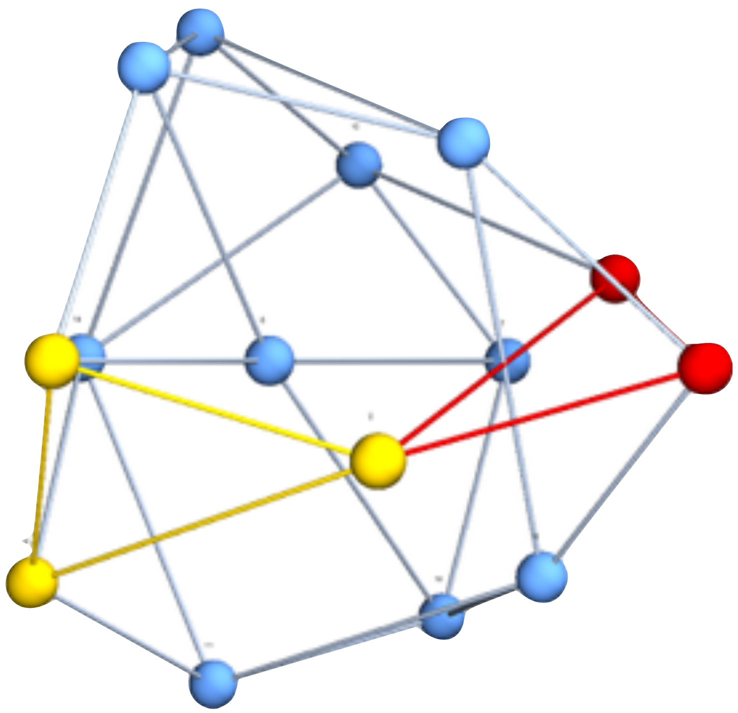
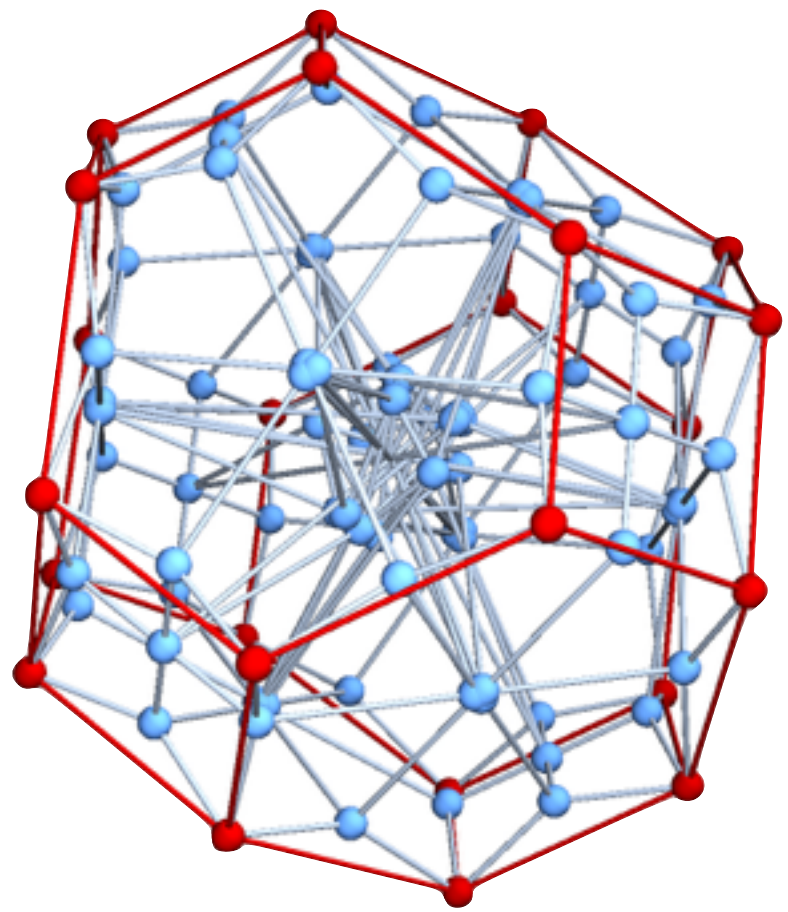
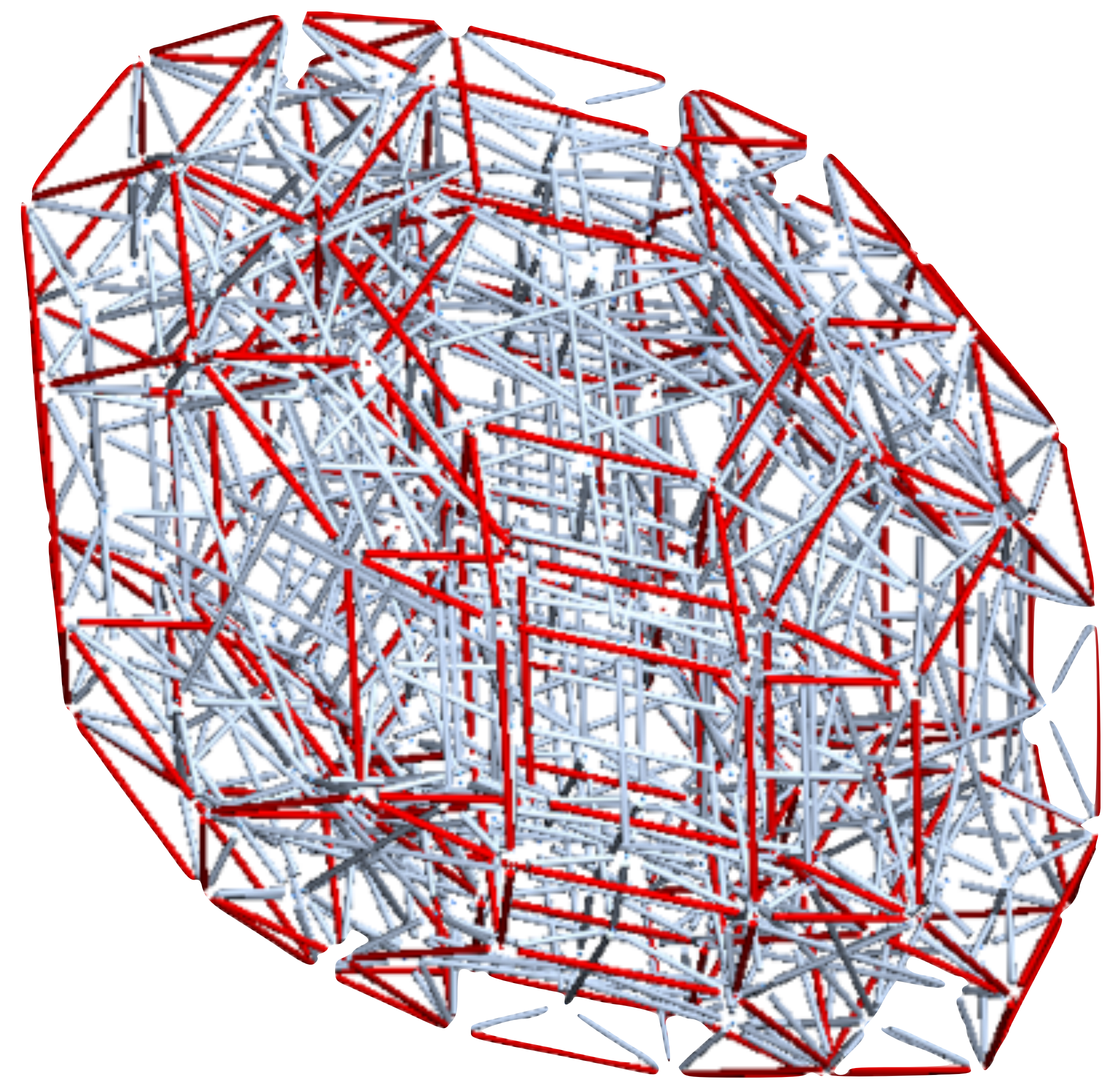
