S-Matrix Theory

Amplitudes 2015 Zürich

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Outline:

- Part I:
- Part II:
- Part III:

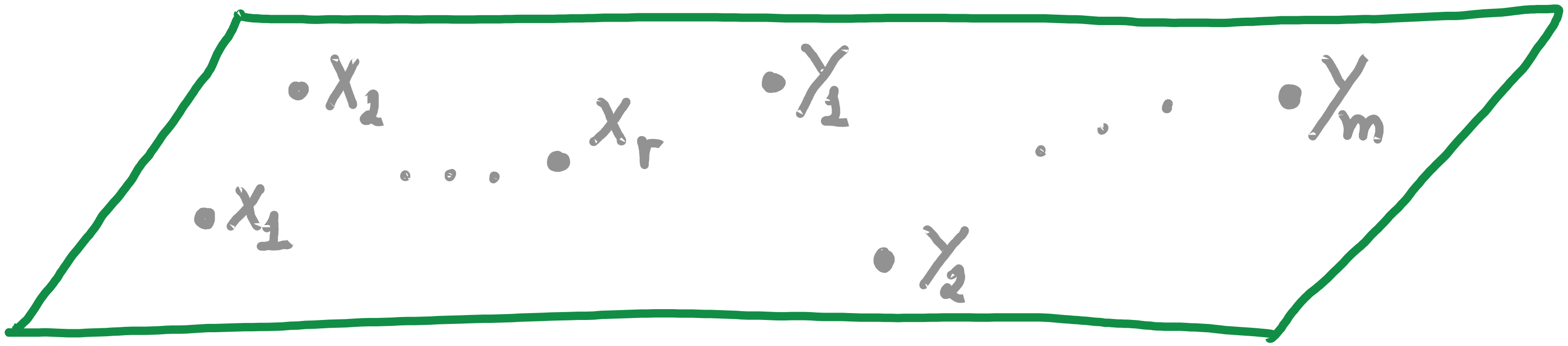
Outline:

- Part I: Speculative.
- Part II: Precise calculations and results.
- Part III: Wildly Speculative.

Part I

Standard Definition of the S-Matrix:

$$\langle T\,\phi(x_1)\cdots\phi(y_m)\rangle = \frac{\int [D\phi]\,\phi(x_1)\cdots\phi(y_m)\,e^{i\int \mathcal{L}(\phi,\partial\phi)\,d^4x}}{\int [D\phi]\,e^{i\int \mathcal{L}(\phi,\partial\phi)\,d^4x}}$$



Standard Definition of the S-Matrix:

$$\int \prod d^4x_i \, e^{ip_i x_i} \int \prod d^4y_i \, e^{-ik_i y_i} \times \frac{\int [D\phi] \, \phi(x_1) \cdots \phi(y_m) \, e^{i\int \mathcal{L}(\phi, \partial\phi) \, d^4x}}{\int [D\phi] \, e^{i\int \mathcal{L}(\phi, \partial\phi) \, d^4x}}$$

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$$\sim \prod \frac{\sqrt{Z}\, i}{p_i^2 - m^2 + i\epsilon} \prod \frac{\sqrt{Z}\, i}{k_i^2 - m^2 + i\epsilon} \times$$

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$$\sim \prod \frac{\sqrt{Z}\, i}{p_i^2 - m^2 + i\varepsilon} \prod \frac{\sqrt{Z}\, i}{k_i^2 - m^2 + i\varepsilon} \times \langle p_1 \cdots p_r | S | k_1 \cdots k_m \rangle$$

Standard Definition of the S-Matrix:

This definition assumes the presence of poles. This can be formally proven for massive particles but here we will assume it is also true for massless particles.
We will be working in perturbation theory where this assumption is valid.

(See Sever's talk)

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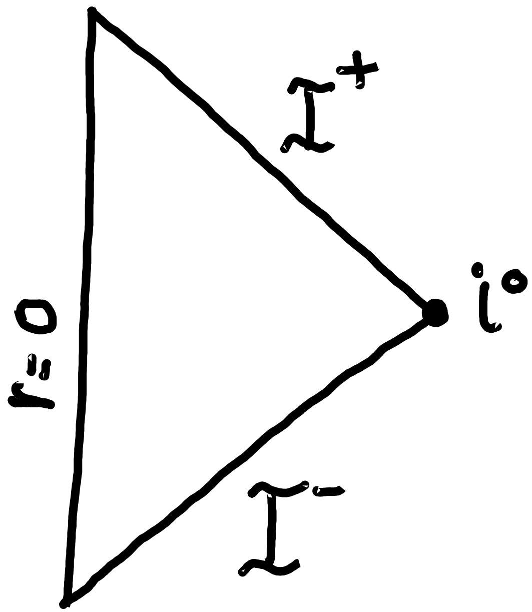
In the rest of this talk we will only consider the scattering of

Massless Particles

A Story of Interactions in a Space-Time

The standard definition of the S-matrix computes it as a sum over all possible interactions that can occur in the interior of space-time.



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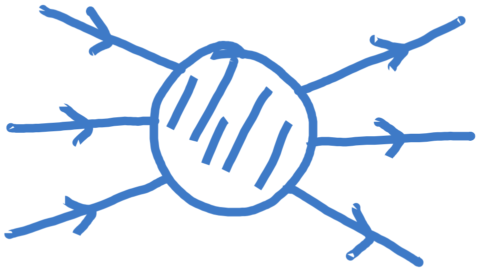

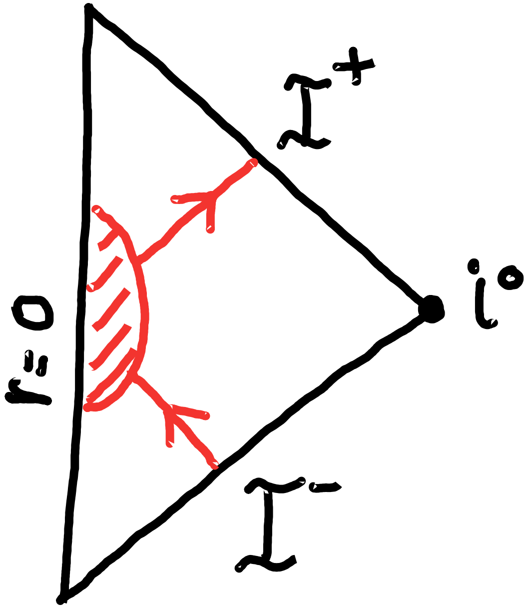

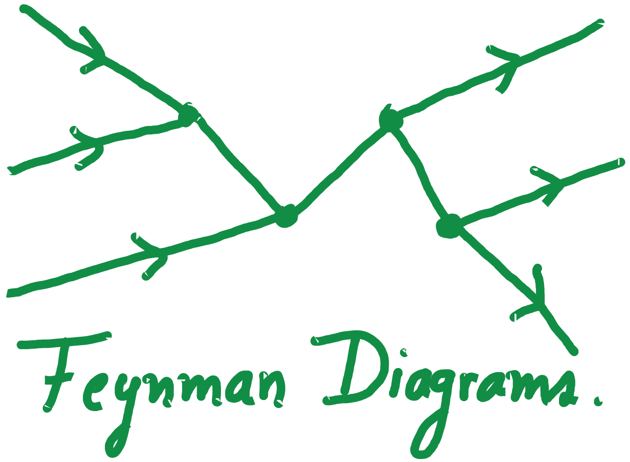

Quantum Field Theory: Locality and Unitarity

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- Since the work of Parke-Taylor and more recently all the explosion of activity sparked by Witten's twistor string theory in 2003, we have learned that there are other formulations for the S-matrix which give rise to more compact expressions at the expense of manifest locality and/or unitarity.

