Large spin systematics in CFT

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Introduction

- In the last few years there has been an increasing interest in conformal field theories in space time dimensions higher than two. Much of this interest is due to the effectiveness of the conformal bootstrap program

  [Rattazzi, Rychkov, Tonni, Vichi, 2008]

- This approach consists in constraining the CFT data by requiring consistency of higher point correlation functions, as crossing-symmetry, together with basic properties of well behaved CFT's, such as unitarity and the structure of the OPE.
- In this talk I will discuss how to study analytically the structure of the dimension and OPE coefficients of operators with spin, in the large spin limit.

Motivations

Why such operators are interesting?

- The cusp anomalous dimension controls IR singularities of scattering amplitudes as well as UV singularities of Wilson loops with cusps
- They appear in deep inelastic scattering processes in QCD
- They play an important role in the AdS/CFT correspondence, expecially in integrability (see Amit's talk)

Operators

- If we consider operators made of scalar fields and derivatives as

$$\text{tr}\left(\varphi\partial_{\mu_1}\ldots\partial_{\mu_\ell}\varphi\right)+\ldots$$

their scaling dimension shows a logarithmic behavior

$$\Delta_\ell-\ell=\Gamma(g)\log\ell+...$$

for large ℓ, $\Gamma(g)$ is the cusp anomalous dimension. Note that this expression is valid at any loop order.

[Alday and Maldacena, 2008, Alday and Zhiboedov, 2015]

- What about the other terms in the large ℓ expansion?

Reciprocity

- Consider operators of twist two, namely $\Delta_\ell - \ell = \gamma_\ell + 2$
- Introduce a function f in such a way

$$\gamma_\ell = f(\ell + \frac{1}{2}\gamma_\ell)$$

and then reciprocity is the statement that

$$f(\ell) = \sum_n \frac{a_n(\log J_b)}{J_b^{2n}}$$

where $J_b^2 = \ell(\ell+1)$

- Firstly it has been observed in QCD and then checked for other theories (also supersymmetric).

[Moch, Vermaseren, Mogt, Gribov, Lipatov, Drell, Levy, Yan, Dokshitzer, Marchesini, Salam, Basso, Korchemski, Beccaria, Forini, Tirziu, Tseytlin...]

Aim of the talk

The aim of this talk is to prove the reciprocity principle, using an approach based on the structure of the conformal partial wave decomposition, analyticity, unitarity and crossing symmetry.

We will see that this approach is valid perturbatively (at any loop order) and non perturbatively and it will provide constraints for the large spin expansion of the anomalous dimension and of the OPE coefficients of high spin operators.

Conformal algebra

The conformal group contains

- Translations: P_μ
- Lorentz transformations: $M_{\mu\nu}$
- Scale transformations: D
- Special conformal transformations: K_μ

Part of the conformal algebra is

$$[D, K_\mu] = -K_\mu$$
$$[D, P_\mu] = P_\mu$$
$$[P_\mu, K_\nu] = \eta_{\mu\nu} D - i M_{\mu\nu}$$

Primary and descendants

The behaviour of a field $\phi(0)$ under dilatations and special conformal transformations is

$$[M_{\mu\nu}, \phi(0)] = \Sigma_{\mu\nu}\phi(0) \to \text{SPIN}$$
$$[D, \phi(0)] = \Delta\phi(0) \to \text{DIMENSION}$$
$$[K_\mu, \phi(0)] = 0 \to \text{PRIMARY FIELD}$$

- By acting with P_μ on a primary → DESCENDANTS
- Operators form an algebra (OPE)

$$\phi_i(x)\phi_j(0) = \sum_k c_{ijk}|x|^{\Delta_k-\Delta_i-\Delta_j}\phi_k(0)$$

on the rhs there are primaries and all the descendants.

2 and 3 pt functions

- All the information of a CFT is encoded in the set of dimensions and structure constants of local operators
- Conformal symmerty fixes the space-time dependence of 2 and 3 point functions. If we consider primary scalar operators:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{\delta_{12}}{x_{12}^{2\Delta}}$$

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{c_{123}}{|x_{12}|^{\Delta_{123}}|x_{23}|^{\Delta_{231}}|x_{13}|^{\Delta_{132}}}$$

where $\Delta_{ijk} = \Delta_i + \Delta_j - \Delta_k$

Four point function

- For the case of four point function, conformal symmetry does not fix the full coordinate dependence.
  The four point function of identical scalar primaries with dimension Δ_ϕ takes this form

