Sunrise integrals and elliptic polylogarithms

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joint work with Luise Adams and Stefan Weinzierl

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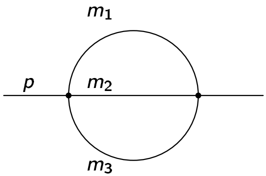


We compute the massive sunrise integral

$$S(D,\,t) = \int \frac{d^D k_1 d^D k_2}{\left(i\pi^{D/2}\right)^2} \frac{1}{\left(-k_1^2+m_1^2\right)\left(-k_2^2+m_2^2\right)\left(-\left(p-k_1-k_2\right)^2+m_3^2\right)}$$

in $D = 2 - 2\epsilon$ and $D = 4 - 2\epsilon$ dimensions:

$$\begin{aligned} S(2-2\epsilon,\,t) &= S^{(0)}(2,\,t) + S^{(1)}(2,\,t)\epsilon + \mathcal{O}\left(\epsilon^2\right), \\ S(4-2\epsilon,\,t) &= S^{(-2)}(4,\,t)\epsilon^{-2} + S^{(-1)}(4,\,t)\epsilon^{-1} + S^{(0)}(4,\,t) + \mathcal{O}(\epsilon) \end{aligned}$$

where

$$\begin{aligned} t &= p^2 \le 0, \\ 0 &< m_1 \le m_2 \le m_3 < m_1 + m_2. \end{aligned}$$

Motivation:

Multiple polylogarithms

$$\mathrm{Li}_{(s_1,\,\ldots,\,s_k)}(z_1,\,\ldots,\,z_k) = \sum_{n_1 > n_2 > \ldots > n_k \geq 1} \frac{z_1^{n_1} \ldots z_k^{n_k}}{n_1^{s_1} \ldots n_k^{s_k}}_{\;n_1},\; s_i \geq 1,\; |z_i| < 1$$

are very useful in the computation of Feynman integrals due to their double nature as **nested sums** and **iterated integrals**. (see Panzer's talk)

Many Feynman integrals can be expressed in terms of these functions, but apparently **not all of them**. (Bauberger, Böhm, Weiglein, Berends, Buza 1994, Caron-Huot, Larsen 2012, Nandan, Paulos, Spradlin, Volovich 2013)

The **massive sunrise** may be the simples example, where multiple polylogarithms are **not sufficient**.

Which functions can we use instead?

Perspective 1: Generalized hypergeometric functions

Berends, Buza, Böhm and Scharf (1994) expressed $S(D, t)$ as a linear combination of type C Lauricella functions

$$F_C\left(\alpha_1 - A_1 D,\, \alpha_2 - A_2 D;\, \beta_1 - B_1 D,\, \beta_2 - B_2 D,\, \beta_3 - B_3 D;\, \frac{m_1^2}{t},\, \frac{m_2^2}{t},\, \frac{m_3^2}{t}\right),$$

with all $\alpha_i,\, \beta_i \in \mathbb{N}$ and $A_i,\, B_i$ half-integers. They are defined by

$$F_C\left(a_1,\, a_2;\, b_1,\, b_2,\, b_3;\, x_1,\, x_2,\, x_3\right) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} \frac{(a_1)_{n_1+n_2+n_3}\,(a_2)_{n_1+n_2+n_3}}{(b_1)_{n_1}\,(b_2)_{n_2}\,(b_3)_{n_3}}\,\frac{x_1^{n_1}x_2^{n_2}x_3^{n_3}}{n_1!n_2!n_3!}$$

Remark: Techniques for the **expansion** of generalized hypergeometric functions today extend to certain Lauricella functions (e.g. Bytev, Kalmykov and Moch 2014), but the expansion of F_C **remains a problem**.
No multiple polylogarithms?