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- Feynman diagrams lead to expressions where locality and unitarity are manifest. This is because they come from the explicitly local interactions of the theory.
- Since the work of Parke-Taylor and more recently all the explosion of activity sparked by Witten's twistor string theory in 2003, we have learned that there are other formulations for the S-matrix which give rise to more compact expressions at the expense of manifest locality and/or unitarity.
- Is this a sign that manifest locality and unitarity are not the basic properties of a formulation of the S-matrix?

More Constraints

- Another very strong constraint on the S-matrix is that it has to be Poincare covariant. Transformations must be consistent with those of asymptotic one-particle states which are irreps. of Poincare.

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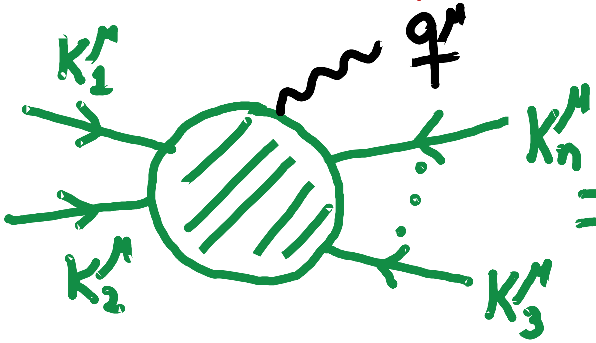
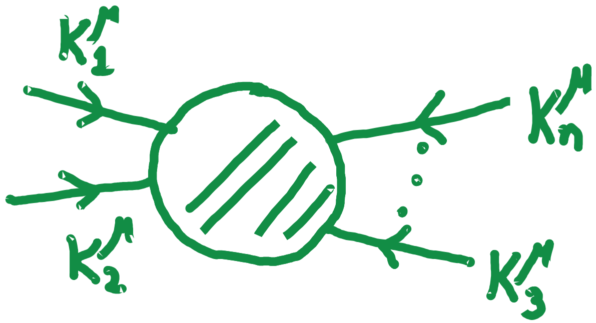
- For massless particles we also have Weinberg's soft theorems (1965):



$$= \left(\sum_{a=1}^{n} \frac{\varepsilon_q^{\mu\nu} k_{a\mu} k_{a\nu}}{q \cdot k_a} \right)$$



Universality of gravitational coupling (Equivalence Principle), Electric charge conservation, no particles with helicities greater then 2. (Weinberg 1965)

More Constraints

- Are there more constraints?
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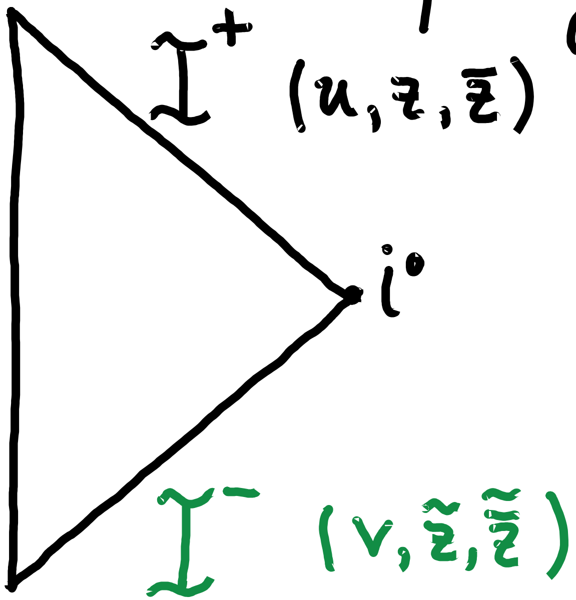
- Specially in the scattering of gravitons one could be looking for symmetries of asymptotically flat spacetimes.
- This is known as the Bondi-van der Burg-Matzner-Sachs (BMS) group.

References: Bondi,van der Burg, Metzner 1962, Sachs 1962 (BMS). Ashtekar 1981, Christodoulou, Klainerman 1993 (CK), Barnish and Troessaert 2009 (BT). Strominger 2014, FC and Strominger 2014 (CS).

(See Plefka's talk)

BMS Group



Topology of $\mathcal{I} = \mathbb{R} \times S^2$

$S^2 \to \mathbb{CP}^1$

$SL(2, \mathbb{C}) = \text{Lorentz}$

$u \longrightarrow u + f(z, \bar{z})$ [analog for v]

Supertranslations

BMS^- & BMS^+

[Strominger 2013]

A New Symmetry?

- Could it be that if $B^{\pm} \in BMS^{\pm}$ then

$$B^{+}S - SB^{-} = 0 \; ?$$

S → S-matrix operator

[Strominger 2013]

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  $B^{+}S - SB^{-} = 0$?

- But BMS^{+} & BMS^{-} do not talk to each other.
- CK (1993) "resolved" i^{0} $\Rightarrow$ Diagonal BMS

(Christodoulou-Klainerman 1993)

A New Symmetry?

[Strominger 2013]

- Ward identity $\Longleftrightarrow$ Weinberg's soft thm.

$$\langle out| :B^{+}S - SB^{-}: |in\rangle = \left(\sum_{a=1}^{n} \frac{\varepsilon_q^{\mu\nu} K_{a\mu} K_{a\nu}}{q\cdot K_a}\right) \langle out| S |in\rangle$$

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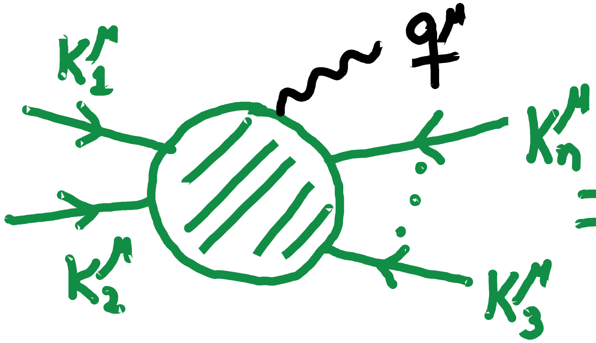

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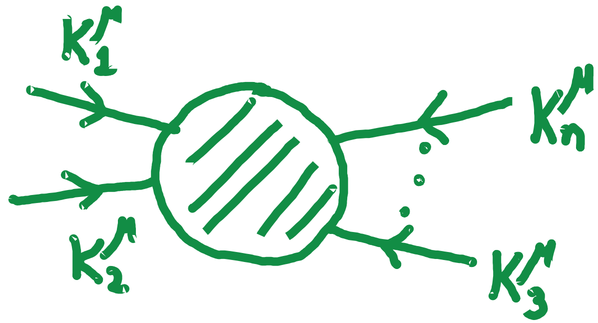