# A *Nice* Approach to Scattering in Non-planar Theories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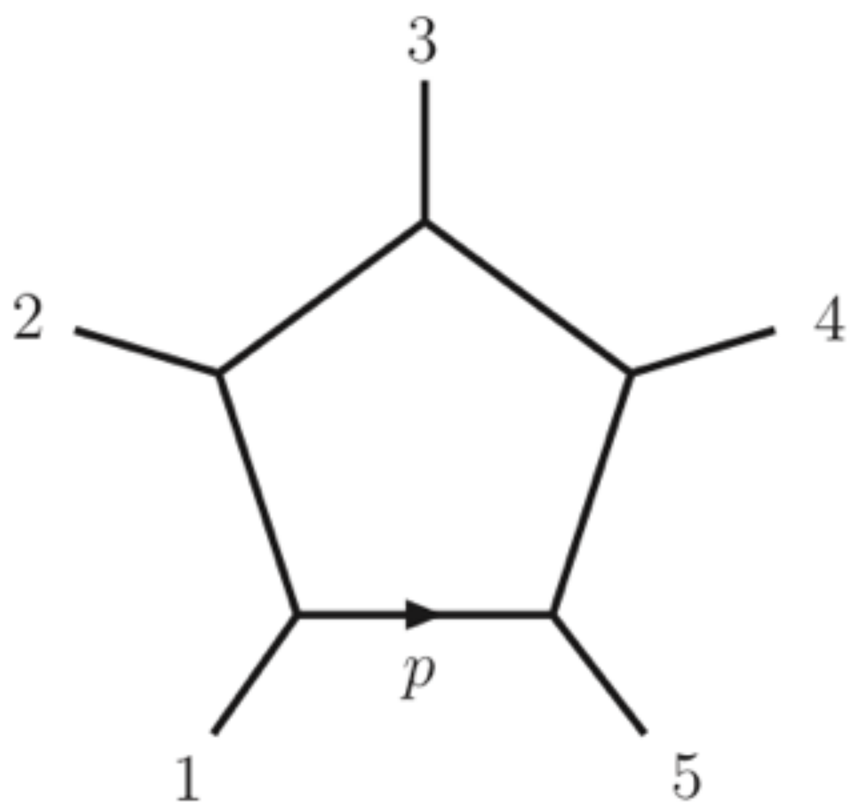
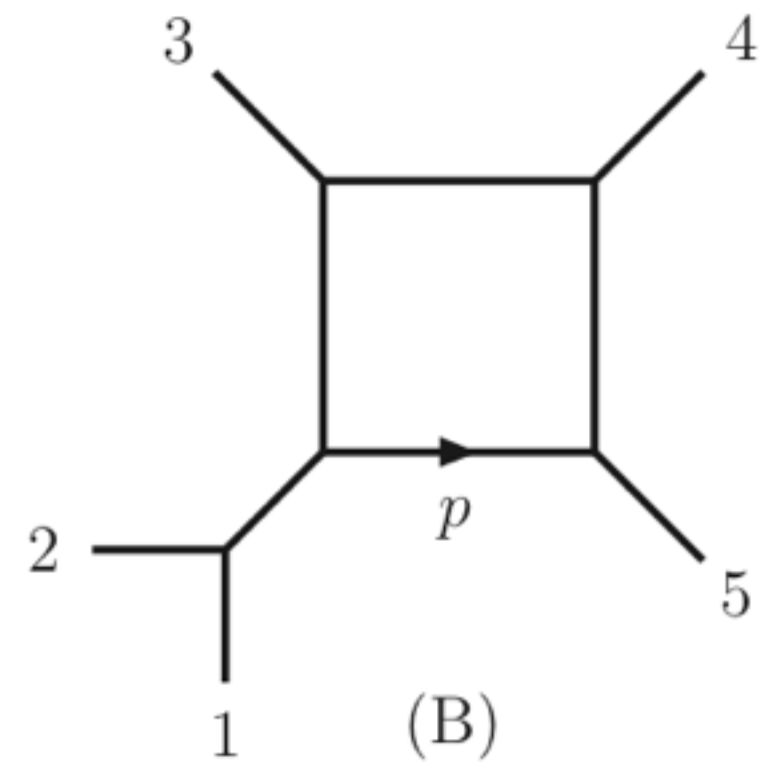
# Five point 1-loop (no triangles, no bubbles)