Perspective 2: Feynman parameters
In $D = 2$ dimensions, the Feynman parametric version of the sunrise integral

$$\begin{aligned} S(2,\, t) &= \int_\sigma \frac{\omega}{\mathcal{F}}, \\ \omega &= x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2 , \\ \sigma &= \left\{ [x_1 : x_2 : x_3] \in \mathbb{P}^2 | x_i \geq 0,\, i = 1,\, 2,\, 3 \right\} \end{aligned}$$

involves the second Symanzik polynomial

$$\mathcal{F} = -x_1 x_2 x_3 t + \left(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2 \right) \left(x_1 x_2 + x_2 x_3 + x_1 x_3 \right) .$$

Remark: $\mathcal{F}$ **fails** the criterion of **linear reducibility**. (Brown 2008)

$\Rightarrow$ Direct iterated integration is not possible in the variables x_1, x_2, x_3

Perspective 3: Differential equations (see talks by Tancredi, Henn, von Manteuffel)

The sunrise integral $S(D, t)$ satisfies an inhomogeneous fourth-order differential equation (Caffo, Czyz, Laporta, Remiddi 1998) in t:

$$\left(P_4 \frac{d^4}{dt^4} + P_3 \frac{d^3}{dt^3} + P_2 \frac{d^2}{dt^2} + P_1 \frac{d^1}{dt^1} + P_0 \right) S\left(D, t\right) = c_{12}\, T_{12} + c_{13}\, T_{13} + c_{23}\, T_{23}$$

where T_{ij} are products of two tadpole integrals of propagators with masses m_i and m_j and where all P_k and c_{ij} are polynomials in m_1^2, m_2^2, m_3^2, t, D.

Each of the ϵ-coefficients $S^{(0)}(2, t)$, $S^{(1)}(2, t)$, $S^{(0)}(4, t)$ satisfies an inhomogeneous differential equation of second or higher order.

Remark: None of these differential operators **factorizes** completely into first order operators. If this would be the case, we could solve simply by iterated integration.

In $D = 2$ dimensions:

Equal mass case: Second order differential equation (Broadhurst, Fleischer, Tarasov 1993);

Solutions Groote, Pivovarov 2000, Laporta, Remiddi 2004, Bloch, Vanhove 2013 ...

Arbitrary masses:
Caffo, Czyz, Laporta, Remiddi (1998): Coupled system of **four** equations of **first order**
Müller-Stach, Weinzierl, Zayadeh (2012): **One** differential equation of **second order**

$$\left(p_2(t)\frac{d^2}{dt^2} + p_1(t)\frac{d}{dt} + p_0(t)\right) S^{(0)}(2,\, t) = p_3(t)$$

$p_0(t)$, $p_1(t)$, $p_2(t)$: polynomials in t and the m_i^2 ; $p_3(t)$: also involving $\ln(m_i^2)$, $i = 1,\, 2,\, 3$.

Standard Ansatz:

$$S^{(0)}(2,\, t) = C_1\psi_1(t) + C_2\psi_2(t) + \int_0^t dt_1 \frac{p_3(t_1)}{p_0(t_1)W(t_1)} \left(-\psi_1(t)\psi_2(t_1) + \psi_2(t)\psi_1(t_1)\right)$$

$\psi_1,\, \psi_2$: solutions of the homogeneous equation; $C_1,\, C_2$: constants;$W(t)$: Wronski determinant.

Underlying geometry:
Second Symanzik polynomial:

$$\mathcal{F} = -x_1 x_2 x_3 t + \left(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2\right)\left(x_1 x_2 + x_2 x_3 + x_1 x_3\right).$$

The variety $\mathcal{F} = 0$ intersects the integration domain at **three points** $P_1 = [1:0:0],\ P_2 = [0:1:0],\ P_3 = [0:0:1]$.
Coosing one of these as **origin** defines an **elliptic curve**.

Transform to Weierstrass normal form $y^2 z - x^3 - g_2(t) x z^2 - g_3(t) z^3 = 0$.

For $z = 1$ define $e_1,\ e_2,\ e_3$ by $y^2 = 4(x-e_1)(x-e_2)(x-e_3)$ with $e_1 + e_2 + e_3 = 0$.