Sub-Leading Soft Theorems

- Proposal that extend the SL(2,C) to a full Virasoro (Barnish-Troessaert 2009)

Previous work (Gross and Jackiw 1968, White 2011,…)
Some extensions (Casali 2014, Bern, Davies and Nohle 2014,…)

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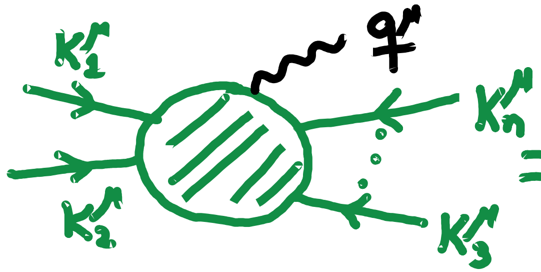
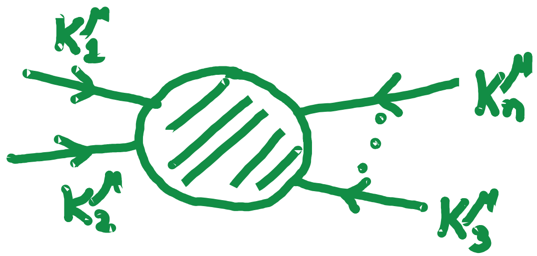
- Sub-leading soft theorems (FC-Strominger 2014) : Einstein Gravity



$$= \left(S^{(0)} + S^{(1)} + \mathcal{O}(q) \right)$$



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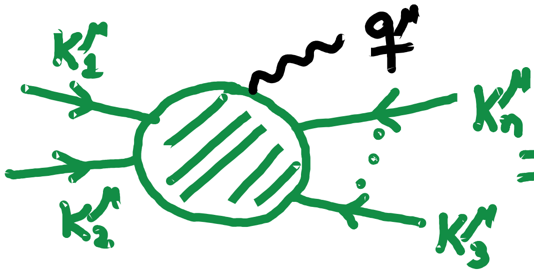

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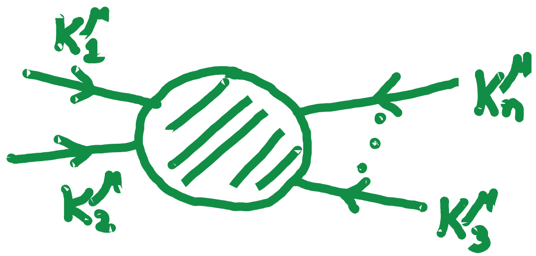


$$S^{(0)} = \sum_{a=1}^{n} \frac{\varepsilon_q^{\mu\nu} K_{a\mu} K_{a\nu}}{q \cdot K_a} \qquad S^{(1)} = -i \sum_{a=1}^{n} \frac{\varepsilon_{\mu\nu} K_a^{\mu} \left(q_\rho J_a^{\rho\nu} \right)}{q \cdot K_a}$$

(Warning: Maybe not be Virasoro. See Lipstein's talk)

Previous work (Gross and Jackiw 1968, White 2011,…)
Some extensions (Casali 2014, Bern, Davies and Nohle 2014,…)

Can the S-Matrix be determined purely from BMS representation theory?

Answer:

Can the S-Matrix be determined purely from BMS representation theory?

Answer: Probably NO but if BMS is extended then maybe YES!

Hints:

- Correlation functions on a sphere. (Witten-RSV, F.C-Geyer, F.C-Skinner-Mason, Skinner 2013,F.C 2013, F.C-He-Yuan, Adamo-Casali-Geyer-Lipstein-Mason-Monteiro-Roehrig-Skinner-Tourkine…)
- On-Shell diagrams. (Arkani-Hamed-Bourjaily-F.C. Caron-Huot-Trnka-Goncharov-Postnikov,Franco-Galloni-Mariotti,Beisert-Broedel-Rosso, Huang-Wen, Bai,Cheung, Hodges…)

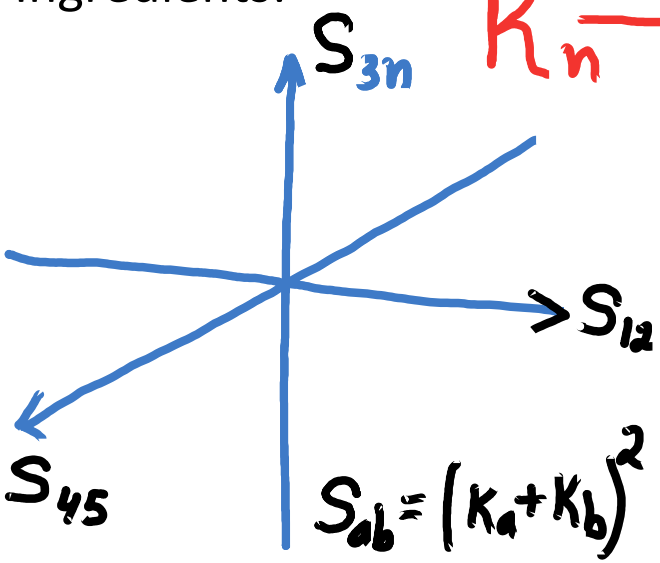
Part II

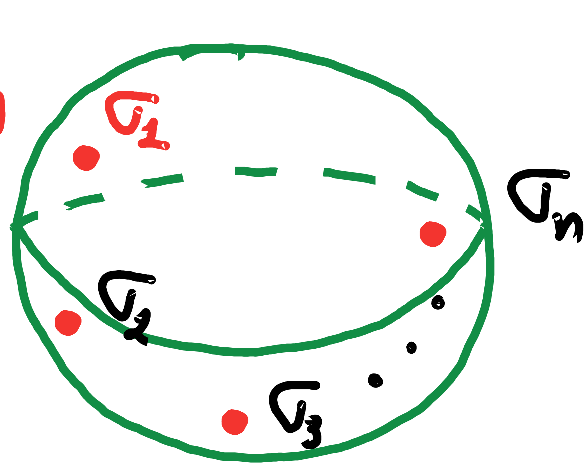
Scattering Equations

Connect the space of kinematic invariants for the scattering of n-massless particles to the moduli space of n-punctured spheres.

