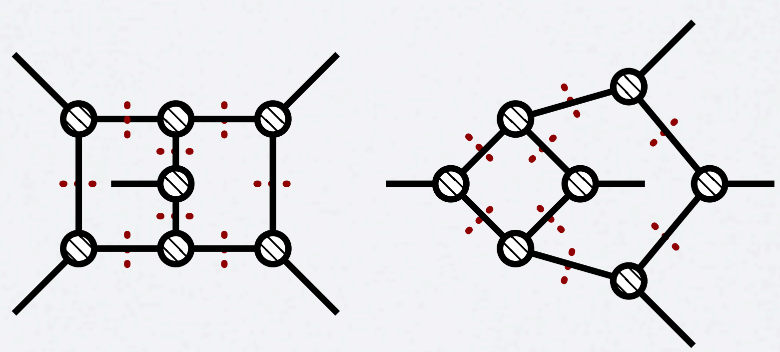
Non-planar integrands for two-loop QCD amplitudes

Simon Badger

6th July 2015

Based on work with Gustav Mogull, Alex Ochirov and Donal O'Connell



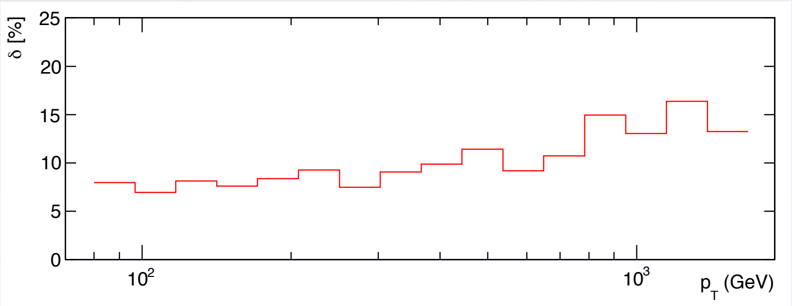
Amplitudes 2015, Zurich

Introduction

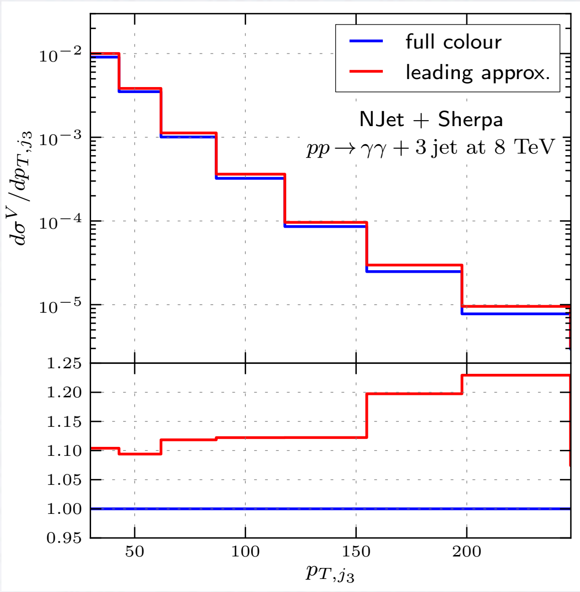
Run II expecting ~ 100 fb^{-1}
measurements reaching ~1% level
accuracy calls for NNLO precision

Quite a lot of success at NLO using
leading colour approximations

e.g. pp→W+5j [Bern et al. (2013)]



gg→gg @ NNLO [Currie et al. (2013)]



[SB, Guffanti, Yundin (2013)]

larger corrections
at high p_T

full colour at NNLO means dealing with
the non-planar sector in the double virtuals
(and a few other things...)

Amplitudes for NNLO

[See Glover's talk]

QCD is going beyond NLO precision

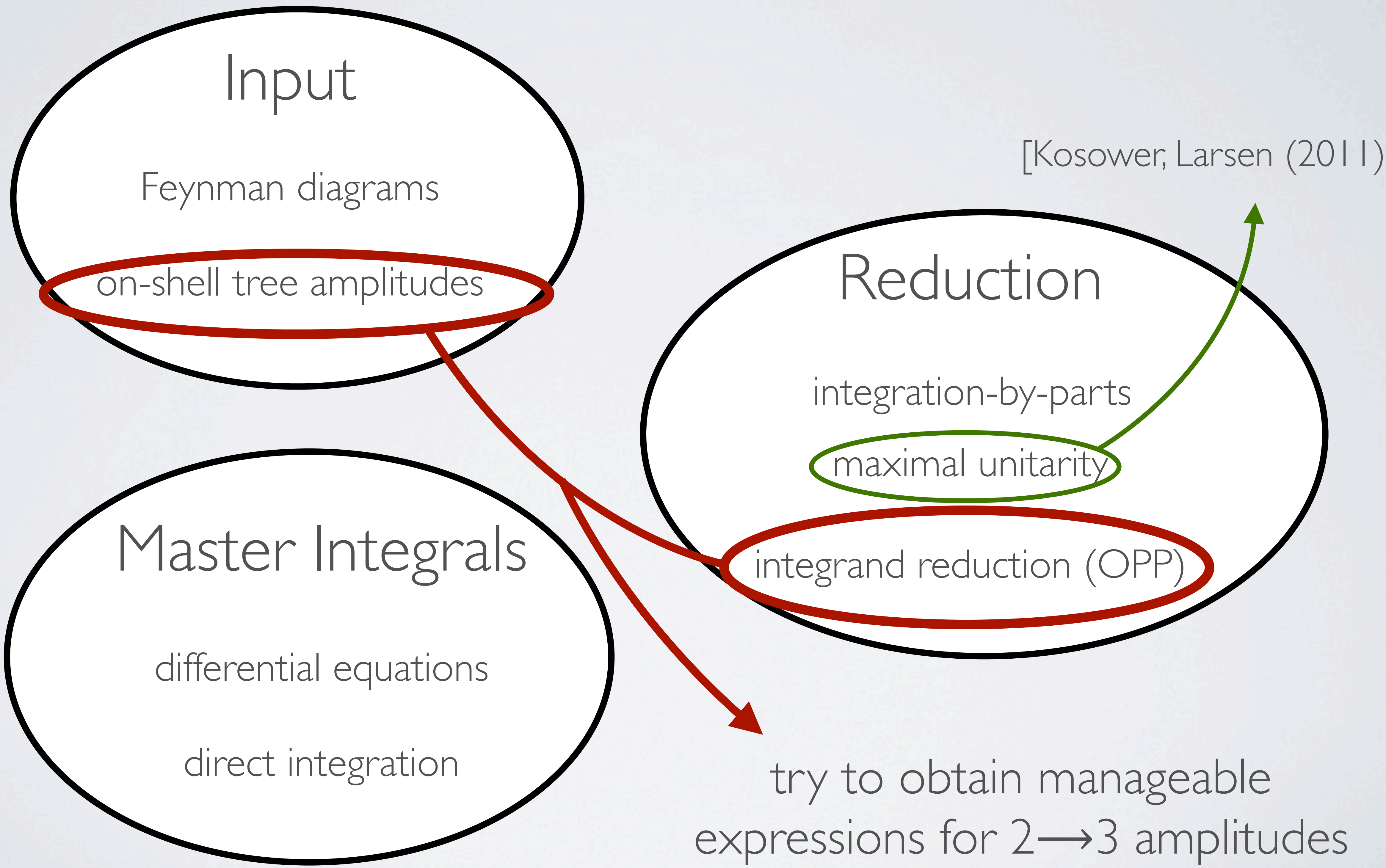
$$\sigma_n^{NNLO} = \int_n (d\sigma^B + d\sigma^V + d\sigma^{VV}) + \int_{n+1} d\sigma^R + d\sigma^{RV} + \int_{n+2} d\sigma^{RR}$$

Traditional approach: Feynman diagrams + integration-by-parts

suitable for 2→2 processes

complexity grows fast with additional legs

Automation for multi-leg NNLO



All-plus helicity amplitudes

useful playground for QCD: simplest helicity

one-loop connection to $\mathcal{N}$=4

$$A^{D,(1)}_{++\cdots+} = \frac{(D-4)(D-3)}{\delta^{(8)}} A^{D+4,(1)}_{\mathcal{N}=4\ \mathrm{MHV}}$$

[Bern, Dixon, Dunbar, Kosower hep-ph/9611127]

vanishes at tree-level
⇒ simple IR structure

(two-loops contributes at N^3LO)

two-loop ++++
amplitude known
for a long time

[Bern, Dixon, Kosower hep-ph/0001001]

more recently shown to obey
colour kinematics duality

[Bern, Davies, Dennen, Huang, Nohle 1303.6605]

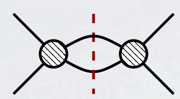
connection to $\mathcal{N}$=4 continues to
some extent at two-loops...

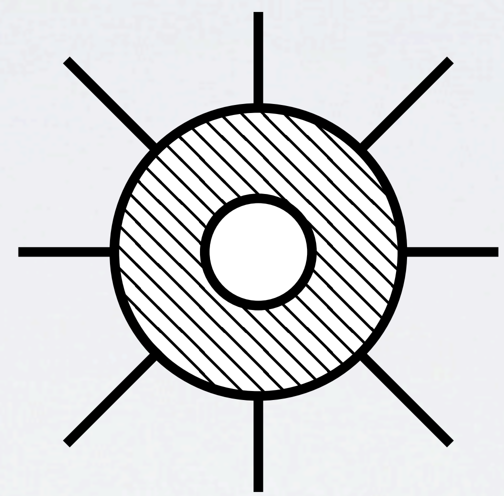
Outline

- Integrand representations of loop amplitudes
- Colour decompositions
    - minimising cut information using **Kleiss-Kuijf** relations
- Further simplifications from colour/kinematics duality
    - non-planar from planar using **BCJ** relations
- Application to $A_5^{(2)}(1^+, 2^+, 3^+, 4^+, 5^+)$ in QCD
    - evaluating the full colour amplitude in the soft region

Integrand reduction and generalized unitarity methods

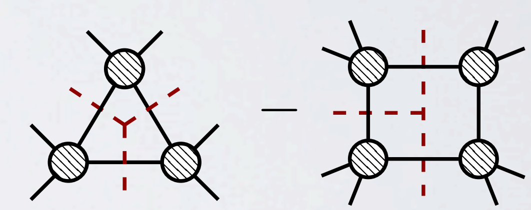
Unitarity: double cuts
[BDDK '94]
[triple cuts BDK '97]