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{\mathcal{G}(u,v)}{|x_{12}|^{2\Delta_\phi}|x_{34}|^{2\Delta_\phi}}$$

  where $\mathcal{G}(u,v)$ is a function of the conformal invariant cross-ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

- Note: the LHS is invariant under the exchange of pairs of points $\left(\mathcal{G}(u,v) = \left(\frac{u}{v}\right)^{\Delta_\phi} \mathcal{G}(v,u)\right)$)

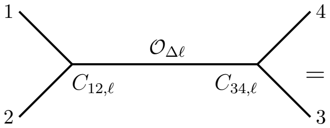
Conformal block decomposition

- Use the OPE

$$\phi(x_1)\phi(x_2) = |x_{12}|^{-2\Delta_\phi}\left(1 + \sum_k C_{\phi\phi k}|x_{12}|^{\Delta_k} O_k(x_1)\right)$$

in $\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$

$$\sum_{\Delta\ell}$$



$$= \mathcal{G}(u,v)$$

$$\mathcal{G}(u,v) = 1 + \sum_{\ell,\Delta} c^2_{\Delta,\ell}\, g_{\Delta,\ell}(u,v)$$

Conformal blocks I

Considering the OPE $\phi(x_1) \times \phi(x_2)$ we can write

$$\mathcal{G}(u,v) = 1 + \sum_{\ell,\Delta} a_{\Delta,\ell} g_{\Delta,\ell}(u,v)$$

- the first term is the contribution of the identity operator, which is always present in the OPE of two identical operators
- the sum runs over the the tower of primaries present in the OPE
- ℓ and Δ denote the spin and the dimension of the intermediate primary
- $a_{\Delta,\ell} = c^2_{\Delta,\ell}$ is the square of the structure constants and is non-negative due to unitarity
- $g_{\Delta,\ell}(u,v)$ are the conformal blocks...

Conformal blocks II

...conformal blocks

- repack the contributions of all descendants of a given primary
- transform under the conformal group in the same way as the four point function
- depend on the spin and the dimension of the intermediate state and on the dimension of the external operator
- are known in a closed form

[Dolan and Osborn, 2005]

$$
\begin{aligned}
G_{\Delta,\ell}(u,v) &= u^{\frac{1}{2}(\Delta-\ell)} g_{\Delta,\ell}(z,\bar{z}) \qquad u = z\bar{z} \quad v = (1-z)(1-\bar{z}) \\
g_{\Delta,\ell}(z,\bar{z}) &= (-1)^{\ell} \frac{1}{z-\bar{z}} \left(k_{\Delta+\ell}(z) k_{\Delta-\ell-2}(\bar{z}) - k_{\Delta+\ell}(\bar{z}) k_{\Delta-\ell-2}(z) \right) \\
k_{\beta}(z) &= z^{\beta/2+1} F_{\beta/2}(z), \qquad F_{\beta/2}(z) = {}_2F_1\left(\frac{\beta}{2}, \frac{\beta}{2}, \beta, z\right)
\end{aligned}
$$

Conformal blocks III

- In order to single out the contribution of leading twist operators $\rightarrow$ small u limit

$$\sum_{\Delta,\ell} a_{\Delta,\ell} G_{\Delta,\ell}(u,v) = \mathcal{G}(u,v)$$

small u limit

$$\sum_{\ell=0,2,\dots} a_\ell u^{\frac{\Delta-\ell}{2}} (1-v)^\ell F_{\frac{\Delta+\ell}{2}}(1-v)$$

- The power of u is controlled by the twist $\Delta - \ell$.
- In perturbation theory $\Delta_\ell - \ell = 2 + \gamma_\ell$ and $\mathcal{G}(u,v) = 1 + uh(\log u.v)$

$$\sum_{\ell=0,2,\dots} a_\ell u^{1+\gamma_\ell/2} (1-v)^\ell F_{\ell+1+\gamma_\ell/2}(1-v) = uh(\log u, v)$$

Casimir operator

- The structure of conformal blocks is fully fixed by conformal symmetry and they are eigenfunctions of Casimir operators
- For small u we can define a Casimir operator

$$\mathcal{D} = (1-v)^2\partial_v - u(1-v)\partial_u + v(1-v)^2\partial_v^2 + vu^2\partial_u^2 - 2uv(1-v)\partial_u\partial_v$$

that satisifes

$$\boxed{\mathcal{D}\left(u^{\frac{\Delta-\ell}{2}}(1-v)^\ell F_{\frac{\Delta+\ell}{2}}(1-v)\right) = J^2\left(u^{\frac{\Delta-\ell}{2}}(1-v)^\ell F_{\frac{\Delta+\ell}{2}}(1-v)\right)}$$

with

$$J^2 = \frac{1}{4}\left(\Delta+\ell\right)\left(\Delta+\ell-2\right)$$

- The idea will be to think about the reciprocity principle as an expansion on the eigenvalue of the full Casimir J^2 vs J_b^2.