Ingredients:

$$\mathcal{K}_n \longrightarrow \mathcal{M}_n$$





Fairlie-Roberts ‘72 (Unpublished), Gross-Mende ’88, Witten ‘04, Fairlie ‘08, Makeenko-Olesen ‘09, F.C. ’12. F.C-He-Yuan ‘13

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$$K_n \rightarrow \mathcal{M}_n$$

$$F(\sigma) = \sum_{a,b=1}^{n} s_{ab} \log|\sigma_a - \sigma_b|$$

$$\boxed{\frac{\partial F(\sigma)}{\partial \sigma_a} = 0 \quad \forall a}$$

$$s_{ab} = (k_a + k_b)^2$$

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Scattering Equations

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Ingredients:

Unitarity & Locality $\rightarrow \mathcal{K}_n \rightarrow \mathcal{M}_n$

$\overline{\mathcal{M}}_n \nearrow \mathcal{M}_n$

Deligne-Mumford

$$\boxed{\frac{\partial F(\sigma)}{\partial \sigma_a} = 0 \quad \forall a}$$

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Poincare requires gauge invariance

- Consider Massless particles of helicity +1 or -1 (e.g. gluons)
- Scattering Data:

For each particle $\{K_a^\mu, \epsilon_a^\mu\}$

Under a general Lorentz transformation

$$\epsilon^\mu_{(\Lambda K, \pm 1)} = e^{\mp i\theta(K,\Lambda)} \left(D^\mu{}_\nu(\Lambda)\, \epsilon^\nu_{(K,\pm 1)} + \Omega(K,\Lambda)\, K^\mu \right)$$

CHY Construction: Yang-Mills

- Integral over the moduli space of n-punctured spheres.
- Integrand must make gauge invariance manifest.
- U(N) color structure.

F.C., Song He and Ellis Yuan arXiv: 1307.2199

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$$A_n = \int \prod_{a=1}^{n} \left[d\sigma_a \, \delta\left(\frac{\partial F(\sigma)}{\partial \sigma_a}\right)\right] P_F \, \Psi(k,\epsilon,\sigma) \left(\frac{\mathrm{tr}(T^{a_1} \cdots T^{a_n})}{(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3)\cdots} + \cdots \right)$$

Tree-Level

F.C., Song He and Ellis Yuan arXiv: 1307.2199

CHY Construction: U(N) color structure

Standard Color Decomposition

$$f^{abc} = \mathrm{tr}(T^a T^b T^c) - \mathrm{tr}(T^a T^c T^b)$$

$$\mathcal{C}(1,2,\ldots,n) = \sum_{\omega \in S_n/\mathbb{Z}_n} \frac{\mathrm{tr}(T^{a_{\omega(1)}} \cdots T^{a_{\omega(n)}})}{(\sigma_{\omega(1)} - \sigma_{\omega(2)}) \cdots (\sigma_{\omega(n)} - \sigma_{\omega(1)})} = \left(\frac{\mathrm{tr}(T^{a_1} \cdots T^{a_n})}{(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3) \cdots} + \cdots \right)$$

Tree-Level

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CHY Construction: Integration measure

- Integral over the moduli space of n-punctured spheres.

$$\prod_{a=1}^{n}\left[d\sigma_a\,\delta\left(\frac{\partial F(\sigma)}{\partial\sigma_a}\right)\right] \equiv \prod_{\substack{a=1\\ a\notin\{i,j,k\}}}^{n} d\sigma_a \prod_{\substack{b=1\\ b\notin\{p,q,r\}}}^{n} \delta\left(\frac{\partial F}{\partial\sigma_b}\right) \times |ijk|\,|pqr|$$

Tree-Level $\quad |ijk| \equiv (\sigma_i-\sigma_j)(\sigma_j-\sigma_k)(\sigma_k-\sigma_i)$

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CHY Construction: Integration measure

- Integral over the moduli space of n-punctured spheres.

The integral localizes to the (n-3)! solutions to the scattering equations.

$$\prod_{a=1}^{n}\left[d\sigma_a\,\delta\!\left(\frac{\partial F(\sigma)}{\partial\sigma_a}\right)\right] \equiv \prod_{\substack{a=1\\ a\notin\{i,j,k\}}}^{n} d\sigma_a \prod_{\substack{b=1\\ b\notin\{p,q,r\}}}^{n}\delta\!\left(\frac{\partial F}{\partial\sigma_b}\right)\times|ijk|\,|pqr|$$

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CHY Construction: Gauge Invariance

$$\mathrm{Pf}\,\Psi(k,\epsilon,\sigma) = \mathrm{Pf}\left[\begin{array}{c|c} \dfrac{k_a\cdot k_b}{\sigma_a-\sigma_b} & \dfrac{k_a\cdot \epsilon_b}{\sigma_a-\sigma_b} \\ \hline \dfrac{\epsilon_a\cdot k_b}{\sigma_a-\sigma_b} & \dfrac{\epsilon_a\cdot \epsilon_b}{\sigma_a-\sigma_b} \end{array}\right]$$

$$(\mathrm{Pfaffian})^2 = \det$$

F.C., Song He and Ellis Yuan arXiv: 1307.2199

CHY Construction: Gauge Invariance

If any polarization vector is replaced by its momentum vector, the matrix reduces its rank and the pfaffian vanishes.

$$\mathrm{Pf}'\Psi(k_a, \epsilon_a, \sigma_a) \xrightarrow[\epsilon_1^\mu \to k_1^\mu]{} 0$$

CHY Construction: Gauge Invariance

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$$\mathrm{Pf}\,\Psi(k_a, \epsilon_a, \sigma_a) \xrightarrow[\epsilon_1^\mu \to k_1^\mu]{} 0$$

The pfaffian is the basic object that transforms correctly under Lorentz tranformations in the massless helicity +1 or -1 representation!

$$\mathrm{Pf}\,\Psi(k_a, \epsilon_a, \sigma_a) \xrightarrow[\Lambda]{} e^{\sum_a h_a \theta(k_a, \Lambda)}\, \mathrm{Pf}\,\Psi$$

CHY Construction: Gravity

We found

$$P_F \Psi \xrightarrow[\Lambda]{} e^{\sum_a h_a \theta(k_a,\Lambda)} P_F \Psi \qquad (h_a = \pm 1)$$

CHY Construction: Gravity

We found

$$\mathrm{Pf}\,\Psi \xrightarrow[\Lambda]{} e^{\sum_a h_a \theta(k_a,\Lambda)}\,\mathrm{Pf}\,\Psi \qquad (h_a = \pm 1)$$

This means that

$$\det \Psi \xrightarrow[\Lambda]{} e^{\sum_a 2 h_a \theta(k_a,\Lambda)}\,\det \Psi$$

$$(h_a = \pm 2)$$

CHY Construction: Gravity

- Gauge invariance is manifest again.

$$\mathcal{A}_n^{\text{Gravitons}} = \int \prod_{a=1}^{n} \left[d\sigma_a \, \delta\!\left(\frac{\partial F(\sigma)}{\partial \sigma_a} \right) \right] \det \Psi_{(k,\epsilon,\sigma)}$$

Tree-Level

CHY Construction: Gravity

- Gauge invariance is manifest again.
- Soft theorems are manifest in both Yang-Mills and Gravity!
- These are the two important ingredients at Null Infinity (BMS).

$$\mathcal{A}_n^{\text{Gravitons}} = \int \prod_{a=1}^{n} \left[d\sigma_a \, \delta\!\left(\frac{\partial F(\sigma)}{\partial \sigma_a} \right) \right] \det \Psi_{(k,\epsilon,\sigma)}$$

Tree-Level

Is this a general framework?