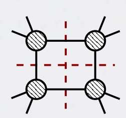


automated techniques ⇒
LHC phenomenology

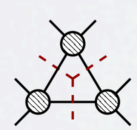
Integrand reduction [OPP '05]

$\Delta_3 =$ 

Generalized unitarity:
quadruple cuts [BCF '04]



triple cuts [e.g. Forde '07]



D-dim. generalized unitarity [GKM '08]

$$A = \sum_i (\text{rational})_i (\text{integral})_i$$

find complex contour to isolate
integral coefficient

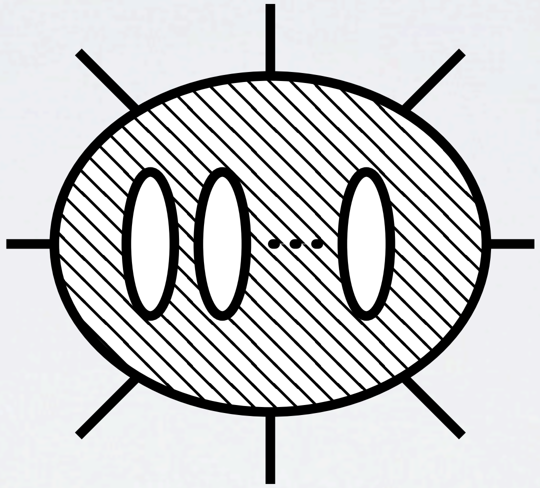
$$A = \int_k \sum_i \frac{\Delta_i(k,p)}{(\text{propagators})_i}$$

explicitly remove poles

Integrand reduction and generalized unitarity methods

Maximal unitarity

[Kosower, Larsen, Johannson, Caron-Huot, Zhang, Søgaard]



Integrand reduction via polynomial division

[Mastrolia, Ossola, SB, Frellesvig, Zhang, Mirabella, Peraro, Malamos, Kleiss, Papadopolous, Verheyen, Feng, Huang]

$$A = \sum_i (\text{rational})_i (\text{integral})_i$$

e.g. IBPs ←

$$A = \int_k \sum_i \frac{\Delta_i(k,p)}{(\text{propagators})_i}$$

Notation

$$\bar{k}_i \cdot p_j,\ \bar{k}_i \cdot \varepsilon_j,\ \bar{k}_i \cdot \bar{k}_j,\ \mu_{ij} = -k_i^{[-2\epsilon]} \cdot k_j^{[-2\epsilon]}\ ..$$

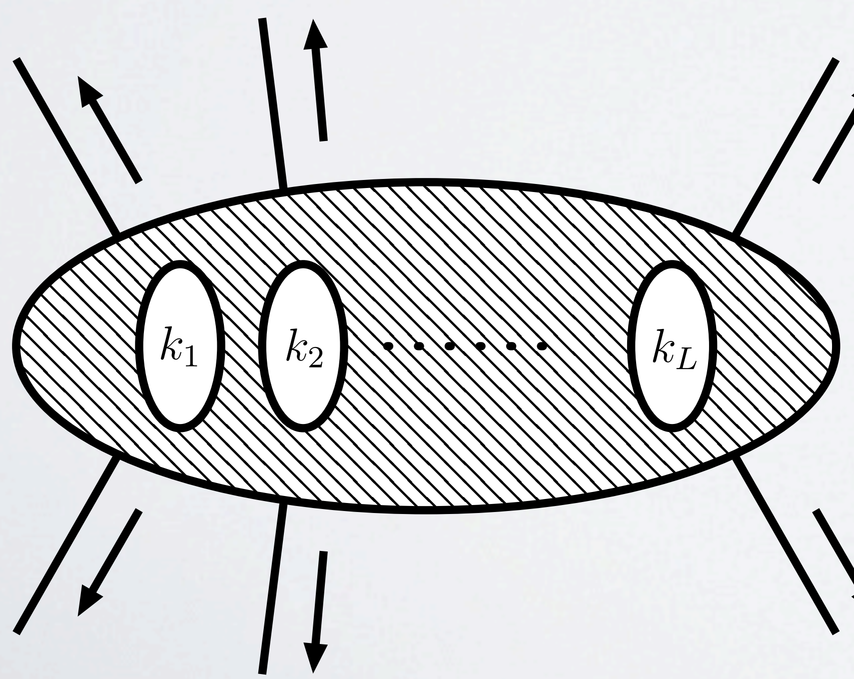
$$A_n^{(L),[D]}(\{p\}) = \int \prod_{i=1}^{L} \frac{d^D k_i}{(2\pi)^D} \frac{N(\{k\},\{p\})}{\prod_{l=1}^{L(L+9)/2} D_l(\{k\},\{p\})}$$

$$= \int \prod_{i=1}^{L} \frac{d^D k_i}{(2\pi)^D} \sum_{c=1}^{L(L+9)/2} \sum_{T \in P_c} \frac{\Delta_{c;T}(\{\bar{k}\cdot v, \mu_{ij}\})}{\prod_{l \in T} D_l(\{k\},\{p\})}$$

$$= \sum_{i \in MI} c_i^{[D]}(\{p\}) I_i(\{k\},\{p\})$$

basis of irreducible scalar products

master integral basis



$$k_i = \bar{k}_i + k_i^{[-2\epsilon]}$$

Integrand reduction strategy

[Mastrolia, Ossola arXiv:1107.6041]

[SB, Frellesvig, Zhang arXiv:1202.2019]

[Zhang arXiv:1205.5707]

[Mastrolia, Mirabella, Ossola, Peraro arXiv:1205.7087]

- top down: start with maximal number of propagators
- identify basis of irreducible scalar products (ISPs)

  spanning basis e.g. Van Neerven-Vermaseren
  $$x_{ij} = k_i \cdot v_j$$

- parametrize integrand using propagators

  $$\Delta = \sum c_i m_i(x_{ij}, \mu_{ij})$$

  Gröbner basis and polynomial division

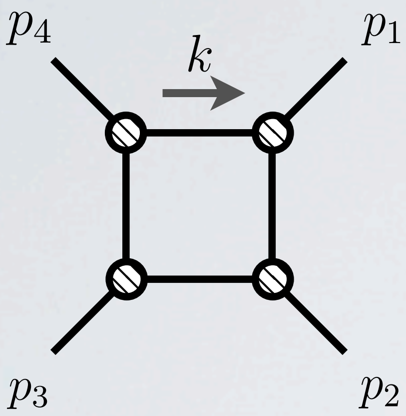
- parameterise on-shell solutions and solve

  $$N(k^{(s)}(\tau_j)) = \Delta(k^{(s)}(\tau_j)) \Rightarrow c_i$$

  primary decomposition

- continue to lower propagator topologies subtracting known singularities

D-dimensional Reduction



$$v^\mu = \{p_1^\mu, p_2^\mu, p_4^\mu, \omega = \varepsilon^{\mu 124}\}$$

$$x_{14} = k \cdot \omega$$

additional ISPs

$$k_i^{[-2\epsilon]} \cdot k_j^{[-2\epsilon]} = -\mu_{ij}$$

$$k_i = \bar{k}_i + k_i^{[-2\epsilon]}$$

box integrand

dimension shifted integrals

$$\Delta_4 = c_0 + c_1 x_{14} + c_2 \mu_{11} + c_3 \mu_{11} x_{14} + c_4 \mu_{11}^2$$

scalar box

spurious

Irreducible numerators

$$\Delta_{c;T}\Big|_{\mathrm{cut}} = \prod_i A_i^{(0)} - \sum_{T'} \frac{\Delta_{c;T'}}{\prod_{l\in T'/T} D_l}\Bigg|_{\mathrm{cut}}$$

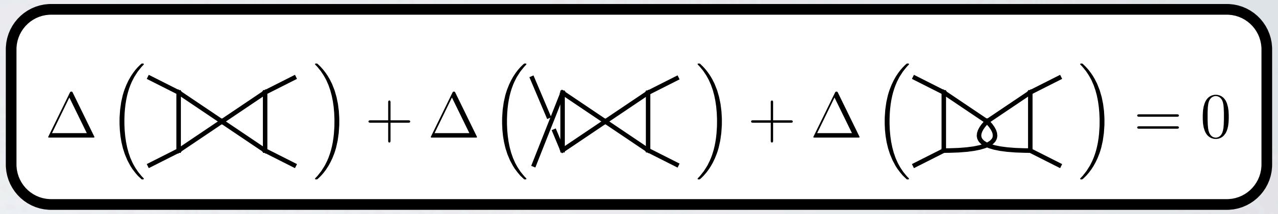
on-shell the numerators can be written as products of tree-level amplitudes

integrand parameterisations not unique - freedom in the choices of ISP monomials

Next step: assemble irreducible numerators into full colour amplitude

Colour decompositions

Eliminate irreducible integrands using KK relations

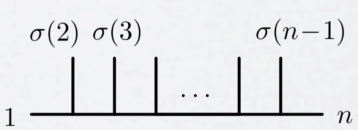


ensure ISP parameterisation satisfies symmetry off-shell

$$\mathcal{A} = \sum_i S_i \frac{C(\Delta_i)\Delta_i}{\prod D_\alpha}$$

Assign colour factors using underlying tree structure

$$\mathcal{A}^{(0)} = \sum_{\sigma\in S_{n-2}}$$

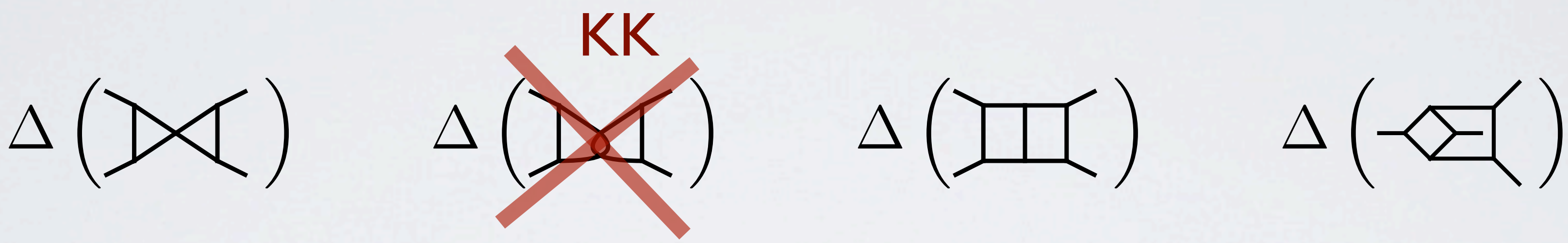


$$A^{(0)}(1,\sigma(2),\ldots,\sigma(n-1),n)$$

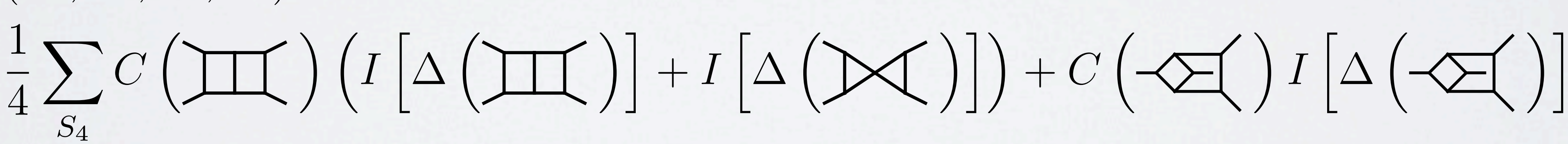
[Dixon, Del Duca, Maltoni (1999)]

