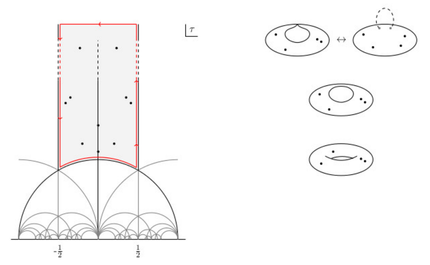


Loop Integrands from the Riemann Sphere

Yvonne Geyer[†], Lionel Mason[†], Ricardo Monteiro[†], Piotr Tourkine[‡]

[†]Mathematical Institute, University of Oxford; [†]DAMTP, University of Cambridge

Introduction

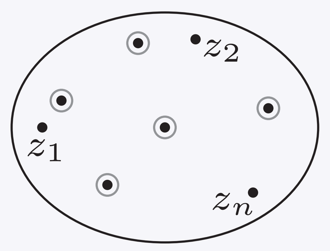
Worldsheet formulations of quantum field theories have had wide ranging impact on the study of amplitudes. However, the mathematical framework becomes very challenging on the higher-genus worldsheets required to describe loop effects. We derive a framework, applicable in such worldsheet models based on the scattering equations, that transforms formulae on higher-genus surfaces to ones on nodal Riemann spheres, and that can potentially be applied quite generally in field theory.

Scattering Equations

The scattering equations underpin the CHY formulae for tree-level scattering amplitudes arising from ambitwistor string theories, and determine n points z_i on a Riemann surface. To define the scattering equations, construct a 1-form $P(z, z_i)$ satisfying

$$\bar\partial P = 2\pi i \sum_i k_i \bar\delta(z - z_i) dz\,.$$

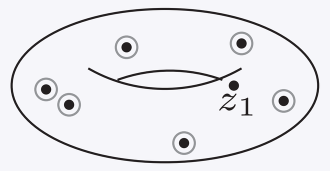
- On the Riemann sphere, this can be achieved for n null momenta k_i via



$$P_0(z) = \sum_{i=1}^n \frac{k_i}{z - z_i}\, dz\,.$$

with scattering equations $\mathrm{Res}_{z_i} P_0^2(z) = 0$.

- On the torus $\Sigma_q = \mathbb{C}/\{\mathbb{Z}\oplus\mathbb{Z}\tau\}$ where $q = e^{2\pi i\tau}$, we introduce $\ell \in \mathbb{R}^d$ to parametrize the zero modes and obtain



$$P = 2\pi i\,\ell dz + \sum_i k_i \left(\frac{\theta_1'(z - z_i)}{\theta_1(z - z_i)} + \sum_{j\neq i} \frac{\theta_1'(z_{ij})}{n\,\theta_1(z_{ij})} \right) dz\,.$$

Using this, the scattering equations are

$$\mathrm{Res}_{z_i} P^2(z) := 2k_i \cdot P(z_i) = 0\,, \qquad P^2(z_0) = 0\,.$$

The ACS proposal [2] for the 1-loop integrand of type II supergravity takes the form

$$\mathcal{M}^{(1)}_{\mathrm{SG}} = \int \mathcal{I}_q\, d^d\ell\, d\tau\, \bar\delta(P^2(z_0)) \prod_{i=2}^n \bar\delta(k_i \cdot P(z_i)) dz_i\,.$$

The formula localizes on a discrete set of solutions to the scattering equations. It was furthermore conjectured [3] that this integral with $\mathcal{I} = 1$ is equivalent to a sum over permutations of n-gons.