We don't know but here are some of the theories for which the formulation exists:

Einstein Gravity   Einstein-Maxwell   Einstein-YM

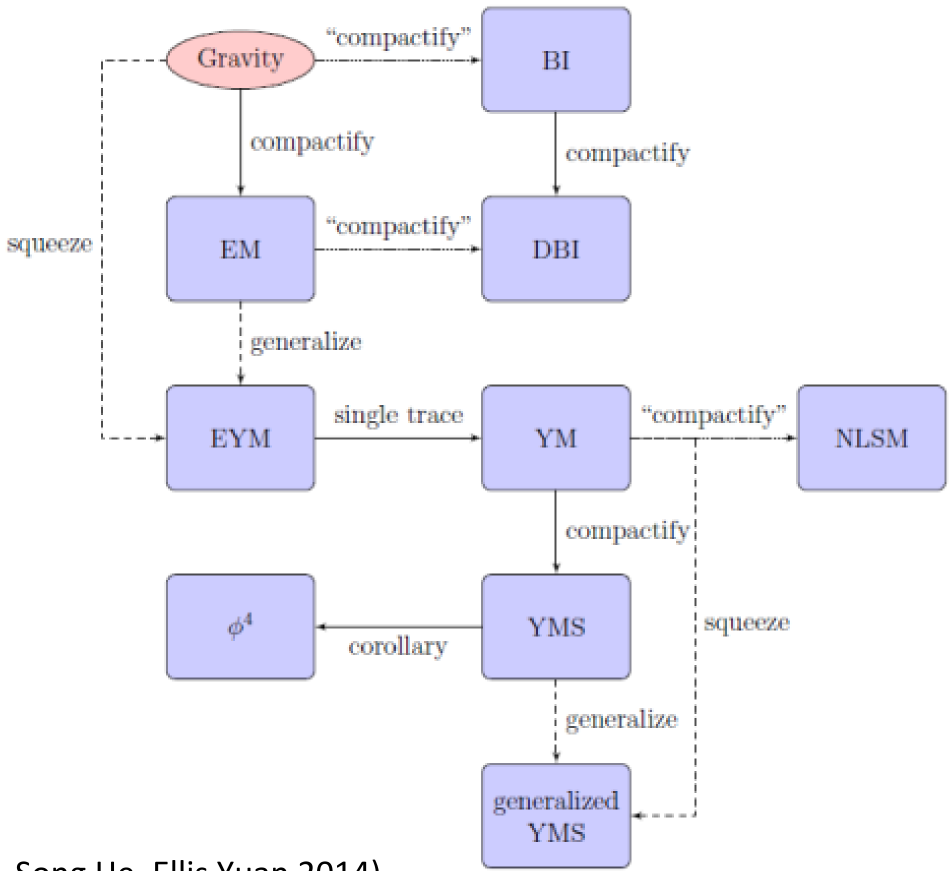
Yang-Mills   Yang-Mills-Scalar

Born-Infeld   DBI

NLSM   Galileons

(FC, Song He, Ellis Yuan 2014)

The theories mentioned in the previous slide are connected by a web of operations. All are natural from the CHY formulation. But some are rather mysterious from a field theory viewpoint.



(FC, Song He, Ellis Yuan 2014)

Operations: Compactification

Start with GR $\rightarrow$ GR + B + Dilaton

$$\det \Psi_{(\kappa,\varepsilon,\sigma)} \longrightarrow \underbrace{P_F\,\Psi_{(\kappa,\varepsilon,\sigma)}}_{\mathcal{I}_{\text{Left}}}\ \underbrace{P_F\,\Psi_{(\kappa,\tilde{\varepsilon},\sigma)}}_{\mathcal{I}_{\text{Right}}}$$

Compactify $\mathbb{R}^{D=d+m}$ down to $\mathbb{R}^d$

Take $K_a = (k_a^0, \dots, k_a^{d-1} \,|\, 0, \dots, 0) \equiv (k_a^\mu \,|\, \vec{0})$

$$\tilde{\mathcal{E}}_a = (\tilde{\epsilon}_a^0, \dots, \tilde{\epsilon}_a^{d-1} \,|\, 0, \dots, 0) \equiv (\tilde{\epsilon}_a^\mu \,|\, \vec{0})$$

$$\mathcal{E}_a = \begin{cases} (\epsilon_a^\mu \,|\, \vec{0}) & \text{if } a \in h \quad \text{(graviton)} \\ (0^\mu \,|\, \vec{e}_a) & \text{if } a \in \gamma \quad \text{(photon)} \end{cases}$$

So, $P_f \Psi_{(\kappa, \tilde{\varepsilon}, \sigma)}$ is unchanged.

$$P_f \Psi_{(\kappa, \varepsilon, \sigma)} \longrightarrow P_f \begin{array}{cc} \begin{bmatrix} \multicolumn{2}{c}{\frac{\kappa_a \cdot \kappa_b}{\sigma_a - \sigma_b}} & \frac{\kappa_a \cdot \epsilon_b}{\sigma_a - \sigma_b} & 0 \\ & & & 0 \\ \multicolumn{2}{c}{\frac{\epsilon_a \cdot \kappa_b}{\sigma_a - \sigma_b}} & \frac{\epsilon_a \cdot \epsilon_b}{\sigma_a - \sigma_b} & 0 \\ 0 & 0 & 0 & X_{ab} \end{bmatrix} & \begin{matrix} \}\, h \\ \}\, \gamma \\ \}\, h \\ \}\, \gamma \end{matrix} \\ \begin{matrix} \underbrace{\quad}_{h} & \underbrace{\quad}_{\gamma} & \underbrace{\quad}_{h} & \underbrace{\quad}_{\gamma} \end{matrix} & \end{array}$$

$$X_{ab} = \frac{\vec{e}_a \cdot \vec{e}_b}{\sigma_a - \sigma_b}$$

Pure Photon Amplitude in Einstein-Maxwell

$$\mathrm{Pf}\left[\begin{array}{c|c} A & 0 \\ \hline 0 & X \end{array}\right]\begin{array}{l} \}\, n \\ \}\, n \end{array} = \mathrm{Pf}\, A \; \mathrm{Pf}\, X$$

$$A = \begin{cases} \dfrac{k_a \cdot k_b}{\sigma_a - \sigma_b} & a \neq b \\ 0 & a = b \end{cases} \qquad X = \begin{cases} \dfrac{1}{\sigma_a - \sigma_b} & a \neq b \\ 0 & a = b \end{cases}$$

$m = 1$

Single $U(1)$

Pure Photon Amplitude in Einstein-Maxwell

$$\mathcal{A}^{Photon}_{EM} = \int \left[\prod_a d\sigma_a \,\delta\!\left(\frac{\delta F}{\delta \sigma_a}\right)\right] \times \left(P_F \, A \, P_F \, X\right) \times P_F \, \Psi_{(k,\tilde{\epsilon},\sigma)}$$

Pure Photon Amplitude in Einstein-Maxwell

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Recall: $\mathrm{Pf}\,\Psi$ & $\mathcal{C}(12\ldots n)$ are "half" integrands

$\Rightarrow$ $\mathrm{Pf}\,A$ & $\mathrm{Pf}\,X$ are "quarter" integrands!

Combining the New Building Blocks

What if ?

$$A \stackrel{?}{=} \int [d\mu_n] \left(P_F A\right)^2 \cdot \left(P_F \Psi(\kappa, \tilde{\epsilon}, \sigma)\right)$$

Hints: Theory of interacting massless vector bosons with <u>no</u> color but higher derivatives.