Two-loop four gluon amplitude

construct full amplitude from all cuts

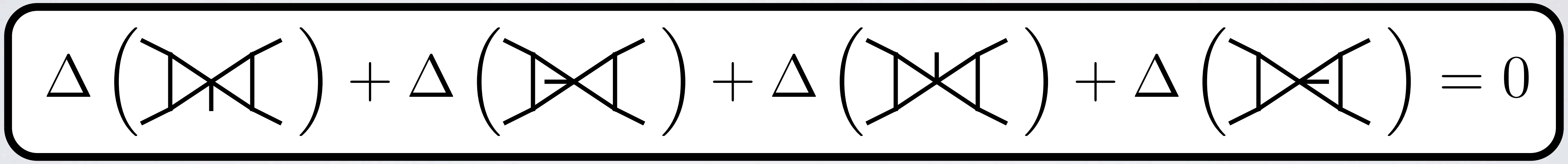


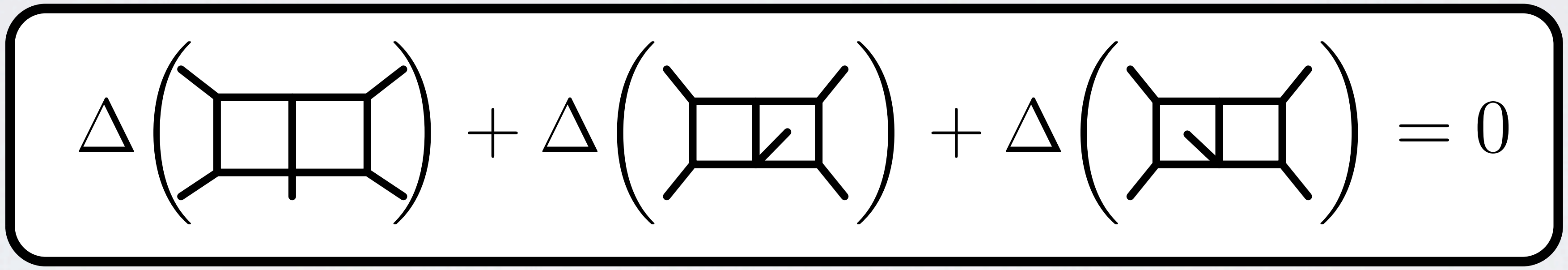
$$\mathcal{A}^{(2)}(1^+, 2^+, 3^+, 4^+) =$$



[in agreement with Bern, Dixon, Kosower (2000)]

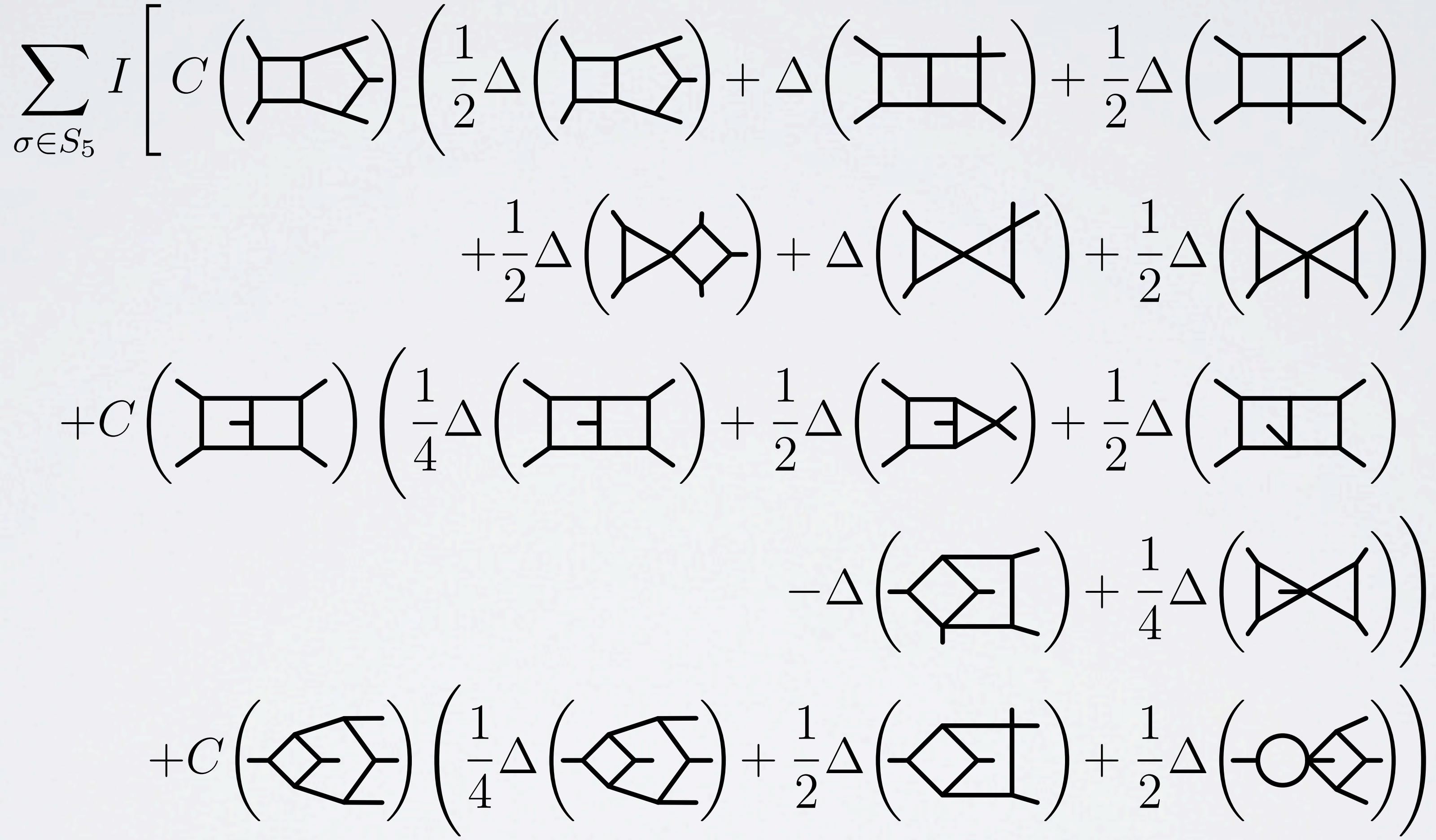
Five gluon decomposition





Five gluon decomposition

$$\mathcal{A}_5^{(2)}(1^+, 2^+, 3^+, 4^+, 5^+) =$$



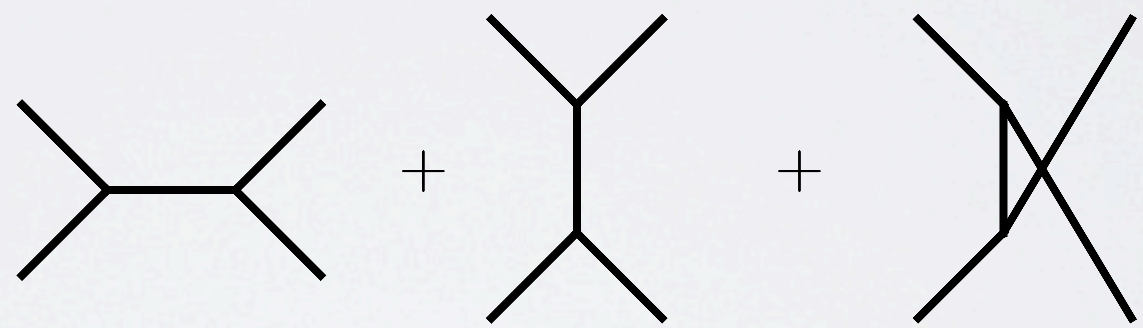
general tree-level DDM colour bases including fermions
[Johansson, Ochirov arXiv:1507.00332]

Non-planar from planar

colour-kinematics duality [Bern, Carrasco, Johansson (2008)]

$$\mathcal{A} = \sum_j \frac{c_j n_j}{\prod_{\alpha_j} p^2_{\alpha_j}}$$

with Jacobi identities for both n and c



$= 0$

$$\Rightarrow A_4(1,2,3,4) = \frac{s_{13}}{s_{12}} A_4(1,3,2,4)$$

powerful identities for loop level cuts - c.f multi-loop $\mathcal{N} = 4$

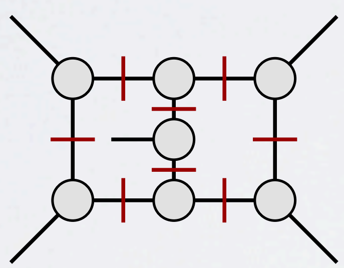
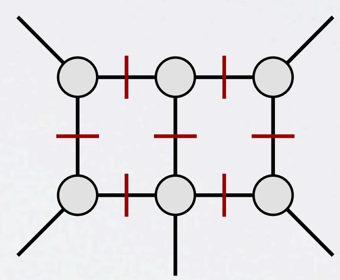
[Carrasco, Johansson, Roiban, Bern, Dixon,...]