$\Rightarrow$ Two **period integrals** of the elliptic curve are

$$\psi_1 = 2\int_{e_2}^{e_3} \frac{dx}{y} = \frac{4}{D^{\frac{1}{4}}} K(k),\ \ \psi_2 = 2\int_{e_1}^{e_3} \frac{dx}{y} = \frac{4i}{D^{\frac{1}{4}}} K(k')$$

with the **complete elliptic integral of the first kind** $K(x) = \int_0^1 dt \frac{1}{\sqrt{(1-t^2)(1-x^2t^2)}}$,
and modulus $k = \sqrt{\frac{e_3-e_2}{e_1-e_2}},\ k' = \sqrt{1-k^2} = \sqrt{\frac{e_1-e_3}{e_1-e_3}}$.

The **period integrals** ψ_1, ψ_2 are **solutions** of the **homogeneous** differential equation.

The constants C_1, C_2 are determined from a simple property of ψ_1, ψ_2 and the limit of $S^{(0)}(2, t)$ at $t = 0$ (Davydychev, Tausk 1996).

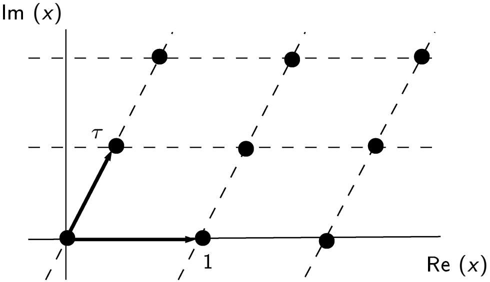
$\Rightarrow$ We obtain $S^{(0)}(2, t)$ as an **integral over** a combination of complete **elliptic integrals** of the first and second type (Adams, C.B., Weinzierl 2013).

Disadvantage: Elliptic integrals are well known in mathematics, but **integrals over elliptic integrals** are not. $\Longleftrightarrow$ No framework for iterated integrals.

Is there an alternative, "closer to" multiple polylogarithms?

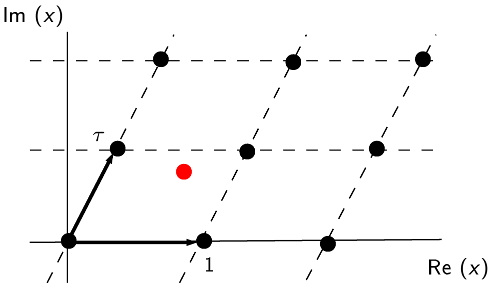
Important step by Bloch and Vanhove (2013) for the **equal mass case**:

New result in terms of an **elliptic dilogarithm**.



Consider the **lattice** $L = \mathbb{Z} + \tau\mathbb{Z}$, $\tau \in \mathbb{C}$ with $\text{Im}(\tau) > 0$.

Elliptic functions f with respect to L: $f(x) = f(x + \lambda)$ for $\lambda \in L$.



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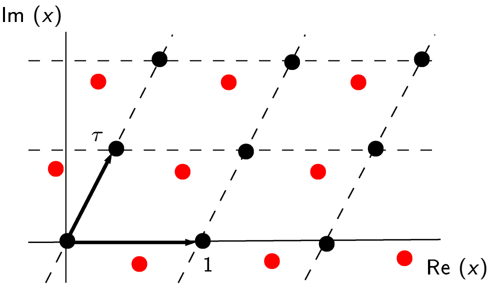

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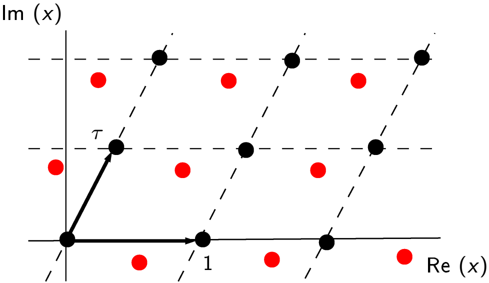

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Elliptic functions f with respect to L: $f(x) = f(x + \lambda)$ for $\lambda \in L$.

Let $\tau = \frac{\psi_1}{\psi_2}$ with ψ_1, ψ_2 the **periods of an elliptic curve** E.