Combining the New Building Blocks

What if ?

$$\mathcal{A}_n^{BI} = \int [d\mu_n] \left(P_F A\right)^2 \cdot \left(P_F \Psi(\kappa, \tilde{\epsilon}, \sigma)\right)$$

This turns out to be Born-Infeld !

$$\mathcal{L} = \sqrt{-\det(\eta_{\mu\nu} + F_{\mu\nu})}$$

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What if?

$$\mathcal{A}_n \overset{?}{=} \int [d\mu_n] \left(\mathrm{Pf}\, A\right)^2 \cdot \left(\frac{\mathrm{tr}(T^{a_1} \cdots T^{a_n})}{(\sigma_1 - \sigma_2) \cdots (\sigma_n - \sigma_1)} + \cdots \right)$$

$\mathcal{C}(12 \cdots n)$

Hints: Theory of scalars with $U(N)$ flavor and higher derivative interactions.

Combining the New Building Blocks

What if ?

$$\mathcal{A}_n^{NLSM} = \int [d\mu_n] \left(\mathrm{Pf}' A \right)^2 \cdot \left(\frac{\mathrm{tr}(T^{a_1} \cdots T^{a_n})}{(\sigma_1 - \sigma_2) \cdots (\sigma_n - \sigma_1)} + \cdots \right) \quad \longleftarrow \ \mathcal{C}(12\cdots n)$$

This turns out to be the chiral lagrangian (NLSM)!

$$\mathcal{L} = \mathrm{tr}\left(\partial_\mu U^\dagger \partial^\mu U \right) \qquad U = \frac{\mathbb{1} + \Phi}{\mathbb{1} - \Phi}$$

Combining the New Building Blocks

What if ?

$$A_n^{?} = \int [d\mu_n] \, (P_F A)^2 * (P_F A)^2$$

Hints: Theory of a single scalar with *many* derivative interactions and very "soft".

Combining the New Building Blocks

What if?

$$A_n^{sGal} = \int [d\mu_n] (P_F A)^2 * (P_F A)^2$$

This turns out to be a special Galileon theory!

$$\mathcal{L} = \sum_{m=2}^{\infty} g_m \mathcal{L}_m \quad \text{with} \quad \mathcal{L}_m = \phi \det\{\partial^{\mu_i} \partial_{\nu_j} \phi\}_{i,j=1}^{m-1}.$$

Combining the New Building Blocks

What if?

$$\mathcal{A}_n^{\text{bAS}} = \int [d\mu_n] \left(\frac{\operatorname{tr}(T^{a_1} \cdots T^{a_n})}{(\sigma_1 - \sigma_2) \cdots (\sigma_n - \sigma_1)} + \cdots \right) \left(\frac{\operatorname{tr}(T^{\hat{a}_1} \cdots T^{\hat{a}_n})}{(\sigma_1 - \sigma_2) \cdots (\sigma_n - \sigma_1)} + \cdots \right)$$

This is a bi-adjoint scalar theory: $f_{abc} f_{\tilde{a}\tilde{b}\tilde{c}} \phi^{a\hat{a}} \phi^{b\tilde{b}} \phi^{c\tilde{c}}$.

Combining the New Building Blocks

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Def : Double partial amplitudes !

$$m(\alpha|\beta) = \int [d\mu_n] \frac{1}{(\sigma_{\alpha_1} - \sigma_{\alpha_2}) \cdots (\sigma_{\alpha_n} - \sigma_{\alpha_1})} \cdot \frac{1}{(\sigma_{\beta_1} - \sigma_{\beta_2}) \cdots (\sigma_{\beta_n} - \sigma_{\beta_1})}$$

Another Operation: Squeezing

- This is a procedure for replacing a set of particles that posses (s) polarization vectors each by a set of particles with (s-1) polarization vectors which interact through a single trace of U(N).

- Using this one can start with Einstein gravity and an amplitude of n gravitons and squeeze m_1 gravitons into m_1 gluons with a single trace.

- Having done that one can squeeze another set of m_2 gravitons into m_2 gluons to get a double trace amplitude and so on.

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- This leads to all amplitudes in Einstein-Yang-Mills!

KLT in CHY

The Kawai-Lewellen-Tye relations express a gravity amplitude as a sum of product of two partial YM amplitudes. (Bern,Dixon,Perelstein,Rozowsky 1999)

$$A_n^{\text{grav.}} = \int [d\mu_n] \underbrace{\mathrm{Pf}\,\Psi_{(k,\epsilon,\sigma)}}_{\mathcal{I}_L} \underbrace{\mathrm{Pf}\,\Psi_{(k,\tilde{\epsilon},\sigma)}}_{\mathcal{I}_R}$$

$$= \sum_{I=1}^{(n-3)!} \mathcal{I}_L^{(I)} \cdot \frac{1}{J^{(I)}} \cdot \mathcal{I}_R^{(I)} = \vec{\mathcal{I}}_L^{\,T} D \vec{\mathcal{I}}_R$$

KLT in CHY

$\vec{I}_L, \vec{I}_R$ are $(n-3)!$ dimensional vectors.

D is a $(n-3)! \times (n-3)!$ diagonal matrix $D_{II} = \frac{1}{J^{(I)}}$.

KLT in CHY

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D is a $(n-3)! \times (n-3)!$ diagonal matrix $D_{II} = \frac{1}{J^{(I)}}$.

Let $\alpha, \beta \in S_n$ then the biadjoint theory is

$$m(\alpha|\beta) = \sum_{I=1}^{(n-3)!} \left(\frac{1}{(\sigma_{\alpha_1}-\sigma_{\alpha_2})\cdots(\sigma_{\alpha_n}-\sigma_{\alpha_1})} \frac{1}{J} \right)^{(I)} J^{(I)} \left(\frac{1}{(\sigma_{\beta_1}-\sigma_{\beta_2})\cdots} \frac{1}{J} \right)^{(I)}$$

KLT in CHY

$\Rightarrow$ m is a $n! \times n!$ matrix $\quad m_{\alpha\beta} = (E^T D^{-1} E)_{\alpha\beta}$

want D^{-1} but $\det m = 0$

$$m(\alpha|\beta) = \sum_{I=1}^{(n-3)!} \overbrace{\left(\frac{1}{(\sigma_{\alpha_1}-\sigma_{\alpha_2}) \cdots (\sigma_{\alpha_n}-\sigma_{\alpha_1})} \times \frac{1}{J} \right)^{(I)}}^{E_\alpha^I} J^{(I)} \overbrace{\left(\frac{1}{(\sigma_{\beta_1}-\sigma_{\beta_2}) \cdots} \times \frac{1}{J} \right)^{(I)}}^{E_\beta^I}$$

KLT in CHY

$\Rightarrow m$ is a $n! \times n!$ matrix $\quad m_{\alpha\beta} = (E^T D^{-1} E)_{\alpha\beta}$

want D^{-1} but $\det m = 0$ $\quad$ (rank $(n-3)!$)