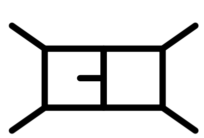
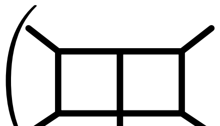
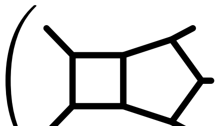
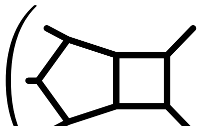
Non-planar from planar

$$A_4(1,2,3,4) = \frac{s_{13}}{s_{12}} A_4(1,3,2,4)$$

factorization

$$\Rightarrow \quad A_3(1,2,-P_{12})A_3(P_{12},3,4) = \underset{s_{12}=0}{\text{Res}}\left(A_4(1,2,3,4)\right) = s_{13}A_4(1,3,2,4)\Big|_{s_{12}=0}$$

$\Rightarrow$  $= (k_1 - P_{123})^2$  $\Big|_{(k_1+k_2+p_3)^2}$

$\Rightarrow$ $\Delta\Big($  $\Big)\Big|_{\text{cut}} = \Big((k_1 - P_{123})^2 \Delta\Big($  $\Big) + \Delta\Big($  $\Big) - \Delta\Big($  $\Big)\Big)\Big|_{\text{cut}}$

Non-planar from planar

$$\Delta_{T_1}\Big|_{\text{cut}_{T_1}} = \prod_{i\in T_1} A_i^{(0)} - \sum_{T'>T_1} \frac{\Delta_{T'}\prod_{\alpha\in T_1} D_\alpha}{\prod_{\alpha\in T'} D_\alpha}\Big|_{\text{cut}_{T_1}} \qquad \Delta_{T_2}\Big|_{\text{cut}_{T_2}} = \prod_{i\in T_2} A_i^{(0)} - \sum_{T'>T_2} \frac{\Delta_{T'}\prod_{\alpha\in T_2} D_\alpha}{\prod_{\alpha\in T'} D_\alpha}\Big|_{\text{cut}_{T_2}} \qquad T_2 \subset T_1$$

use BCJ relations to connect different integrands

$$\prod_{i\in T_2} A_i^{(0)} \overset{\text{BCJ}}{=} f(k_i,p_i) \prod_{i\in T_2} A_i^{(0)}$$

(in general a sum over sub-topologies)

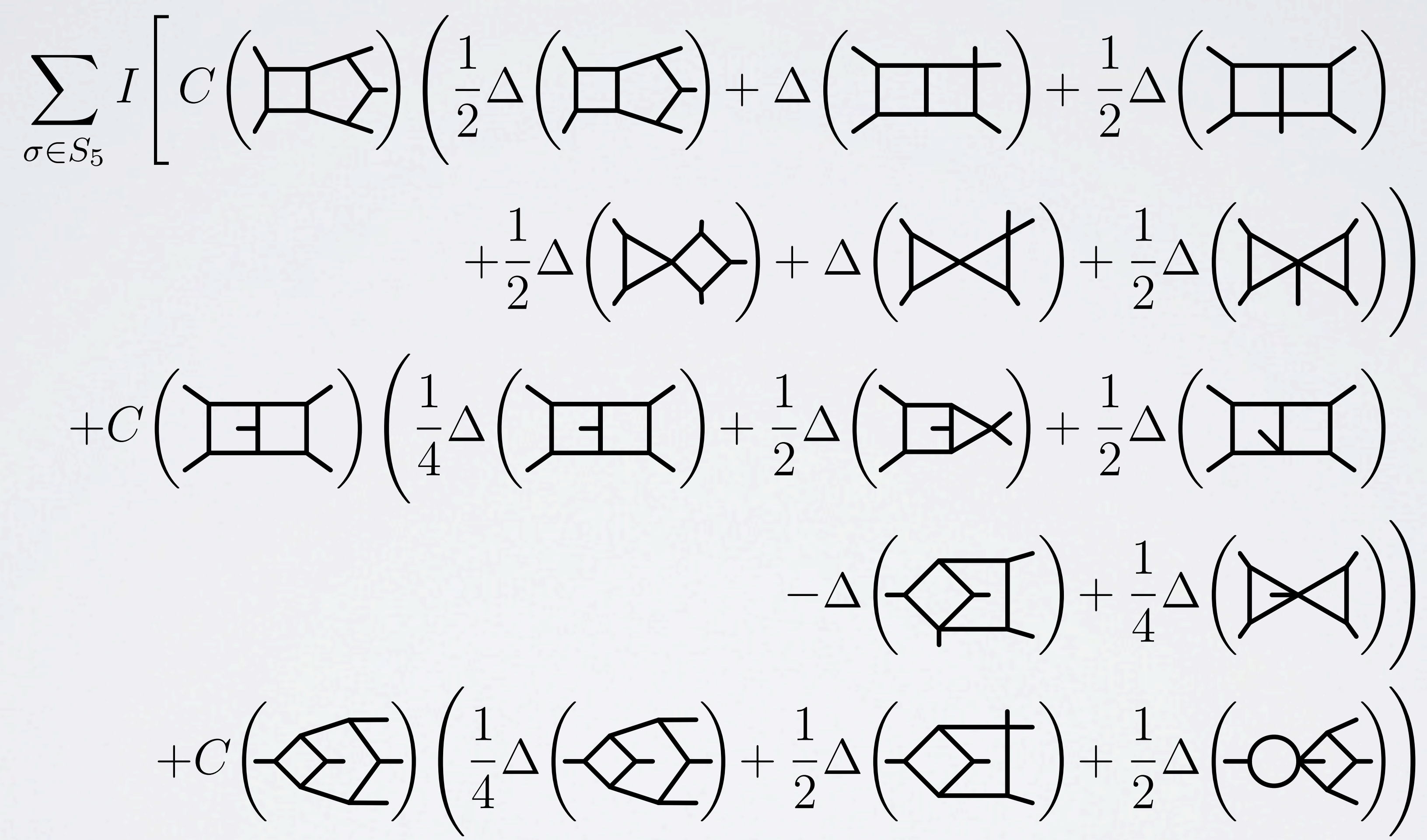
↗ propagators

$$\Rightarrow \quad \Delta_T\Big|_{\text{cut}_{T_1}} = \left\{ f(k_i,p_i)\left(\Delta_{T_2} + \sum_{T'>T_2} \frac{\Delta_{T'}\prod_{\alpha\in T_2} D_\alpha}{\prod_{\alpha\in T'} D_\alpha}\right) - \sum_{T'>T_1} \frac{\Delta_{T'}\prod_{\alpha\in T_1} D_\alpha}{\prod_{\alpha\in T'} D_\alpha} \right\}\Big|_{\text{cut}_{T_1}}$$

Application to the two-loop five-gluon amplitude in QCD

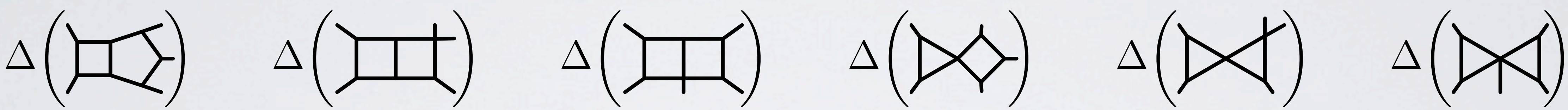
Full colour amplitude

$$\mathcal{A}_5^{(2)}(1^+,2^+,3^+,4^+,5^+) =$$



Planar integrand

[SB, Frellesvig, Zhang arXiv:1310.1051]



- D-dimensional integrand reduction

[Zhang arXiv:1205.5707]

BasisDet Mathematica package
http://www.nbi.dk/~zhang/BasisDet.html

- 6-d spinor helicity formalism (with scalars for full D_s dependence)

[Cheung, O'Connell (2009)]

[Bern, Carrasco, Dennen, Huang, Ita (2011)]

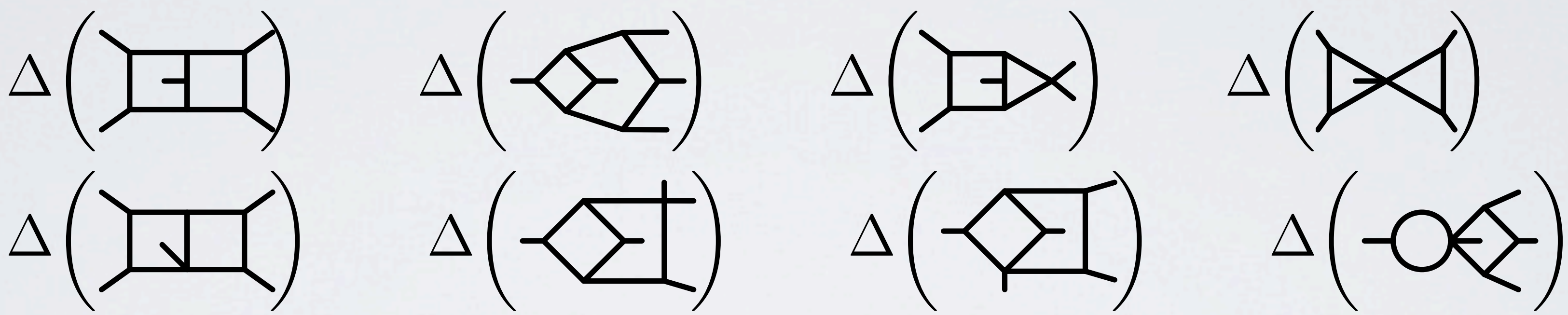
[Davies (2012)]

- Momentum twistor parameterisation to deal with five-point kinematics

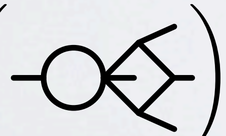
[Hodges (2009)]

Non-planar results

[SB, Mogull, Ochirov, O'Connell arXiv:1507.xxxxx]



- All topologies related to planar cuts via BCJ
- Off-shell symmetries imposed so all KK relations satisfied
- Compact analytic expressions for all cases

$\Delta\Big($  $\Big) = \dfrac{i\,\mathrm{tr}_+(1345)F_3}{2\langle 12\langle\rangle 23\rangle\langle 34\langle\rangle 45\rangle\langle 51 s\rangle_{13} s_{45}} \times$

$$\left(s_{12}s_{23} + 2s_{12}k_1\cdot\omega_{123} + (s_{45}-s_{12})(k_1-p_1)^2 + (s_{45}-s_{23})(k_1-p_{12})^2\right)$$