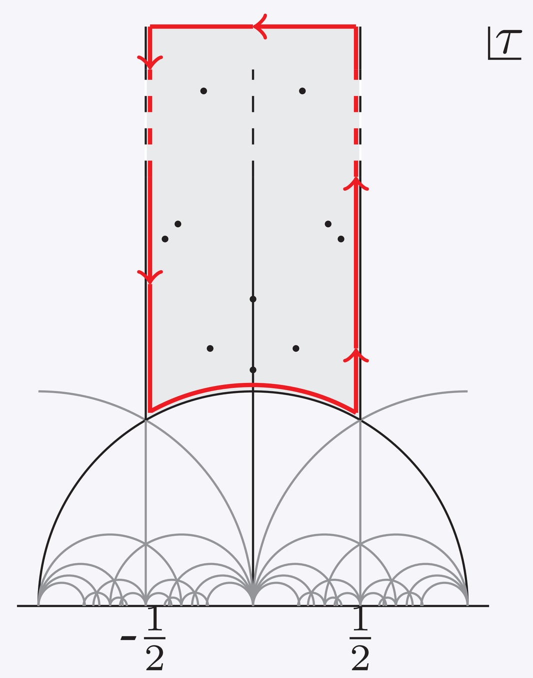
From Torus to Riemann Surface

Using a residue theorem and assuming holomorphicity of the integrand, we can reduce the formula from the elliptic curve to a nodal Riemann spere at $q = 0$. The residue theorem is equivalent to a contour integral argument in the fundamental domain, which equates the sum over residues of $1/P^2(z_0, \dots|q)$ with the contour indicated. Contributions from the sides and the unit circle cancel due to modular invariance, so it localizes on $q = 0$. Mapping the fundamental domain to the Riemann sphere $\sigma = \mathrm{e}^{2\pi i(z - \tau/2)}$, we obtain

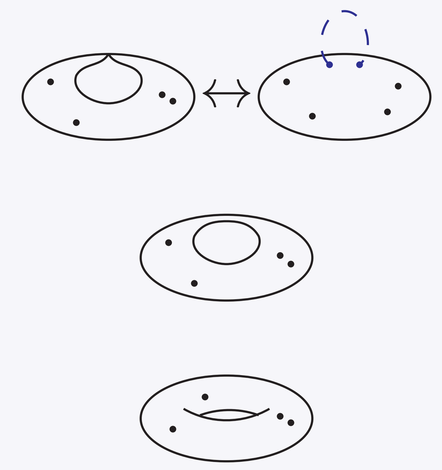
$$P(z) = P(\sigma) = \ell\, \frac{d\sigma}{\sigma} + \sum_{i=1}^n \frac{k_i\, d\sigma}{\sigma - \sigma_i}\,.$$

Setting $S = P^2 - \ell^2\, \mathrm{d}\sigma^2/\sigma^2$, the vanishing of the residues of S gives the *off-shell scattering equations*

$$0 = \mathrm{Res}_{\sigma_i} S = k_i \cdot P(\sigma_i) = \frac{k_i \cdot \ell}{\sigma_i} + \sum_{j\neq i} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j}\,,$$



Contour argument in the fundamental domain



Using this, the 1-loop formula becomes

$$\mathcal{M}^{(1)} = -\int \mathcal{I}_0\, d^d\ell\, \frac{1}{\ell^2} \prod_{i=2}^n \bar\delta(k_i \cdot P(\sigma_i)) \frac{d\sigma_i}{\sigma_i^2}\,,$$

which is our new proposal for the supergravity 1-loop integrand, with $\mathcal{I}_0$ the $q = 0$ limit of the ACS correlator.

The n-gon conjecture

Following the framework derived above, the n-gon conjecture becomes

$$\mathcal{M}^{(1)}_{n\text{-gon}} = -\int d^{2n+2}\ell\, \frac{1}{\ell^2} \prod_{i=2}^n \bar\delta(k_i \cdot P(\sigma_i)) \frac{d\sigma_i}{\sigma_i^2}\,,$$

which can be checked numerically to give $\mathcal{M}^{(1)}_n = \frac{(-1)^n}{\ell^2} \sum_{\sigma\in S_n} \prod_{i=1}^{n-1} \frac{1}{\ell\cdot\sum_{j=1}^i k_{\sigma_i} + \frac{1}{2}\left(\sum_{j=1}^i k_{\sigma_i}\right)^2}$.
Using partial fraction identities and shifts in the loop momentum, this is indeed equivalent to the sum over permutations of n-gons.

Supergravity 1-loop integrand

For supergravity, $\mathcal{I}_q \equiv \mathcal{I}(k_i, \epsilon_i, z_i|q) = \mathcal{I}_q^L\, \mathcal{I}_q^R$ [2]. At $q = 0$, this becomes

$$\mathcal{I}_0^L = 16\,(\mathrm{Pf}(M_2) - \mathrm{Pf}(M_3)) - 2\,\partial_{q^{1/2}} \mathrm{Pf}(M_3)\,,$$

see [1,2] for details. The 1-loop supergravity integrand is thus given by

$$\mathcal{M}^{(1)} = -\int \mathcal{I}_0^L \mathcal{I}_0^R\, \frac{1}{\ell^2} \prod_{i=2}^n \bar\delta(k_i \cdot P(\sigma_i)) \frac{d\sigma_i}{\sigma_i^2}\,.$$