Def: $\hat{m}$ as a $(n-3)! \times (n-3)!$ non-singular submatrix

$\hat{m} = E^T D^{-1} E \quad \Rightarrow \quad \boxed{D = E^T \hat{m}^{-1} E}$

KLT in CHY

$$\mathcal{A}_n^{\text{grav.}} = \int [d\mu_n] \underbrace{\mathrm{Pf}'\Psi_{(k,\epsilon,\sigma)}}_{\mathcal{I}_L} \underbrace{\mathrm{Pf}'\Psi_{(k,\tilde{\epsilon},\sigma)}}_{\mathcal{I}_R} = \vec{\mathcal{I}}_L^{\,T} D \, \vec{\mathcal{I}}_R$$

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$$= \left(E\vec{\mathcal{I}}_L\right)^T \hat{m}^{-1} \left(E\vec{\mathcal{I}}_R\right)$$

$$\boxed{\left(E\vec{\mathcal{I}}_R\right)_\beta = \sum_{\mathcal{I}=1}^{(n-3)!} \frac{\mathrm{Pf}\,\Psi^{(\mathcal{I})}}{(\sigma_{p_1}-\sigma_{p_2})\cdots} \cdot \frac{1}{J^{(\mathcal{I})}}}$$

KLT in CHY follows from linear algebra!

$$A_n^{\text{grav.}} = \int [d\mu_n]\, \underbrace{\mathrm{Pf}\,\Psi_{(k,\varepsilon,\sigma)}}_{\mathcal{I}_L}\, \underbrace{\mathrm{Pf}\,\Psi_{(k,\tilde{\varepsilon},\sigma)}}_{\mathcal{I}_R} = \vec{\mathcal{I}}_L^{\,T} D\, \vec{\mathcal{I}}_R$$

$$= (E\vec{\mathcal{I}}_L)^T\, \hat{m}^{-1}\, (E\vec{\mathcal{I}}_R)$$

$$= \sum_{\alpha,\beta=1}^{(n-3)!} A^{\text{YM}}_{(\alpha)}\, (\hat{m}^{-1})_{\alpha\beta}\, A^{\text{YM}}_{(\beta)}$$

$$\boxed{(E\vec{\mathcal{I}}_R)_\beta = \sum_{J=1}^{(n-3)!} \frac{\mathrm{Pf}\,\Psi^{(J)}}{(\sigma_{\beta_1}-\sigma_{\beta_2})\cdots} \cdot \frac{1}{J^{(J)}}}$$

KLT in CHY: Examples

- KLT (YM , YM) = Gravity + B-field + Dilaton
- KLT (YM , NLSM) = Born-Infeld
- KLT (NLSM , NLSM) = special Galileon

Part III

Observations:

- Scattering Amplitudes of a variety of theories can be expressed as:

$$\mathcal{A}_n = \sum_{I=1}^{(n-3)!} \mathcal{I}_L^{(I)} \frac{1}{J^{(I)}} \mathcal{I}_R^{(I)} = \vec{\mathcal{I}}_L^{\,T} D \vec{\mathcal{I}}_R$$

- Each one of the (n-3)! solutions to the scattering equations knows many physical features but not all.

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- Each one of the (n-3)! solutions to the scattering equations knows many physical features but not all.

    They know about: Soft limits, Poincare, Global symmetries, etc.

    They do not know about: Locality in spacetime. (Each solution has no meaning as a story of local interactions in spacetime.)

Irreducible Representations?

- Consider a single kind of massless particles and a single free parameter that trivially multiplies each amplitude (coupling constant).
- Construct the corresponding CHY integrand. So far all examples contain a left and a right integrand

$$\vec{\mathcal{I}}_L , \vec{\mathcal{I}}_R \in \mathbb{C}^{(n-3)!}$$

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- Are these vectors irreducible representations of some extension of the BMS group? Let's call it the group Z.

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$$\vec{\mathcal{I}}_L \, , \, \vec{\mathcal{I}}_R \in \mathbb{C}^{(n-3)!}$$

- Are these vectors irreducible representations of some extension of the BMS group? Let's call it the group Z. (Zurich)

Irreducible Representations?

- Could it be that the diagonal matrix D is an invariant tensor of the group Z?
- If this is true, then scattering amplitudes are "partial inner products".
- In other words, to construct scattering amplitudes we trace over the "solution space" part and leave the rest.
- But then, what's the meaning of theories that contain several kinds of particles?
- What's the meaning of KLT?

Irreducible Representations?

- But then, what's the meaning of theories that contain several kinds of particles?

Combining particles $\Longleftrightarrow$ Direct sum of representations

- What's the meaning of KLT?

KLT $\Longleftrightarrow$ Tensor product of representations

Other Hints

- Clearly, Poincare is in Z and covariance is a very strong constraint when we consider particles with non-vanishing helicity.
- When particles have zero helicity Poincare loses its power.

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- Clearly, Poincare is in Z and covariance is a very strong constraint when we consider particles with non-vanishing helicity.
- When particles have zero helicity Poincare loses its power.
- For massless scalar particles we still have the soft limits.
- Cheung-Kampf-Novotny-Trnka 2014 proposed a classification in D=4 based on two integers. One of them is the power of the vanishing using a single soft limit.
- Are these numbers also part of the labeling of irreps of Z?

Is S-Matrix theory nothing but representation theory?

Other Symmetries:

- Planar N=4 Super-Yang Mills enjoys an infinite dimensional symmetry:

PSL(4|4) Yangian

Superconformal algebra	Level 0
Super-dualconformal algebra	Level 1

- Is there a framework that makes these symmetries manifest?

(Arkani-Hamed, Bourjaily, FC, Goncharov, Postnikov, Trnka 2012)

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- Planar N=4 Super-Yang Mills enjoys an infinite dimensional symmetry:

  PSL(4|4) Yangian

Superconformal algebra	Level 0
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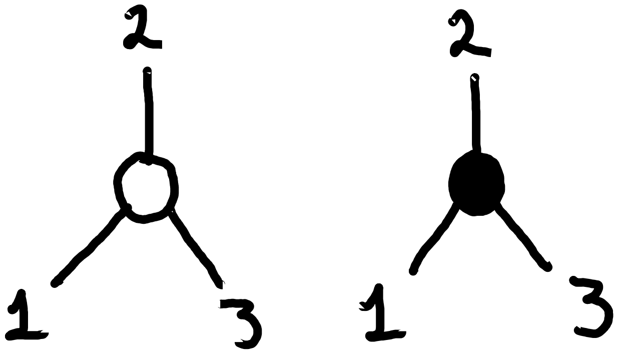
- Is there a framework that makes these symmetries manifest?
- The answer is yes!
- The framework is called on-shell diagrams.