Non-planar results

Connection with $\mathcal{N} = 4$ continues for non-planar sector

$$F_1(k_1^{[-2\epsilon]}, k_2^{[-2\epsilon]}) \equiv (D_s - 2)\big(\mu_{11}\mu_{22} + (\mu_{11} + \mu_{22})^2 + (\mu_{11} + \mu_{22})\mu_{12}\big) + 16(\mu_{12}^2 - \mu_{11}\mu_{22})$$
$$F_2(k_1^{[-2\epsilon]}, k_2^{[-2\epsilon]}) \equiv 2(D_s - 2)(\mu_{11} + \mu_{22})\mu_{12} = F_1(k_1^{[-2\epsilon]}, k_2^{[-2\epsilon]}) - F_1(k_1^{[-2\epsilon]}, -k_2^{[-2\epsilon]})$$
$$F_3(k_1^{[-2\epsilon]}, k_2^{[-2\epsilon]}) \equiv (D_s - 2)^2\mu_{11}\mu_{22}$$

$$A^{(2)}_{+++++} = \frac{F_1}{\delta^{(8)}} A^{(2)}_{\mathcal{N}=4} + A^{(2)}_{(\text{one-loop})^2} \qquad \Delta_{(\text{one-loop})^2} = AF_2 + BF_3$$

[5-point $\mathcal{N}$=4 BCJ numerator
Carrasco, Johansson arXiv:1106.4711]

Colour decomposition is in agreement with
Carrasco-Johansson numerator representation

Infrared behaviour

five-point two-loop integrals 'unknown'... **[see Henn's Talk]**

Check universal IR pole structure in planar case numerically

Mellin-Barnes and Sector decomposition

[**Fiesta** Smirnov, Smirnov, Tentyukov]

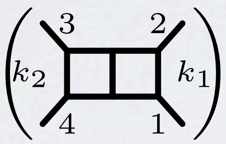
[**SecDec** Borowka, Carter, Heinrich]

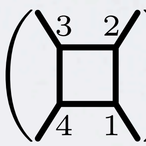
$$\mathcal{A}^{(2)}(1^+,2^+,3^+,4^+,5^+) = \sum_{i>j} \frac{c_\Gamma}{\epsilon^2}\left(\frac{\mu_R^2}{-s_{ij}}\right)^{\epsilon} T_i \cdot T_j \circ \mathcal{A}^{(1)}(1^+,2^+,3^+,4^+,5^+) + \frac{11N_c}{3}\mathcal{A}^{(1)}(1^+,2^+,3^+,4^+,5^+) + \mathcal{O}(\epsilon^0)$$

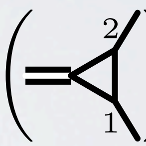
Integrals in the soft limit

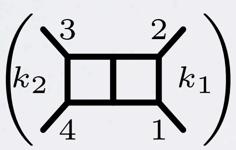
$F_1 = (D_s - 2)(2\mu_{11}\mu_{22} + \mu_{11}^2 + \mu_{22}^2 + \mu_{12}(\mu_{11} + \mu_{22})) + 16(\mu_{12}^2 - \mu_{11}\mu_{22})$

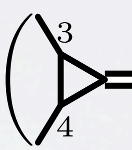
$$\lim_{k_1 \to 0} F_1 = (D_s - 2)\mu_{22}^2$$

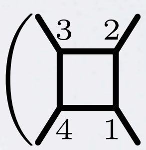
$I^{4-2\epsilon}$ 

$[F_1] \overset{k_1 \to 0}{\to} (D_s - 2) I^{4-2\epsilon}$ 

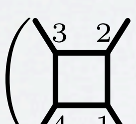
$[\mu_{22}^2]\, I^{4-2\epsilon}$ 

$I^{4-2\epsilon}$ 

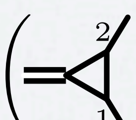
$[F_1] \overset{k_2 \to 0}{\to} (D_s - 2) I^{4-2\epsilon}$ 

$I^{4-2\epsilon}$ 

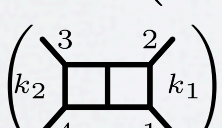
$[\mu_{11}^2]$

$I^{4-2\epsilon}$ 

$[\mu_{11}^2] = -\frac{1}{6} + \mathcal{O}(\epsilon)$

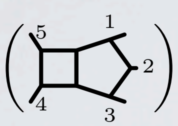
$I^{4-2\epsilon}$ 

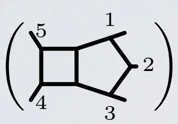
$= \frac{c_\Gamma}{\epsilon^2}(-s_{12})^{-1-\epsilon} = -\frac{1}{(4\pi)^2 s_{12}\epsilon^2} + \mathcal{O}(\epsilon^{-1})$

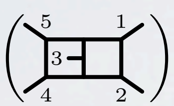
$\Rightarrow I^{4-2\epsilon}$ 

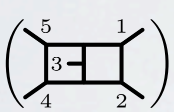
$[F_1] = \frac{D_s - 2}{(4\pi)^2 3\epsilon^2 s_{12}} + \mathcal{O}(\epsilon^{-1})$

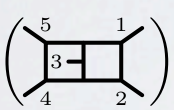
Integrals in the soft limit

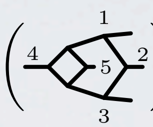
$I^{4-2\epsilon}\Big($  $\Big)[F_1] = -\frac{D_s-2}{(4\pi)^2 3 s_{12}s_{23}\epsilon^2} + \mathcal{O}(\epsilon^{-1})$

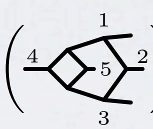
$I^{4-2\epsilon}\Big($  $\Big)[F_1 2k_1\cdot p_5] = -\frac{(D_s-2)(2s_{15}+s_{25})}{(4\pi)^2 6 s_{12}s_{23}\epsilon^2} + \mathcal{O}(\epsilon^{-1})$

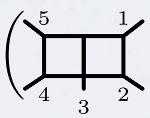
$I^{4-2\epsilon}\Big($  $\Big)[F_1] = \mathcal{O}(\epsilon^{-1})$

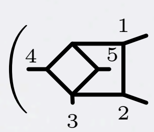
$I^{4-2\epsilon}\Big($  $\Big)[F_1\, 2k_1\cdot(p_5-p_4)] = \mathcal{O}(\epsilon^{-1})$

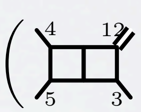
$I^{4-2\epsilon}\Big($  $\Big)[F_1\, 2(k_1-k_2)\cdot p_3] = \mathcal{O}(\epsilon^{-1})$

$I^{4-2\epsilon}\Big($  $\Big)[F_1] = -\frac{D_s-2}{(4\pi)^2 3 s_{12}s_{23}\epsilon^2} + \mathcal{O}(\epsilon^{-1})$

$I^{4-2\epsilon}\Big($  $\Big)[F_1 2k_1\cdot(p_5-p_4)] = -\frac{(D_s-2)(s_{15}-s_{14}+s_{34}-s_{35})}{(4\pi)^2 6 s_{12}s_{23}\epsilon^2} + \mathcal{O}(\epsilon^{-1})$

$I^{4-2\epsilon}\Big($  $\Big)[F_1] = -\frac{D_s-2}{(4\pi)^2 6\epsilon^2}\left(\frac{1}{s_{12}}+\frac{1}{s_{45}}\right) + \mathcal{O}(\epsilon^{-1})$

$I^{4-2\epsilon}\Big($  $\Big)[F_1] = -\frac{D_s-2}{(4\pi)^2 6 s_{12}\epsilon^2} + \mathcal{O}(\epsilon^{-1})$

$I^{4-2\epsilon}\Big($  $\Big)[F_1] = -\frac{D_s-2}{(4\pi)^2 6 s_{45}\epsilon^2} + \mathcal{O}(\epsilon^{-1})$

full colour amplitude correctly
reproduces the expected behaviour

Open problems/work in progress

- Can we find a complete BCJ numerator (i.e. satisfy BCJ off-shell)

[Mogull, O' Connell (in progress)]

- Minimal missing information from IBPs?
  - Though we avoided the need for additional simplifications here it will be important for the more general amplitudes

- New developments exploiting algebraic geometry coming all the time

[“residues with doubled propagators”, Søgaard, Zhang 1403.2463]

[“massive internal states”, Søgaard, Zhang 1412.5577]

[“cross-order relations in maximal unitarity”, Johansson, Kosower, Larsen, Søgaard 1503.06711]

[“IBPs from differential geometry”, Zhang 1400.4004]

Conclusions

- multi-loop amplitudes from tree-amplitudes
  - KK and BCJ relations can be applied systematically to decompose amplitudes into minimal set of irreducible numerators
  - simple colour decompositions using the DDM basis
- First non-trivial application
  - two-loop five-gluon amplitude in QCD with all positive helicities