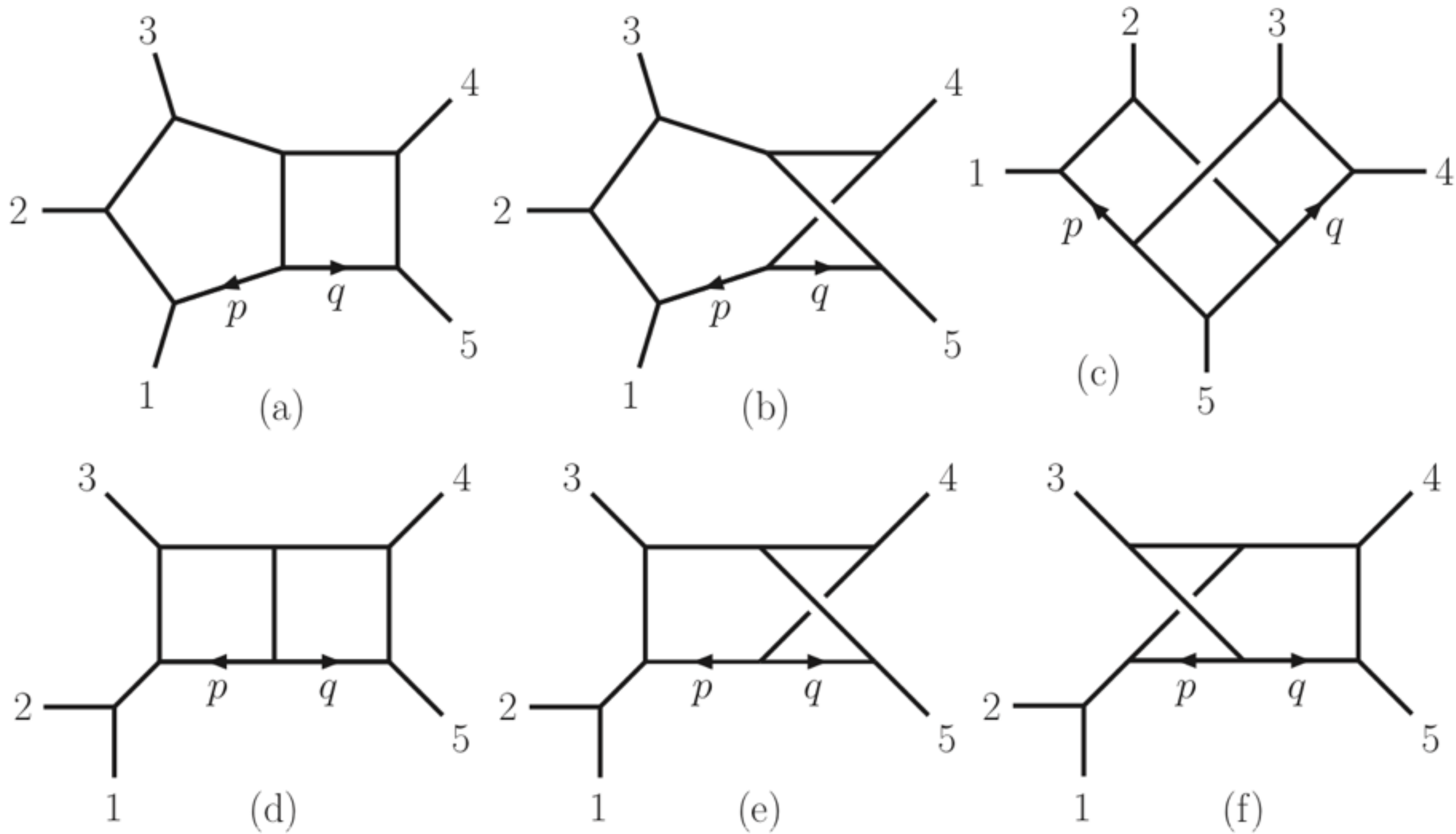



(P)