Perturbative CFT: systematics I

At tree level:

- $\gamma_\ell = 0$ and $a_\ell = a_\ell^{(0)}$ and

$$\sum_{\ell=0,2,\ldots} a_\ell^{(0)} (1-v)^\ell F_{\ell+1}(1-v) = \frac{1}{v} + 1$$

- we can solve for $a_\ell^{(0)}$ and obtain

$$a_\ell^{(0)} = \frac{2\Gamma\left(\ell+1\right)^2}{\Gamma\left(2\ell+1\right)}$$

- Each term in the sum diverges (logarithmically) as $v \to 0$
- The large ℓ behavior of $a_\ell^{(0)}$ is fixed by the divergence $\frac{1}{v}$

Perturbative CFT: systematics II

- In perturbation theory, the four point function $\mathcal{G}(u,v)$ admits an expansion

$$\mathcal{G}(u,v) = \mathcal{G}^{(0)}(u,v) + \boxed{g}\,\mathcal{G}^{(1)}(u,v) + ... \to 1 + uh(\log u, v)$$

  where g is the coupling constant

- Leading twist intermediate primary operators have dimension

$$\Delta_\ell = \ell + 2 + \boxed{\gamma_\ell}$$

  with $\ell = 0, 2, 4, \ldots$

Perturbative CFT: systematics III

- For small v we expect

$$h(\log u, v) \sim \frac{1}{v} h_0(\log u, \log v) + h_1(\log u, \log v) + ...$$

  - $h(\log u, v)$ contains a divergence as v becomes small $\rightarrow$ sum an infinite number of terms!
  - The divergence will come solely from the region $\ell \gg 1 \rightarrow \sum_{\ell=0}^{\infty}$ and $\sum_{\ell=\ell_0}^{\infty}$ produces the same singularity
  - This structure is implied at any order in perturbation theory by analiticity of the tree-level result.
  - The rhs contains ONLY integer powers of v!

Perturbative CFT: systematics IV

In perturbation theory:

- we want to understand how to evaluate the divergence in v of the sum at small v
- the divergence will come from the region of large ℓ

1. Introduce $v = \epsilon \to$ expand in powers of ϵ and

$$\ell = \frac{x}{\epsilon^{1/2}}, \qquad \sum_{\ell} \to \frac{1}{2}\int_0^\infty dx$$

2. Use the integral representation for the hypergeometric function

$$F_{\ell+1+\frac{\gamma_\ell}{2}}(1-v) = \frac{\Gamma(2\ell+\gamma_\ell+2)}{\Gamma(\ell+\gamma_\ell/2+1)^2}\int_0^1 \frac{(t(1-t))^{\ell+\gamma_\ell/2}}{(1-t(1-v))^{\ell+\gamma_\ell/2+1}}dt$$

with $t \to 1 - t\epsilon^{1/2}$.

Perturbative CFT: systematics V

3. Rescale the OPE coefficient in this way

$$a_\ell = \frac{2\Gamma\left(\ell+1\right)^2}{\Gamma\left(2\ell+1\right)}\hat{a}_\ell$$

4. Introduce the rescaled eigenvalue of the Casimir operator

$$\boxed{\frac{j^2}{\epsilon} = \left(\frac{x}{\epsilon^{1/2}} + \gamma_\ell/2\right)\left(\frac{x}{\epsilon^{1/2}} + 1 + \gamma_\ell/2\right)}$$

5. Interpret the anomalous dimension and rescaled structure constants as a functions of j

Perturbative CFT: systematics VI

6. Integrate in t order by order in $\epsilon \to$ this expansion reproduces all the divergent terms!

$$h(\log u, v)|_{v=\epsilon} = \frac{1}{\epsilon} 4 \int_0^\infty \hat{a}(j) u^{\gamma(j)/2} j K_0(2j) dj - \\ - \frac{1}{\sqrt{\epsilon}} 2 \int_0^\infty \hat{a}(j) u^{\gamma(j)/2} j K_0(2j) \gamma'(j) dj + \ldots$$