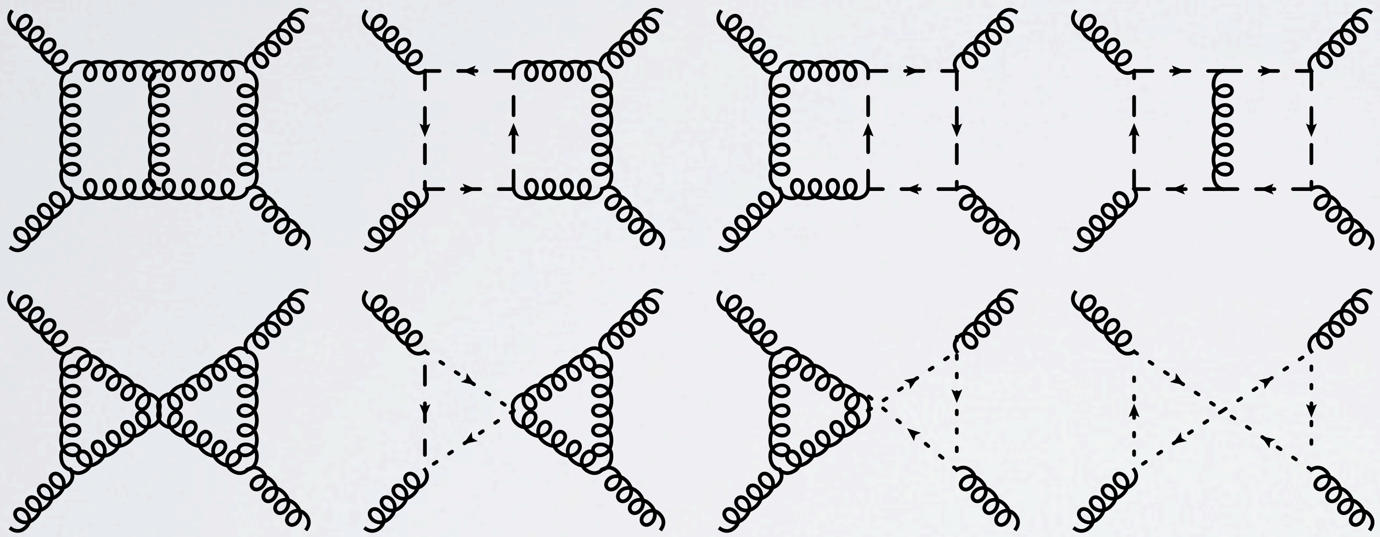
[SB, Mogull, Ochirov, O'Connell arXiv:1507.xxxxx]

Backup Slides

Numerator construction

FDH scheme at two-loops

[Bern, De Freitas, Dixon, Wong (2002)]



$$g^{\mu}_{\mu} = D_s$$

Feynman rules + Feynman gauge and ghosts (scalars)

Tree-amplitudes using
six-dimensional helicity method

need to capture $\mu_{11}, \mu_{22}, \mu_{12}$

[Cheung, O'Connell (2009)]

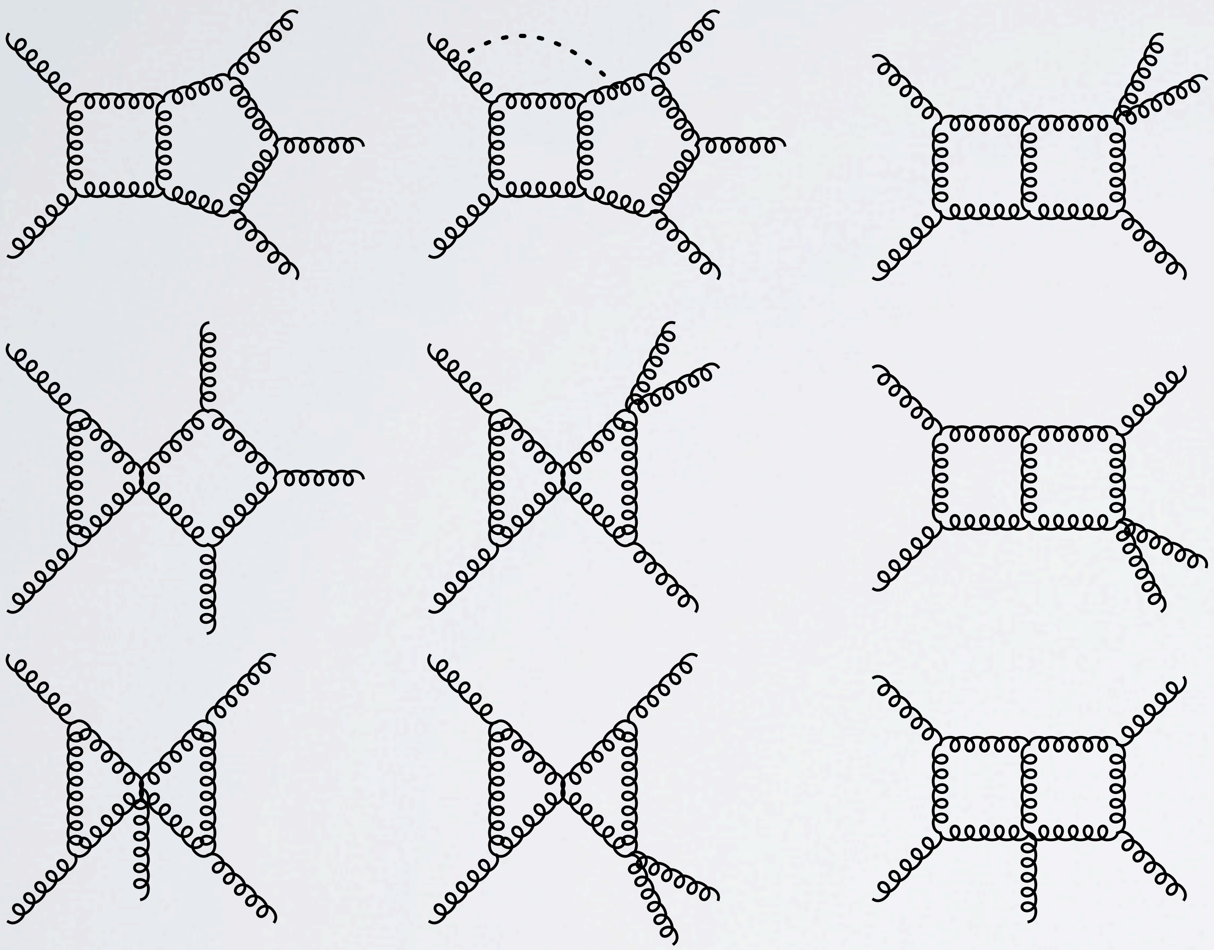
[Bern, Carrasco, Dennen, Huang, Ita (2011)]

[Davies (2012)]

use momentum twistors to deal with the complicated kinematics at 2→3

planar five-gluon integrand representation

[SB, Frellesvig, Zhang (2013)]



only ≥6 propagator topologies

$$
\begin{aligned}
c_{431} &= -\frac{s_{12}s_{23}s_{34}s_{45}^2s_{15}}{\mathrm{tr}_5}\,, & c_{431}^T &= -\frac{s_{12}s_{23}s_{45}\,\mathrm{tr}_+(1345)}{\mathrm{tr}_5}\,,\\
c_{331;M_1} &= -\frac{s_{34}s_{45}^2\,\mathrm{tr}_+(1235)}{\mathrm{tr}_5}\,, & c_{331;M_2} &= -\frac{s_{15}s_{45}^2\,\mathrm{tr}_-(1234)}{\mathrm{tr}_5}\,,\\
c_{331;5L} &= \frac{s_{12}s_{23}s_{34}s_{45}s_{15}}{\mathrm{tr}_5}\,, & c_{430} &= -\frac{s_{12}\,\mathrm{tr}_+(1345)}{2s_{13}s_{45}}\,,\\
c_{330;M_1} &= -\frac{(s_{45}-s_{12})\,\mathrm{tr}_+(1345)}{2s_{13}s_{45}}\,, & c_{330;M_2} &= -\frac{(s_{45}-s_{23})\,\mathrm{tr}_+(1345)}{2s_{13}s_{45}}\,,\\
c_{330;5L}^b &= \frac{\mathrm{tr}_+(1235)}{2s_{35}s_{12}}\,, & c_{330;5L}^c &= \frac{\mathrm{tr}_+(1345)}{2s_{13}s_{45}}\,,
\end{aligned}
$$

$$
c_{330;5L}^a = -\frac{1}{2}\left(\mathrm{tr}_+(1245) - \frac{\mathrm{tr}_+(1235)\,\mathrm{tr}_+(1345)}{s_{13}s_{35}}\right),
$$

$$
c_{330;5L}^d = c_{330;5L}^a\frac{s_{12}+s_{45}}{s_{12}s_{45}} - s_{12}c_{330;5L}^b - s_{45}c_{330;5L}^c - s_{15}\,,
$$

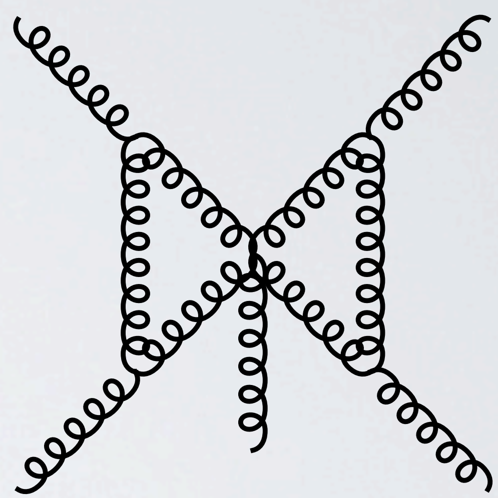
+ spurious terms

choice of basis important to find simplest form

double-box type topologies are $\mathcal{N}$ = 4 x $(\mu_{11}\mu_{22}+\mu_{22}\mu_{33}+\mu_{33}\mu_{11}) + 4(\mu_{12}^2 - 4\mu_{11}\mu_{22})$

Choices of integrand basis

$$\Delta_{330;5L}(1^+,2^+,3^+,4^+,5^+) = -\frac{i}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\times$$
$$\Bigg(\frac{1}{2}\left(\mathrm{tr}_+(1245) - \frac{\mathrm{tr}_+(1345)\mathrm{tr}_+(1235)}{s_{13}s_{35}}\right)\Bigg(2(D_s-2)(\mu_{11}+\mu_{22})\mu_{12}$$
$$+ (D_s-2)^2\mu_{11}\mu_{22}\frac{4(k_1\cdot p_3)(k_2\cdot p_3) + (k_1+k_2)^2(s_{12}+s_{45}) + s_{12}s_{45}}{s_{12}s_{45}}\Bigg)$$
$$+ (D_s-2)^2\mu_{11}\mu_{22}\Bigg[(k_1+k_2)^2 s_{15}$$
$$+ \mathrm{tr}_+(1235)\left(\frac{(k_1+k_2)^2}{2s_{35}} - \frac{k_1\cdot p_3}{s_{12}}\left(1 + \frac{2(k_2\cdot\omega_{453})}{s_{35}} + \frac{s_{12}-s_{45}}{s_{35}s_{45}}(k_2-p_5)^2\right)\right)$$
$$+ \mathrm{tr}_+(1345)\left(\frac{(k_1+k_2)^2}{2s_{13}} - \frac{k_2\cdot p_3}{s_{45}}\left(1 + \frac{2(k_1\cdot\omega_{123})}{s_{13}} + \frac{s_{45}-s_{12}}{s_{12}s_{13}}(k_1-p_1)^2\right)\right)\Bigg]\Bigg)$$



important to identify spurious direction for each loop integral

these are reducible but with this choice five propagator cuts vanish

Q: how to find the best basis? chiral numerators?