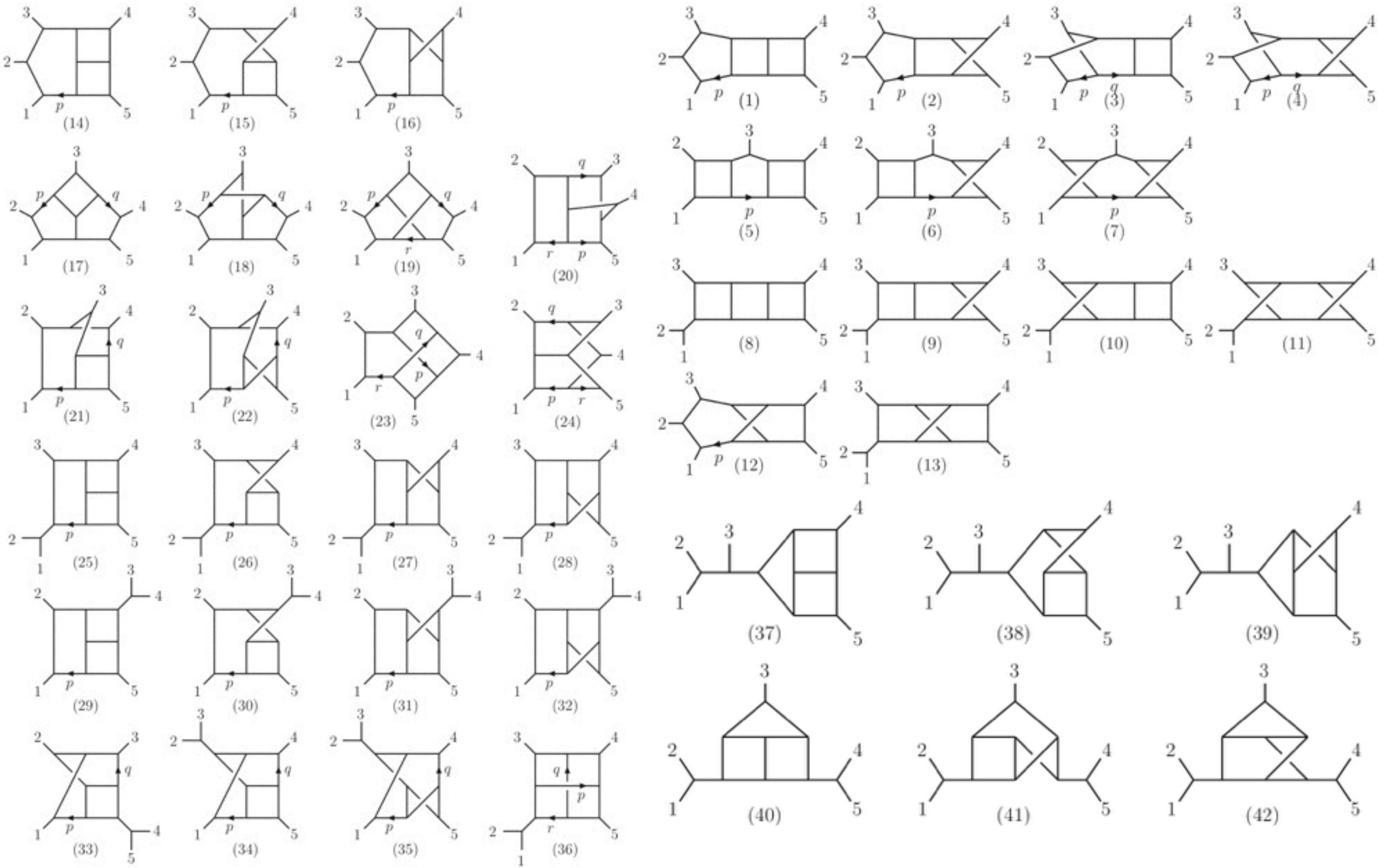

(B)