$\Rightarrow$ E is isomorphic to a cell of L. $\Rightarrow$Consider f as a function **defined on** E.

Change variables to $z = e^{2\pi ix} \in \mathbb{C}^\star$

$\Rightarrow$Ellipticity $f(x) = f(x+\lambda)$ means $\tilde{f}(z) = \tilde{f}(z \cdot q_\lambda)$, $q_\lambda \in e^{2\pi i\lambda}$ for $\lambda \in L$.

Particularly: $q = e^{2\pi i\tau}$.

Basic concept: For some function g **construct** an elliptic function of the type $f(z, q) = \sum_{n\in\mathbb{Z}} g(z \cdot q^n)$

E.g. Brown, Levin 2011 consider **elliptic polylogarithms** $\sum_{n\in\mathbb{Z}} u^n \mathrm{Li}_m(z \cdot q^n)$,

elliptic multiple polylogarithms and a framework of **iterated integrals**

(Also see previous definitions in Bloch 1977, Beilinson, Levin 1994, Levin 1997, Levin, Racinet 2007, ...)

Adams, C.B., Weinzierl 2014: Generalizing $\mathrm{Li}_n(x) = \sum_{j=1}^{\infty} \frac{x^j}{j^n}$ we define

$$\mathrm{ELi}_{n;m}(x;y;q) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{x^j}{j^n}\frac{y^k}{k^m} q^{jk} = \sum_{k=1}^{\infty} y^k \mathrm{Li}_n(q^k x),$$

$$\mathrm{E}_{n;\,m}(x;\,y;\,q) =$$

$$\begin{cases} \frac{1}{i}\left(\frac{1}{2}\mathrm{Li}_2(x) - \frac{1}{2}\mathrm{Li}_2(x^{-1}) + \mathrm{ELi}_{2;\,0}(x;\,y;\,q) - \mathrm{ELi}_{2;\,0}(x^{-1};\,y^{-1};\,q)\right) & ,\ n+m \text{ even}, \\ \frac{1}{2}\mathrm{Li}_2(x) + \frac{1}{2}\mathrm{Li}_2(x^{-1}) + \mathrm{ELi}_{2;\,0}(x;\,y;\,q) + \mathrm{ELi}_{2;\,0}(x^{-1};\,y^{-1};\,q) & ,\ n+m \text{ odd}. \end{cases}$$

With these definitions we have for example

$$\mathrm{E}_{2;\,0}(x;\,y;\,q) = \sum_{n\in\mathbb{Z}} u^n \mathrm{Li}_2\left(z\cdot q^n\right) + \text{sum over squared logarithms and } \zeta(2).$$

With this function, we obtain

$$S^{(0)}(2,\,t) = \frac{\psi_1(q)}{\pi}\sum_{i=1}^{3} \mathrm{E}_{2;\,0}(w_i(q);\,-1;\,-q) \text{ where } q = e^{\pi i \frac{\psi_1}{\psi_2}}.$$

The arguments w_1, w_2, w_3 are obtained from the intersection points P_1, P_2, P_3 by above transformations of the elliptic curve.

$$S(2-2\epsilon,\, t) = S^{(0)}(2,\, t) + S^{(1)}(2,\, t)\epsilon + \mathcal{O}\left(\epsilon^2\right),$$
$$S(4-2\epsilon,\, t) = S^{(-2)}(4,\, t)\epsilon^{-2} + S^{(-1)}(4,\, t)\epsilon^{-1} + S^{(0)}(4,\, t) + \mathcal{O}(\epsilon)$$

Using Tarasov's method (1996, 1997), we express $S^{(0)}(4,\, t)$ as linear combination of

$$S^{(0)}(2,\, t),\ S^{(1)}(2,\, t),\ \frac{\partial}{\partial m_i^2} S^{(0)}(2,\, t),\ \frac{\partial}{\partial m_i^2} S^{(1)}(2,\, t),\ i = 1,\, 2,\, 3.$$

$S^{(1)}(2,\, t)$ satisfies a differential equation

$$L_{1,a} L_{1,b} L_2 S^{(1)}(2,\, t) = I_1(t).$$

This can be solved for $L_2 S^{(1)}(2,\, t)$ and gives

$$L_2 S^{(1)}(2,\, t) = I_2(t).$$

Solving this equation, we obtain **results** for $S^{(1)}(2,\, t)$ and $S^{(0)}(4,\, t)$. (Adams, C.B., Weinzierl 2015)

In these results, we find the functions $\mathrm{E}_{1;0}$, $\mathrm{E}_{2;0}$, $\mathrm{E}_{3;1}$ and a more complicated quadruple sum.