7. The large j expansion of the anomalous dimension and of the rescaled OPE coefficients is

$$\gamma(j) = p_0(\log j^2/\epsilon) + \frac{p_1(\log j^2/\epsilon)}{j} \epsilon^{1/2} + \frac{p_2(\log j^2/\epsilon)}{j^2} \epsilon + \cdots$$

$$\hat{a}(j) = q_0(\log j^2/\epsilon) + \frac{q_1(\log j^2/\epsilon)}{j} \epsilon^{1/2} + \frac{q_2(\log j^2/\epsilon)}{j^2} \epsilon + \cdots$$

Perturbative CFT: systematics VII

8. Plug these expansions in the expansion of $h(\log u, v)$ and impose half integer powers of v to vanish

$$\frac{1}{\sqrt{\epsilon}} \int_0^\infty j u^{\frac{1}{2}p_0} \left(2q_1 - q_0 p_0' + q_0 p_1 \log u\right) K_0(2j) dj = 0$$

9. At any loop order, $P(\log j^2/\epsilon)$ is a polynomial of any degree and then

$$\int_0^\infty P(\log j^2/\epsilon) K_0(2j) dj = 0 \quad \rightarrow \quad P(\log j^2/\epsilon) = 0$$

giving

$$\boxed{p_1 = 0, \quad q_1 = \frac{1}{2} q_0 p_0'}$$

This can be trusted provided that the power of ϵ is negative!

Is there a way out?

Perturbative CFT: systematics VIII

Yes!

9. Apply the Casimir operator on both sides of the CPWA equation!
10. This will bring down a power of $\frac{j^2}{\epsilon}$ and will allow to explore more orders in the expansion of the anomalous dimension and of the rescaled structure constant.
11. Notice that the action of the Casimir operator will not influence the fact that only integer power of v appear.

Perturbative CFT: results

The large J expansion of

$$\gamma(J) \quad \text{and} \quad \hat{a}(J)\left(1 - \frac{\sqrt{1+4J^2}}{4J}\gamma'(J)\right)$$

contains only even powers of J!

Comments:

- These results are valid at any loop order in perturbation theory.
- The expansion for γ is equivalent to the reciprocity principle.

Non-perturbative CFT: systematics I

- Consider the four-point function of four identical real scalar operators $\mathcal{O}$ of dimension $\Delta_{\mathcal{O}}$
- τ_{min} is the twist of the minimal twist operator appearing in the OPE of $\mathcal{O} \times \mathcal{O}$
- Crossing symmetry implies the existence of a tower of double trace operators of twist

$$\Delta_\ell - \ell = 2\Delta_{\mathcal{O}} + \gamma_\ell, \qquad \gamma_\ell = -\frac{c}{\ell^{\tau_{min}}} + ...$$

Non-perturbative CFT: systematics II

- Now the rescaled Casimir is

$$J^2 = (\ell + \Delta_{\mathcal{O}} + \gamma_\ell/2)(\ell + \Delta_{\mathcal{O}} + \gamma_\ell/2 - 1)$$

- the leading behavior at large J is fixed by the divergence $v^{\frac{\tau_{min}}{2} - \Delta_{\mathcal{O}}}$ to be

$$\gamma_\ell = \frac{c_1}{J^{\tau_{min}}} + \ldots \qquad \hat{a}_\ell = 1 + \frac{d_1}{J^{\tau_{min}}} + \ldots$$

[Alday and Maldacena, 2008][Komargodski and Zhiboedov, 2012][Fitzpatrick, Kaplan, Poland and Simmons-Duffin,2012]

Non-perturbative CFT: systematics III

- In order to analyze the other orders in J we need to specify τ_{min}. For $\tau_{min} = 2$ we expect

$$\begin{aligned} \gamma_\ell &= \frac{c_1}{J^2} + \frac{c_2}{J^3} + \frac{c_3}{J^4} + \frac{c_4}{J^5} + ... \\ \hat{a}_\ell &= 1 + \frac{d_1}{J^2} + \frac{d_2}{J^3} + \frac{d_3}{J^4} + \frac{d_4}{J^5} + ... \end{aligned}$$

- We can go through the same steps as for the perturbative case and we obtain again that
  - The expansion of $\gamma(J)$ for large J contains only even powers of $1/J$.
  - The expansion of $\hat{a}(J)\left(1 - \frac{\sqrt{1+4J^2}}{4J}\gamma'(J)\right)$ for large J contains only even powers of $1/J$.