Super Yang-Mills 1-loop integrand

This naturally leads to a conjecture for super Yang-Mills scattering amplitudes at 1 loop;

$$\mathcal{M}^{(1)}(1, \dots, n) = \int \mathcal{I}_0^L\, PT_n \prod_{i=2}^n \bar\delta(k_i \cdot P(\sigma_i)) \frac{d\sigma_i}{\sigma_i}\,.$$

Here, the supergravity factor $\mathcal{I}_0^R$ has been replaced by a cyclic sum over Parke-Taylors factor running through the loop,

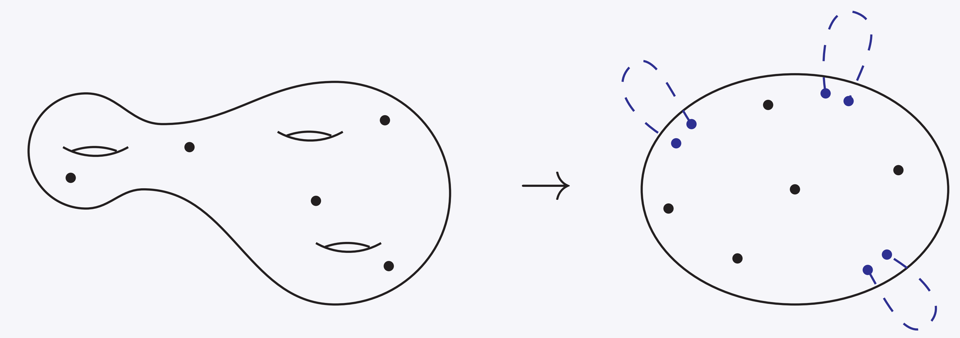
$$PT_n = \sum_{i=1, i \bmod n}^n \frac{\sigma_{0\,\infty}}{\sigma_{0\,i}\sigma_{i\,i+1}\sigma_{i+1\,i+2}\cdots\sigma_{i+n\,\infty}}\,.$$

Conclusion

- framework to derive formulae for loop integrands on a nodal Riemann sphere using residue theorems
- new, off-shell scattering equations that depend on the loop momenta
- new formulae for supergravity, super YM and n-gon integrands at 1 loop
- proposal for the all-loop integrands in supergravity, SYM and biadjoint scalar theories

Outlook: All-loop Integrands

Sarting from the natural extensions of the ACS proposals to Riemann surfaces of genus g, we can again use residue theorems to localize on boundary components of the moduli space by contracting g a-cycles to obtain a nodal Riemann sphere.



This fixes g moduli, with the remaining $2g - 3$ now associated with $2g$ new marked points. The 1-form P is then given by

$$P = \sum_{r=1}^g \ell_r \omega_r + \sum_i k_i \frac{d\sigma}{\sigma - \sigma_i}\,,$$

where ω_r is a basis of g global holomorphic 1-forms on the nodal Riemann sphere. Setting $S(\sigma) := P^2 - \sum_{r=1}^g \ell_r^2 \omega_r^2$, the multiloop off-shell scattering equations are

$$\mathrm{Res}_{\sigma_i} S = 0\,, \quad i = 1, \dots, n + 2g,.$$

This leads to the following proposal for the all-loop integrand;

$$\mathcal{M}^{(g)}_{SG} = \int_{(\mathbb{CP}^1)^{n+2g}} \frac{\mathcal{I}_0^L \mathcal{I}_0^R}{\mathrm{Vol}\, G} \prod_{r=1}^g \frac{1}{\ell_r^2} \prod_{i=1}^{n+2g} \bar\delta(\mathrm{Res}_{\sigma_i} S(\sigma_i))\,,$$

$$\text{where } \mathcal{I}_0 = \begin{cases} \mathcal{I}_0^L \mathcal{I}_0^R, & \text{gravity} \\ \mathcal{I}_0^L PT_n, & \text{Yang-Mills} \\ PT_n PT_n', & \text{biadjoint scalar} \end{cases}.$$

Remarkably, this suggests that n-point g-loop integrands have a similar complexity to tree amplitudes with $n + 2g$ particles.

References

[1] Y. Geyer, L. Mason, R. Monteiro, P. Tourkine. Loop Integrands from the Riemann Sphere, arXiv:1507.00321

[2] T. Adamo, E. Casali, D. Skinner. JHEP 1404, 104 (2014).

[3] E. Casali, P. Tourkine. JHEP 1504, 013 (2015).

Contact

Yvonne Geyer, yvonne.geyer@maths.ox.ac.uk

Mathematical Institute, University of Oxford, UK