The functions $\mathrm{E}_{1;\,0}$, $\mathrm{E}_{2;\,0}$, $\mathrm{E}_{3;\,1}$ defined by

$$\mathrm{ELi}_{n;m}(x;y;q) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{x^j}{j^n}\frac{y^k}{k^m} q^{jk}$$

$$\mathrm{E}_{n;\,m}(x;\,y;\,q) =$$

$$\begin{cases} \frac{1}{i}\left(\frac{1}{2}\mathrm{Li}_2(x) - \frac{1}{2}\mathrm{Li}_2(x^{-1}) + \mathrm{ELi}_{2;\,0}(x;\,y;\,q) - \mathrm{ELi}_{2;\,0}(x^{-1};\,y^{-1};\,q)\right) & ,\, n+m \text{ even}, \\ \frac{1}{2}\mathrm{Li}_2(x) + \frac{1}{2}\mathrm{Li}_2(x^{-1}) + \mathrm{ELi}_{2;\,0}(x;\,y;\,q) + \mathrm{ELi}_{2;\,0}(x^{-1};\,y^{-1};\,q) & ,\, n+m \text{ odd}. \end{cases}$$

can be seen as generalizations of Clausen and Glaisher functions:

$$\mathrm{Cl}_n(\varphi) = \begin{cases} \frac{1}{2i}\left(\mathrm{Li}_n\left(e^{i\varphi}\right) - \mathrm{Li}_n\left(e^{-i\varphi}\right)\right) \\ \frac{1}{2}\left(\mathrm{Li}_n\left(e^{i\varphi}\right) + \mathrm{Li}_n\left(e^{-i\varphi}\right)\right) \end{cases} \quad \mathrm{Gl}_n(\varphi) = \begin{cases} \frac{1}{2}\left(\mathrm{Li}_n\left(e^{i\varphi}\right) + \mathrm{Li}_n\left(e^{-i\varphi}\right)\right) & ,n \text{ even}, \\ \frac{1}{2i}\left(\mathrm{Li}_n\left(e^{i\varphi}\right) - \mathrm{Li}_n\left(e^{-i\varphi}\right)\right) & ,n \text{ odd}, \end{cases}$$

$$\begin{aligned} \lim_{q\to 0}\mathrm{E}_{1;\,0}\left(e^{i\varphi};\,y;\,q\right) &= \mathrm{Cl}_1(\varphi), \\ \lim_{q\to 0}\mathrm{E}_{2;\,0}\left(e^{i\varphi};\,y;\,q\right) &= \mathrm{Cl}_2(\varphi), \\ \lim_{q\to 0}\mathrm{E}_{3;\,1}\left(e^{i\varphi};\,y;\,q\right) &= \mathrm{Gl}_3(\varphi). \end{aligned}$$

Conclusions:

An **elliptic curve**, defined by the second Symanzik polynomial $\mathcal{F}$ is very useful in the computation of the sunrise integral.

In $D = 2$ we obtain an **elliptic generalization** $\mathrm{E}_{2;0}$ of the dilogarithm $\mathrm{Li}_2(z)$ with arguments obtained from the elliptic curve.

In $D = 4 - 2\epsilon$ we obtain a result furthermore involving $\mathrm{E}_{1;0}$, $\mathrm{E}_{2;0}$, $\mathrm{E}_{3;1}$.

A further investigation of these functions and their relation with elliptic iterated integrals (Brown, Levin 2010, Broedel, Mafra, Mattthes, Schlotterer 2014) will be interesting.