# Five point 2-loop (no triangles, no bubbles)

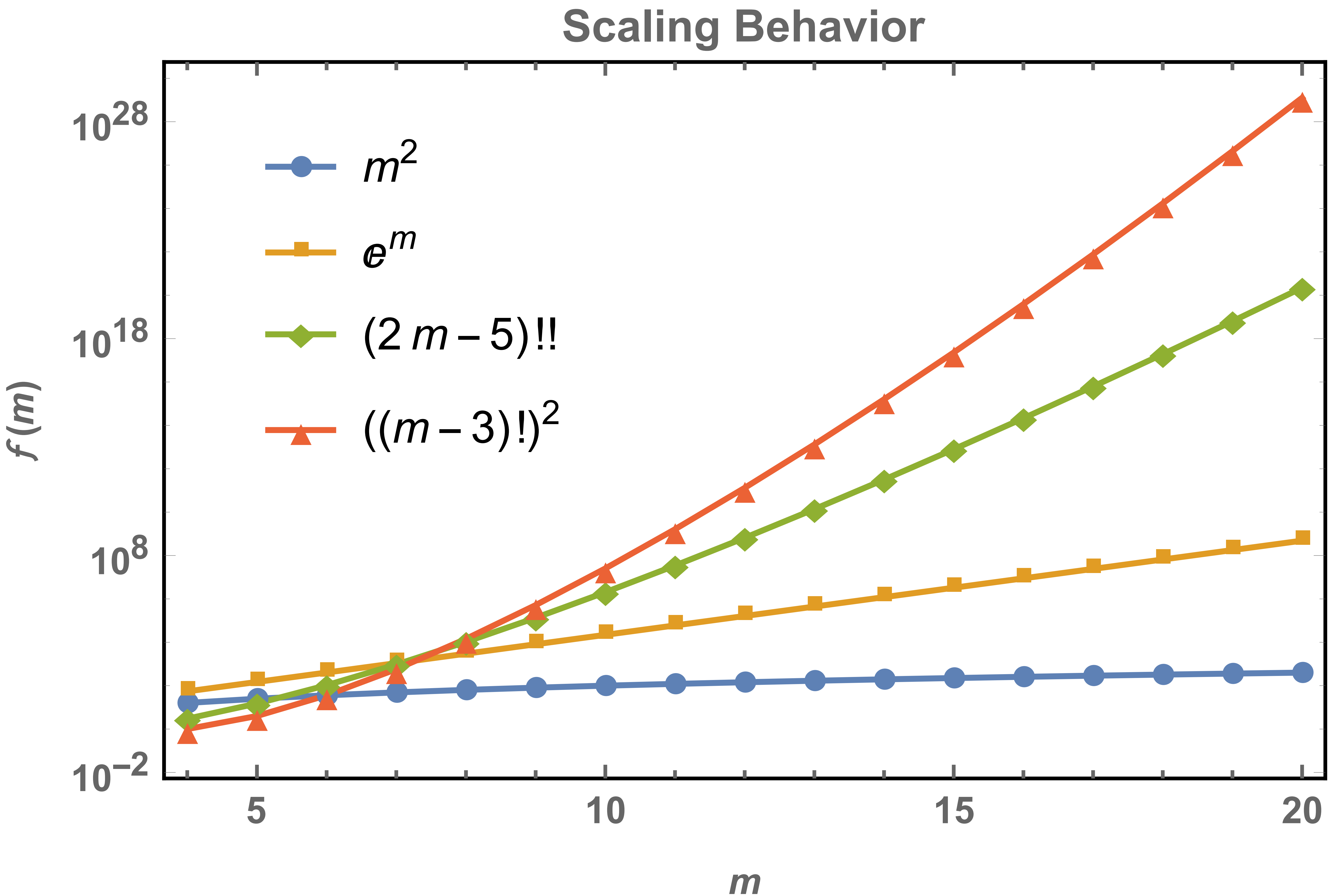


# Five point 3-loop (no bubbles, no triangles)

JJMC, Johansson (to appear)