Radical ideals

Definition:

for a field $k[\mathbf{x}] = k[x_1, \ldots, x_n]$ and an ideal $I \in k[\mathbf{x}]$
the radical of I is $\sqrt{I} = \{f \in k[\mathbf{x}] \,|\, f^m \in I, m \in \mathbb{N}\}$

An ideal is a ***radical ideal*** if $\sqrt{I} = I$

Algorithms to compute the radical of an ideal are available in Macaulay2

sketch proof that D-dimensional propagator ideals are radical

at 2-loops there are $P-3$ linear relations
leading to $m = 11 - P$ ISPs of the form x_{ij}

$$I = \langle \mu_{11} - f_1(x_1, \dots x_m), \quad \mu_{12} - f_2(x_1, \dots x_m), \quad \mu_{22} - f_3(x_1, \dots x_m) \rangle$$

we have an isomorphism

$$\phi : \mathbb{C}[x_1, \dots x_m, \mu_{11}, \mu_{12}, \mu_{22}]/I \to \mathbb{C}[x_1, \dots x_m]$$

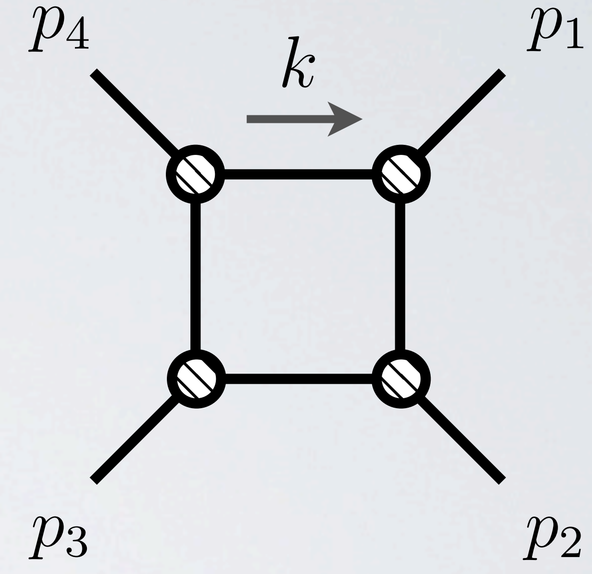
with $\mu_{11} \mapsto f_1(x_1, \dots x_m)$, $\mu_{12} \to f_2(x_1, \dots x_m)$ and $\mu_{22} \to f_3(x_1, \dots x_m)$

$\mathbb{C}[x_1, \dots x_m]$ is a domain $\Rightarrow I$ is a prime ideal

prime ideal are radical ideals

One-loop box example

$$P = \langle x_{14}^2 - \mu_{11} - stu, x_{11}, x_{12}, x_{13} \rangle$$



$$\Delta_4 = c_0 + c_1 x_{14} + c_2 \mu_{11} + c_3 \mu_{11} x_{14} + c_4 \mu_{11}^2$$

$$\bar{k}^\mu = \frac{s(1+\tau)}{4\langle 4|2|1]}\langle 4|\gamma^\mu|1] + \frac{s(1-\tau)}{4\langle 1|2|4]}\langle 1|\gamma^\mu|4]$$

$$x_{14} = \frac{st}{2}\tau \qquad \mu_{11} = -\frac{st}{4u}(1-\tau^2)$$

$$\Delta_4(k(\tau)) = \sum_{i=0}^{4} d_i \tau^i$$

$$\begin{pmatrix} 1 & -\frac{t}{2} & 0 & 0 & 0 \\ 0 & t & -\frac{st}{u} & \frac{st^2}{2u} & 0 \\ 0 & 0 & \frac{st}{u} & -\frac{3st^2}{2u} & \frac{s^2t^2}{u^2} \\ 0 & 0 & 0 & \frac{st^2}{u} & -\frac{2s^2t^2}{u^2} \\ 0 & 0 & 0 & 0 & \frac{s^2t^2}{u^2} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}$$

continue reduction with subtractions

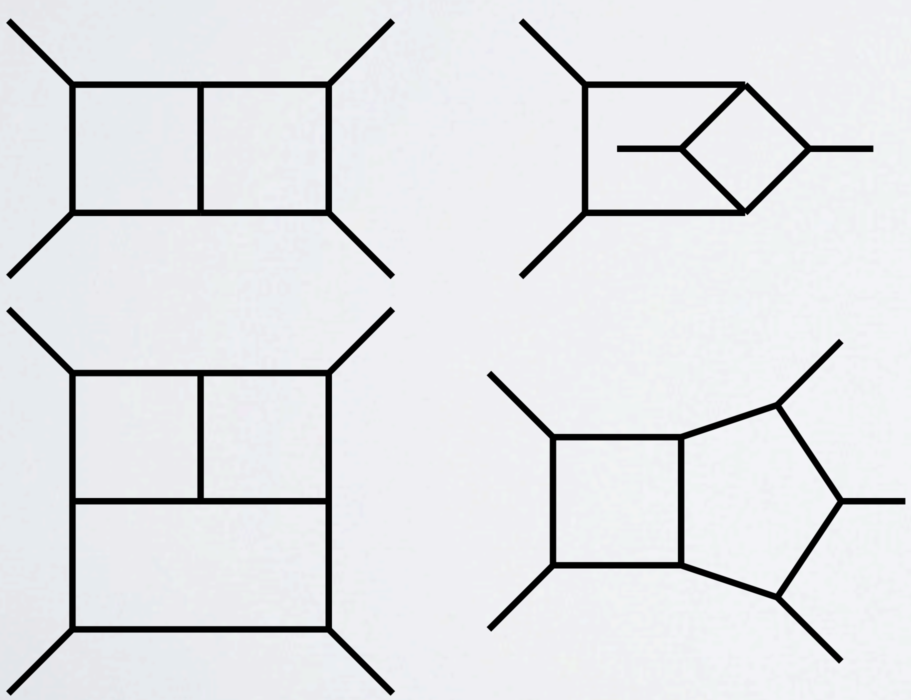
$$\Delta_{3;123}(k(\tau_1,\tau_2)) = N(k(\tau_1,\tau_2), p_1, p_2, p_3, p_4) - \frac{\Delta_4(k(\tau_1,\tau_2))}{(k(\tau_1,\tau_2)+p_4)^2}$$

Multi-loop integrand parametrization

automated computation of integrand basis for each topology

[Zhang arXiv:1205.5707]

BasisDet Mathematica package
http://www.nbi.dk/~zhang/BasisDet.html



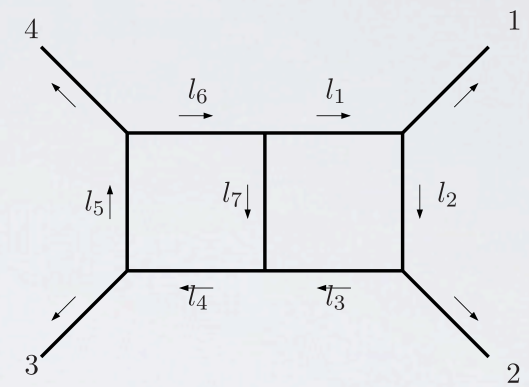
determination of all on-shell branches using primary decomposition

Macaulay2: http://www.math.uiuc.edu/Macaulay2/

complex multi-loop structures investigated in [Huang, Zhang arXiv:1302.1023]

4D Examples

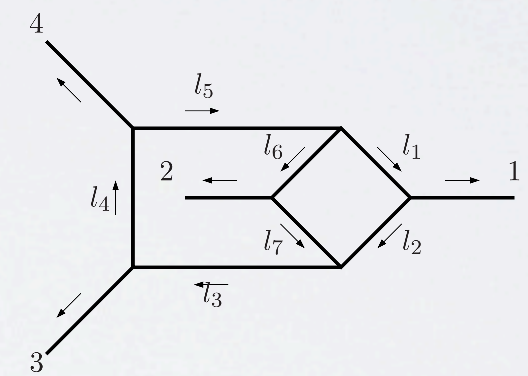
SB,Frellesvig,Zhang
[1202.2019], [1207.2976]



M ~ 38 x 32

2 MIs

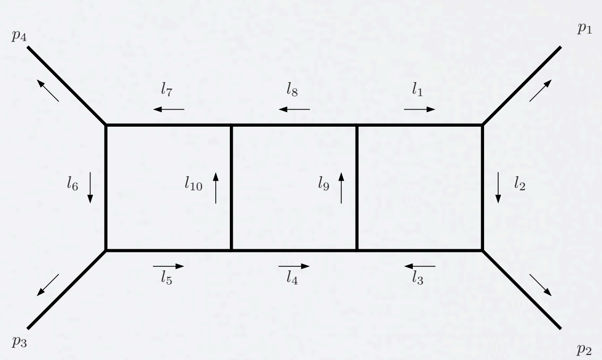
planar and non-planar
hepta-cuts at two loops



M ~ 48 x 38

2 MIs

planar triple box at
three loops



M ~ 622 x 398

3 MIs

also with maximal unitarity : Kosower, Larsen [1108.1180], Caron-Huot, Larsen [1205.0801], Kosower Larsen, Johansson [1208.1754, 1308.4632], Søgaard [1306.1496] Zhang, Søgaard [1310.6006, 1406.5044]

D-dimensional reduction

Is the integrand system well defined? will there linear system always have a solution?

complications 4-d

an ISP monomial vanishes on all on-shell solutions

i.e. ideal is not *radical*

different on-shell solutions have different dimensions

i.e. integrand systems with different numbers of propagators may need to be solved simultaneously

in D-d

all propagator ideals are *radical*

all integrands have exactly one on-shell branch

More examples

Both the order of the polynomial division and choice of spanning basis affect the simplicity of the representation