Checks

We have checked several examples

	γ	a_ℓ
$\mathcal{N}=4$ SYM	3 loops	3 loops
$\mathcal{N}=0,1,2$ SYM	2 loops	
quark transversity distribution QCD	2 loops	
$\mathcal{N}=4$ SYM large N	$1/N$	
critical $O(N)$ models	$1/N^2$	

Conclusions

- Infinite number of constraints on the large spin expansions on the large spin limit of
  - anomalous dimensions $\rightarrow$ proof of reciprocity for leading twist
  - OPE coefficients $\rightarrow$ new reciprocity priciple for OPE coeff.
- This approach is only based on generic properties of CFT such as analyticity, unitarity, crossing symmetry and the structure of the conformal partial wave expansion
- All the results are valid for leading twist operators, they do not rely on planar limit and they are valid at any loop order in perturbation theory
- Can we use these results to compute/constrain three point functions in CFTs?
- Can we check our prediction for OPE coefficient in the case of QCD?

EXAMPLES

Example: $\mathcal{N} = 4$ SYM

- In the case of $\mathcal{N} = 4$ SYM there is a $SU(4)_R$ R-symmetry group and then the fields in the theory transform as
  - scalars transform in the $\mathbf{6}$ representation
  - fermions transform in the $\mathbf{4}$ and $\bar{\mathbf{4}}$ representation
  - gauge bosons transform in the $\mathbf{1}$
- We can start with the four point function of superconformal primary scalar operators $\mathcal{O}$ of protected dimension $\Delta_{\mathcal{O}} = 2$ and which transforms in the $\mathbf{20}'$
- Hence this four point function will decompose into the various representations contained in $\mathbf{20}' \times \mathbf{20}'$.

Example: $\mathcal{N}=4$ SYM

- Twist-two operators will contribute to the following representations

$$\begin{aligned} Tr\varphi\partial_{\mu_1}\cdots\partial_{\mu_\ell}\varphi &\quad\rightarrow\quad \mathbf{1}+\mathbf{15}+\mathbf{20}' \\ Tr\bar{\psi}\gamma_{(\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_\ell)}\psi &\quad\rightarrow\quad \mathbf{1}+\mathbf{15} \\ TrF_{\nu(\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_{\ell-1}}F_{\mu_\ell)\nu} &\quad\rightarrow\quad \mathbf{1} \end{aligned}$$

- If we project in the $\mathbf{20}'$, only non-degenerate twist-two operators of the form $Tr\varphi^{(i}\partial^\ell\varphi^{j)}$ contribute
- The anomalous dimensions and the OPE coefficients have been computed, up to three loops, and their expansions in J satisfy our conditions.

Example: non-conformal theories

- Reciprocity principle has been extended also to non-conformal theories in [Basso and Korchemsky, 2007]
- In this case there is a non vanishing beta function which will enter in the function f
- The way to do that is by using the dimensional regularization scheme in $d = 4 - 2\epsilon$, the beta function of the coupling is simply

$$\beta_\epsilon(g) = -2\epsilon + \beta(g)$$

  where $\beta(g)$ is the beta function of the four dimensional theory.
- Then we note that $\beta_\epsilon(g)$ vanishes at $\epsilon_{cr} = \beta(g)/2$ and hence the gauge theory is conformal in $d_{cr} = 4 - 2\epsilon_{cr}$ dimensions

Example: non-conformal theories

In our approach

- We need to study the conformal blocks in generic dimensions. The small u limit, independently on the number of space time dimension, is

$$G_{\Delta,\ell}(u,v) \sim u^{\frac{1}{2}(\Delta-\ell)}(1-v)^{\ell}\ {}_2F_1(\frac{1}{2}(\Delta+\ell),\frac{1}{2}(\Delta+\ell),\Delta+\ell;1-v)$$

- The fact that we are working in a different dimension is reflected in a shift in the Casimir operator

$$J_\beta^2 = (\ell+\gamma_\ell/2-\beta/2)(\ell+1+\gamma_\ell/2-\beta/2)$$

- Hence the expansion in $\frac{1}{J_\beta}$ contains only even powers.
- We checked this up to 2 loops for anomalous dimensions of two-loop quark transversity distribution in QCD and for the analogues in $\mathcal{N}=0,1,2$ SYM theories.