(Arkani-Hamed, Bourjaily, FC, Goncharov, Postnikov, Trnka 2012)

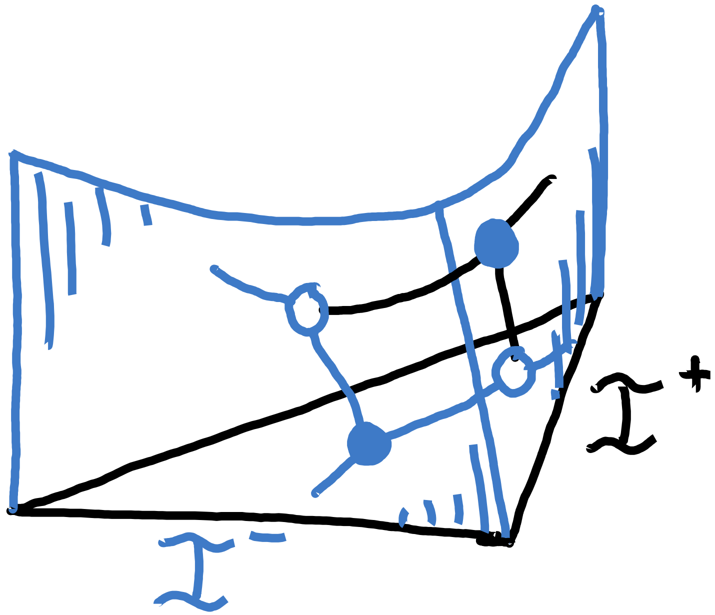
On-Shell Diagrams

All planar amplitudes at all loop orders are given by interactions of purely on-shell particles. All interactions take place in a complexified version of null infinity. Again no need for interactions in space-time.

Basic building blocks:

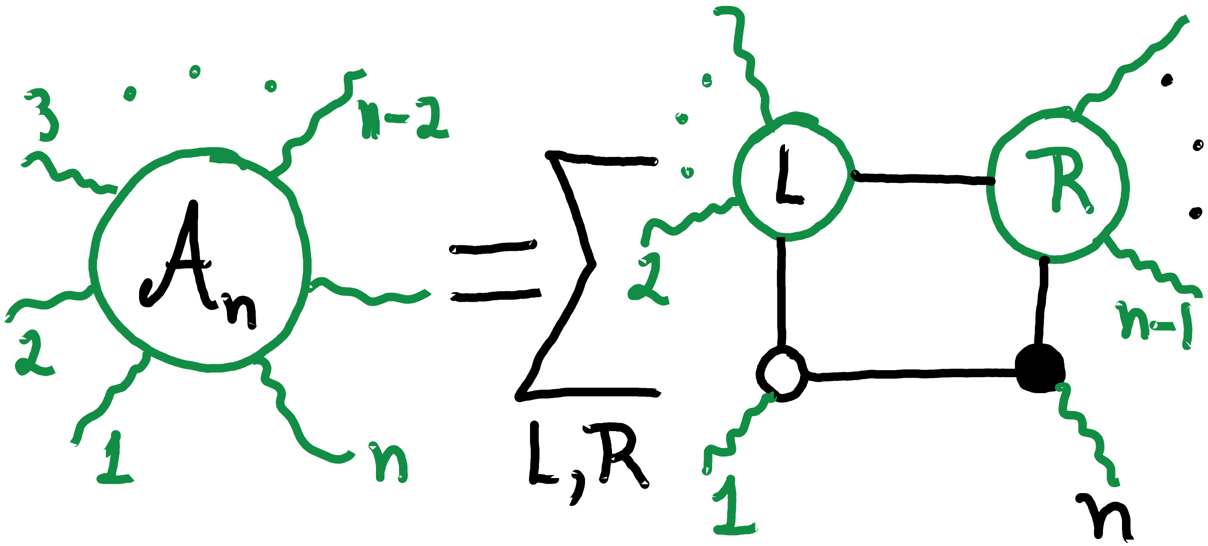


Three-Point Amplitudes



All Loop Recursion Relation

A scattering amplitude at any loop order and any number of particles can be obtained in terms of on-shell diagrams as:



Tree-Level
BCFW

(Britto, FC, Feng, Witten 2005)

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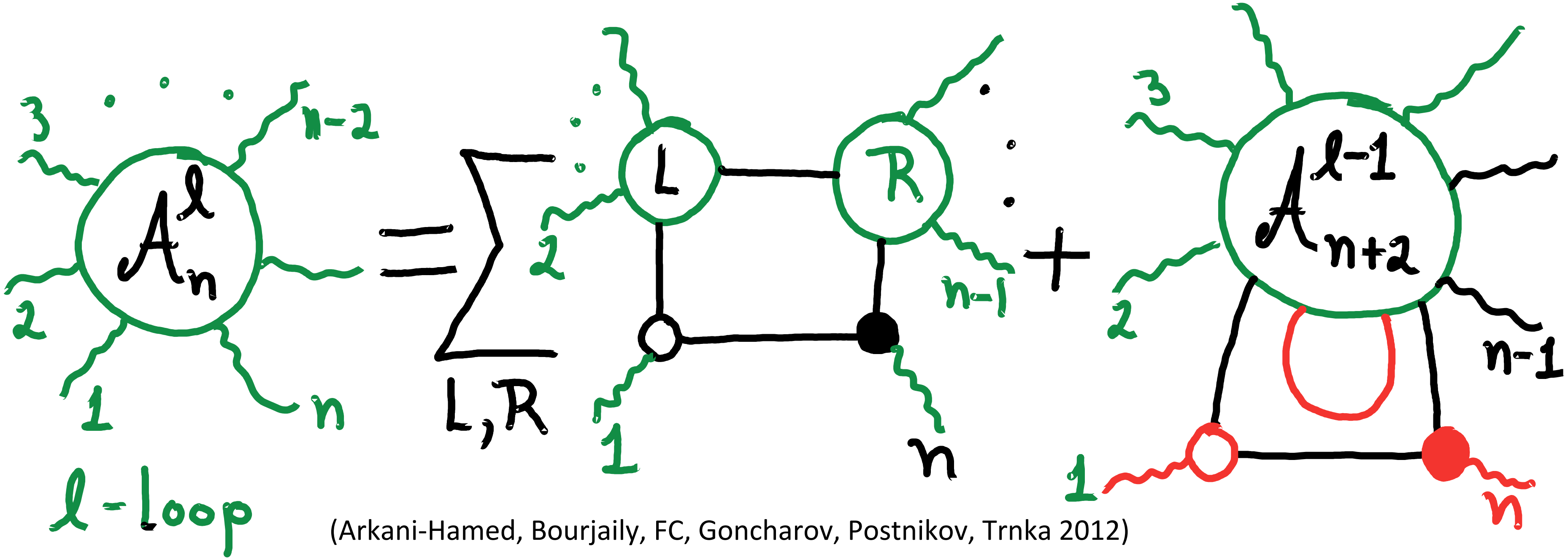

(Arkani-Hamed, Bourjaily, FC, Goncharov, Postnikov, Trnka 2012)

Conclusions

- S-matrices not only relate states at null infinity but seem to be described purely in terms of “boundary data and boundary interactions”.

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Conclusions

- S-matrices not only relate states at null infinity but seem to be described purely in terms of “boundary data and boundary interactions”.
- There seems to be a connection between symmetries of null infinity and the CHY formulation. Perhaps ambitwistor string ideas will make the connection clear. (Mason, Skinner, et.al 2014)
- The connection of on-shell diagrams and the Yangian symmetry, which is non-local, shows that “boundary descriptions” are useful and perhaps fundamental.

Is there a Holographic S-Matrix Theory?