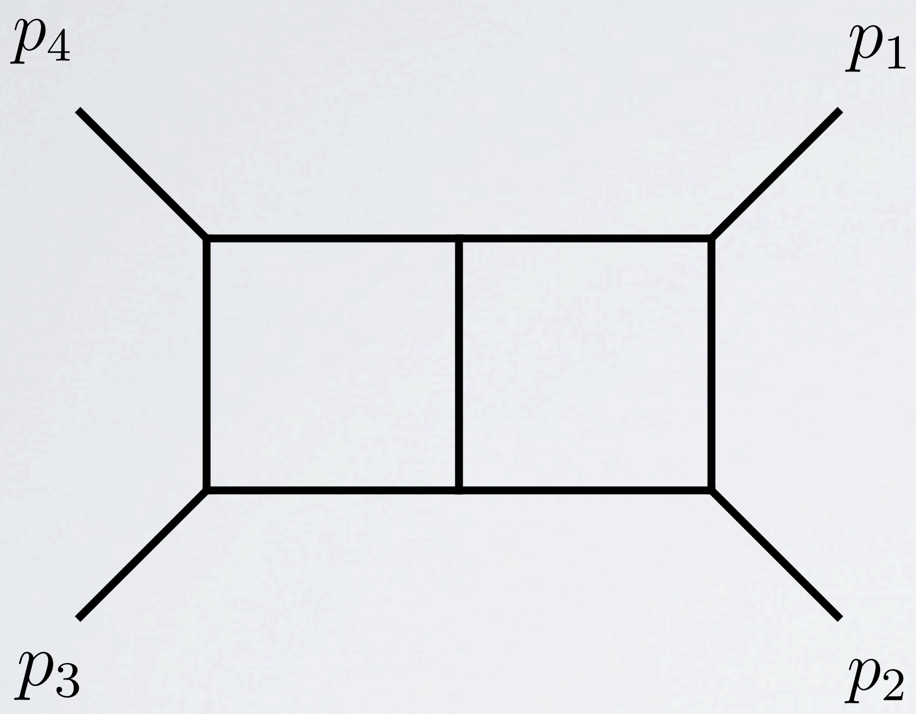
Dimension shifted integrals e.g. one-loop pentagon

$$I = \langle \mu_{11} - \text{ const} \rangle \Rightarrow \Delta_5 = c_0 \text{ or } \Delta_5 = c_0 \mu_{11} \qquad I_5[\mu_{11}] = \mathcal{O}(\epsilon)$$

Vanishing integrals: e.g. one-loop triangles

$$I = \langle \mu_{11} + (k_1 \cdot \omega_1)^2 + (k_1 \cdot \omega_1)^2 - \text{ const} \rangle \Rightarrow I_3[(k_1 \cdot \omega_1)^2 - (k_1 \cdot \omega_2)^2] = 0$$

Two-loop example



$$v = \{p_1, p_2, p_4, \omega_{124}\}$$
$$\text{ISP} = \{x_{13}, x_{21}, x_{14}, x_{24}, \mu_{11}, \mu_{12}, \mu_{22}\}$$

32 spurious terms

38 non-spurious terms

However: $k_1 \leftrightarrow k_2$ symmetry leaves 22 independent integrals

remaining IBPs from shift invariance

only 17 remain as $D \to 4$

$\mathcal{O}(\epsilon^{-4})$	8
$\mathcal{O}(\epsilon^{-2})$	4
$\mathcal{O}(\epsilon^{-1})$	4
$\mathcal{O}(1)$	1
$\mathcal{O}(\epsilon)$	5

“all-plus” amplitudes in QCD

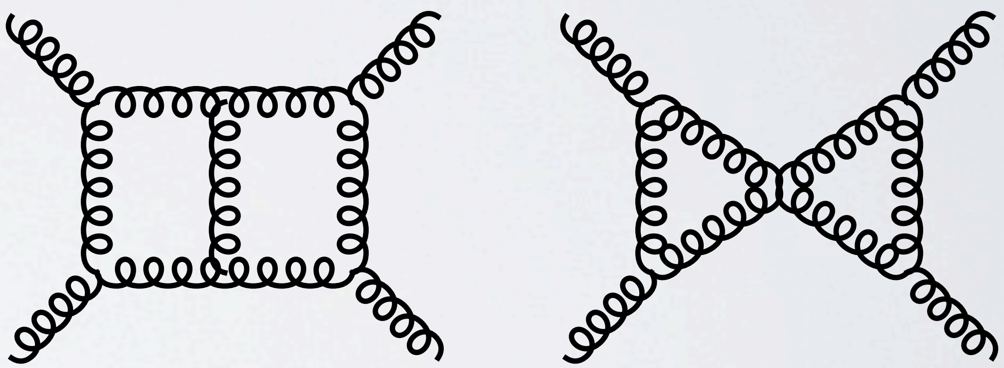
one-loop amplitudes only contain boxes. e.g.

$$A_4^{(1)}(1^+,2^+,3^+,4^+) = \frac{i\mathrm{tr}_+(1234)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} I_{4;1234}[(D_s-2)\mu_{11}^2]$$

$$\mathrm{tr}_+(1234) = [12]\langle 23\rangle[34]\langle 41\rangle$$

[Bern, Dixon, Dunbar, Kosower (1996)]

two-loop four-point also has simple form



$$A_4{}^{(2)}(1^+,2^+,3^+,4^+) = \frac{-i}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\Big($$
$$s^2 t\, I_{7;12*34*}[(\mu_{11}\mu_{22}+\mu_{22}\mu_{33}+\mu_{33}\mu_{11}) + 4(\mu_{12}^2 - 4\mu_{11}\mu_{22})]$$
$$+ t\, I_{6;12*34}[(D_s-2)(\mu_{11}+\mu_{22})\mu_{12}s + (D_s-2)^2\mu_{11}\mu_{22}((k_1+k_2)^2+s)/s]\Big)$$

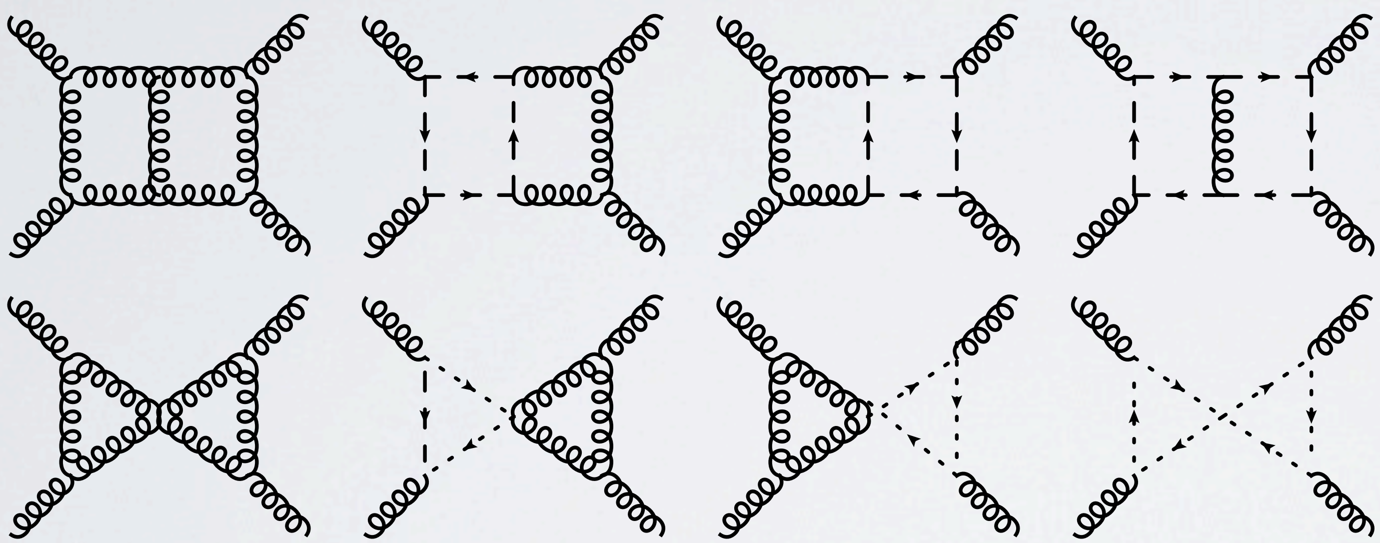
$$\mu_{33} = \mu_{11}+\mu_{22}+\mu_{12}$$

[Bern, Dixon, Kosower (2000)]

Numerator construction

FDH scheme at two-loops

[Bern, De Freitas, Dixon, Wong (2002)]



$$g^{\mu}_{\mu} = D_s$$

Feynman rules + Feynman gauge and ghosts (scalars)

Tree-amplitudes using **six-dimensional** helicity method

[Cheung, O'Connell (2009)]

[Bern, Carrasco, Dennen, Huang, Ita (2011)]

[Davies (2012)]

need to capture $\mu_{11}, \ \mu_{22}, \ \mu_{12}$

whichever way we choose we need a good way to deal with complicated kinematics

Momentum twistors at higher multiplicity

$$Z = \begin{pmatrix} 1 & 0 & f_1 & f_2 & f_3 & \dots & f_{n-3} & f_{n-2} \\ 0 & 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 0 & 0 & \frac{x_{n-1}}{x_2} & x_n & \dots & x_{2n-6} & 1 \\ 0 & 0 & 1 & 1 & x_{2n-5} & \dots & x_{3n-11} & 1 - \frac{x_{3n-10}}{x_{n-1}} \end{pmatrix}$$

$$f_i = \sum_{k=1}^{i} \frac{1}{\prod_{l=1}^{k} x_l}$$

$$x_i = \begin{cases} s_{12} & i = 1 \\ -\frac{\langle i\, i{+}1\rangle\langle i{+}2\, 1\rangle}{\langle 1\, i\rangle\langle i{+}1\, i{+}2\rangle} & i = 2, \dots, n-2 \\ \delta_{n,4} + (1-\delta_{n,4})\frac{s_{23}}{s_{12}} & i = n-1 \\ -\frac{[2|P_{2,i-n+4}|i{-}n{+}5\rangle}{[21]\langle 1|\, i{-}n{+}5\rangle} & i = n, \dots, 2n-6 \\ \frac{\langle 1|P_{23}P_{2,i-2n+9}|i{-}2n{+}10\rangle}{s_{23}\langle 1|\, i{-}2n{+}10\rangle} & i = 2n-5, \dots, 3n-11 \\ \frac{s_{123}}{s_{12}} & i = 3n-10 \end{cases}$$

We can find an (**invertible**) representation for arbitrary number of massless particles