Example: theories with gravity dual

- The most well studied conformal field theory with gravity is $\mathcal{N}=4$ SYM in the large N limit
- Consider the four-point function of 2-2 dilaton scattering
- In this case, there is a tower of double trace operators of the form $\mathcal{O}\partial^{\ell}\mathcal{O}$
- $\mathcal{O}$is the operator dual to the dilaton and has dimension four
- The dimension of these double-trace operators was shown to be

$$\Delta_{\ell}-\ell=8-\frac{96}{N^2}\frac{1}{(\ell+1)(\ell+6)}$$

- Setting $\Delta_{\mathcal{O}}=4$ in the Casimir we can obtain

$$\frac{1}{(\ell+1)(\ell+6)}=\frac{1}{J_0^2-6}$$

which contains only even powers of $1/J_0$ as expected from our relations

Example: critical $O(N)$ models

- In the $O(N)$ we consider the four point function of spin field operators σ
- Among the intermediate states we have higher-spin states
  - transforming in the singlet representation of $O(N)$, of the form $\sigma_i \partial^\ell \sigma_i$
  - transforming in the symmetric traceless representation of $O(N)$, of the form $\sigma_{(i} \partial^\ell \sigma_{j)}$
- We will study critical $O(N)$ models in two regimes:
  - In dimensions $d = 4 - \epsilon$
  - At large N in d dimensions

Example: critical $O(N)$ models- ϵ expansion

- The anomalous dimension of singlet and symmetric traceless operators have been computed in an ϵ expansion and it is given by

$$\gamma_{\sigma_i \partial^\ell \sigma_i} = 2\gamma_\sigma - \epsilon^2 \frac{3(N+2)}{(N+8)^2} \frac{1}{\ell(\ell+1)}$$

$$\gamma_{\sigma_{(i} \partial^\ell \sigma_{j)}} = 2\gamma_\sigma - \epsilon^2 \frac{3(N+6)}{(N+8)^2} \frac{1}{\ell(\ell+1)}$$

- Consider that $\Delta_{\mathcal{O}} = 1 + \gamma_\sigma \approx 1$ in four dimensions
- Expanding the anomalous dimensions in terms of the Casimir, we see that they behave as $1/J_0^2$

Example: critical $O(N)$ models- large N expansion

- The dimension of symmetric traceless operator is

$$\gamma_{\sigma_{(i}\partial^{\ell}\sigma_{j)}} - 2\gamma_{\sigma} \sim \gamma_{\sigma}\frac{1}{(d+2\ell-4)(d+2\ell-2)}$$

  where $\gamma_{\sigma} \sim \frac{1}{N}$
- The intermediate operator with the lowest twist is σ^2, it has twist two
- The large ℓ expansion is in perfect agreement with our relation, where $\Delta_{\mathcal{O}} = \frac{1}{2}(d-2)$ is the dimension of the spin field in d dimensions

Example: critical $O(N)$ models- large N expansion

- The anomalous dimension of operators in the singlet is

$$\gamma_{\sigma_i \partial^\ell \sigma_i} = \frac{8\gamma_\sigma}{(d+2\ell-4)(d+2\ell-2)}\left((d+\ell-2)(\ell-1) - \frac{\Gamma(d+1)\Gamma(\ell+1)}{4(d-1)\Gamma(d+\ell-3)}\right)$$

For large spin $\gamma_{\sigma_i \partial^s \sigma_i} - 2\gamma_\sigma$ is given by

$$\gamma_\sigma \frac{d(2-d)}{2}\left(\frac{1}{\ell^2} + \frac{3-d}{\ell^3} + \frac{7+3/4d(d-6)}{\ell^4} + \frac{(d-3)(d^2-6d+10)}{\ell^5} + \ldots\right)$$

$$+\gamma_\sigma \frac{\Gamma(d+1)}{2-2d}\left(\frac{1}{\ell^{d-2}} - \frac{1}{2}\frac{(d-3)(d-2)}{\ell^{d-1}} + \ldots\right)$$

- It corresponds to the presence of σ^2, which has twist two. In terms of the Casimir it is proportional to $1/J_0^2$
- It corresponds to the presence of conserved currents with twist $d-2$. In terms of the Casimir it takes the form:

$$\frac{\Gamma\left(\frac{1}{2}\left(\sqrt{1+4J_0^2}+5-d\right)\right)}{J_0^2\Gamma\left(\frac{1}{2}\left(\sqrt{1+4J_0^2}-3+d\right)\right)}$$