by transforming to cylinder: \( z = e^{i\omega} \Rightarrow \mathcal{F}(w) = (\sin \frac{\alpha w}{2})^{-2h_L} \)

Consider the conical defect metric

\[ ds^2 = \frac{\alpha^2}{\cos^2 \rho} \left( \frac{d\rho^2}{\alpha^2} + d\tau^2 + \sin^2 \rho d\phi^2 \right) \quad w = \phi + i\tau \]

simple computation:

\[ e^{-mL} \propto \left| \sin \frac{\alpha \omega}{2} \right|^{-4h_L} = |\mathcal{F}(w)|^2 \]

\[ m^2 = 4h_L(h_L - 1) \approx 4h_L^2 \]

L = regulated geodesic length

geodesic in conical defect background
Bulk version of general heavy-light block combines this with our understanding of global case

Take all operators to be scalars \( h = \bar{h} \)

background is conical defect dressed with a scalar field

\[
\varphi_p(y') = (\cos \rho')^{2h_p} {}_2F_1\left( h_p + \frac{H_{12}}{\alpha}, h_p - \frac{H_{12}}{\alpha}, 2h_p; \cos^2 \rho' \right) e^{-2H_{12} \tau'}
\]

We then integrate over geodesic

\[
W(w, \bar{w}) = \int_{-\infty}^{\infty} d\lambda' \varphi_p(y(\lambda')) G_{b\partial}(w_1 = 0, y(\lambda')) G_{b\partial}(w_2 = w, y(\lambda'))
\]

\[
= \left( \sin \frac{\alpha w}{2} \right)^{2h_p - 2h_{L1} - 2h_{L2}} \int_{-\infty}^{\infty} d\lambda' e^{-2h_{12} \lambda'} (\cosh \lambda')^{-2h_p}
\]

\[
\times {}_2F_1\left( h_p + \frac{H_{12}}{\alpha}, h_p - \frac{H_{12}}{\alpha}, 2h_p; \frac{\sin^2 \frac{\alpha w}{2}}{\cosh^2 \lambda'} \right)
\]
With some effort, integral can be done:

\[
W(w, \overline{w}) \propto \left| \sin \frac{\alpha w}{2}^{h_p - h_{L1} - h_{L2}} \times {}_2F_1 \left( h_p + h_{12}, h_p - \frac{H_{12}}{\alpha}, 2h_p; 1 - e^{i\alpha w} \right) \right|^2
\]

precisely the FKW result in cylinder coordinates

Since we considered scalar fields, result is the absolute square of the chiral Virasoro block

Interesting to instead derive just the chiral part. Can be achieved by working with higher spin gauge fields propagating in a conical defect dressed with higher spin fields.
heavy-light Virasoro blocks have a simple bulk description. Nontrivial bulk solutions “emerge” from CFT.

Semiclassical Virasoro block is the leading term in a $1/c$ expansion. Subleading correction can be computed in CFT, at least in a series expansion in the cross ratio. These should map to quantum corrections in the bulk, which would be interesting to reproduce. CFT result actually give predictions nonperturbative in $c$. 

Comments
Recall the conical defect solution

\[ ds^2 = \frac{\alpha^2}{\cos^2 \rho} \left( \frac{d\rho^2}{\alpha^2} + d\tau^2 + \sin^2 \rho d\phi^2 \right) \quad w = \phi + i\tau \]

\[ \alpha = \sqrt{1 - \frac{24h_H}{c}} \]

Virasoro blocks are expressed in terms of \( e^{i\alpha w} \)

this solution is the bulk dual of heavy operators

FKW point out that for \( h_H > \frac{c}{24} \) parameter \( \alpha \) becomes imaginary, and correlators are periodic in imaginary time \( \tau \approx \tau + \frac{2\pi}{|\alpha|} \)

an attractive interpretation is that a pure state appears effectively thermal in this regime
Conclusion

Geodesic Witten diagrams are an efficient method for computing AdS correlators. Will be interesting to see how far this can be pushed: loop diagrams, etc.

Bulk representation of semiclassical Virasoro blocks in the heavy-light limit. Wealth of data available regarding 1/c corrections.