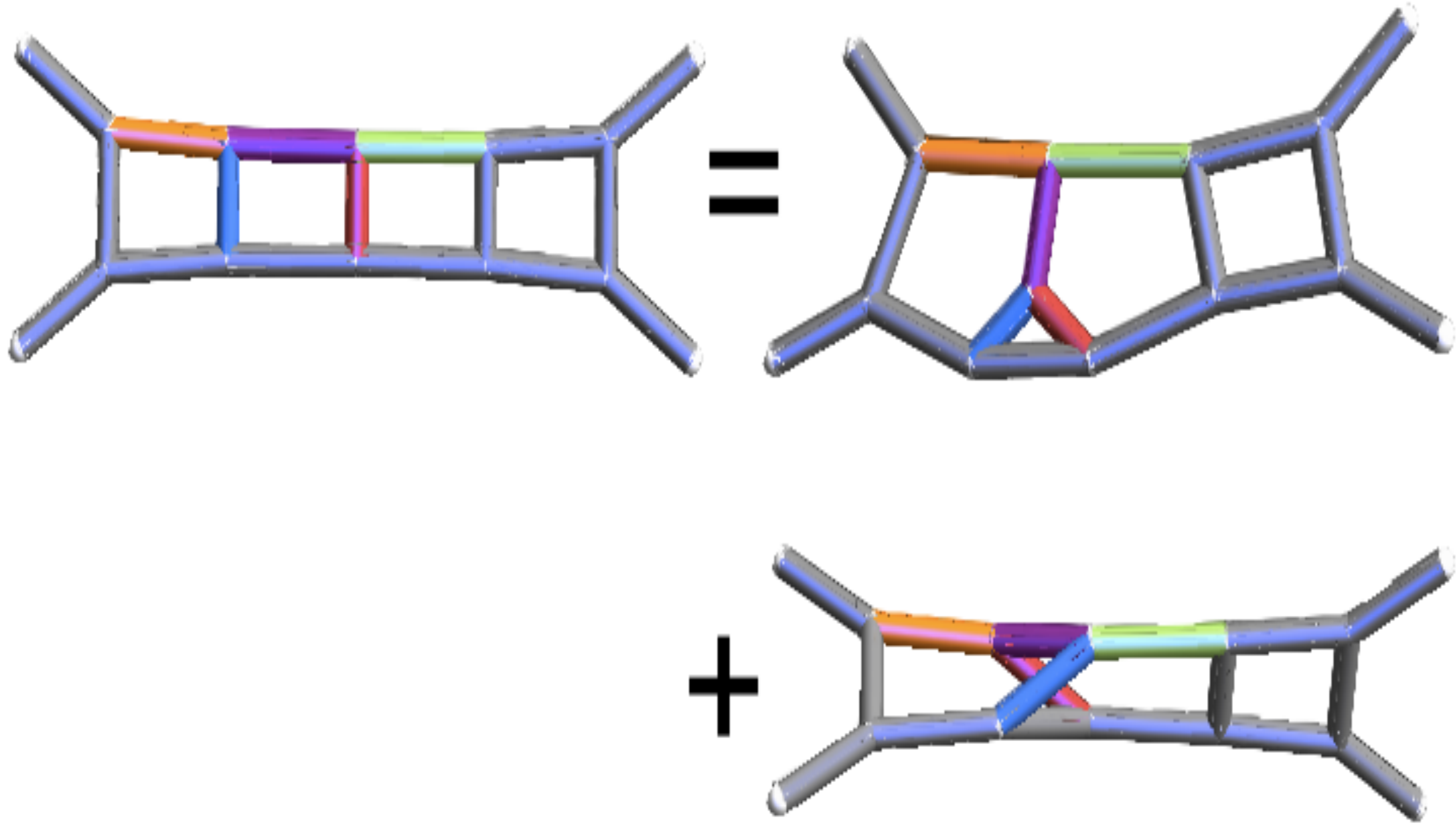


# Number of graphs grows factorially:

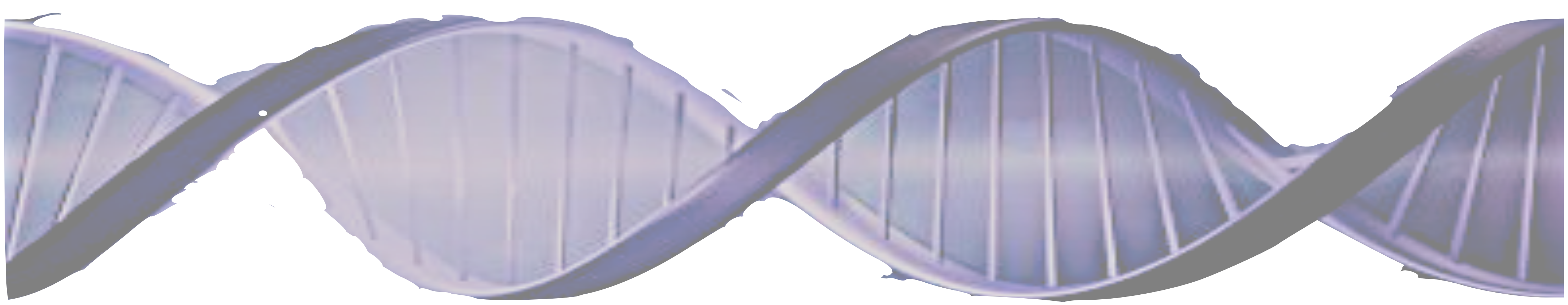


Bern, JJMC, Johansson (`08, `10)

# Color and Kinematics dance together.

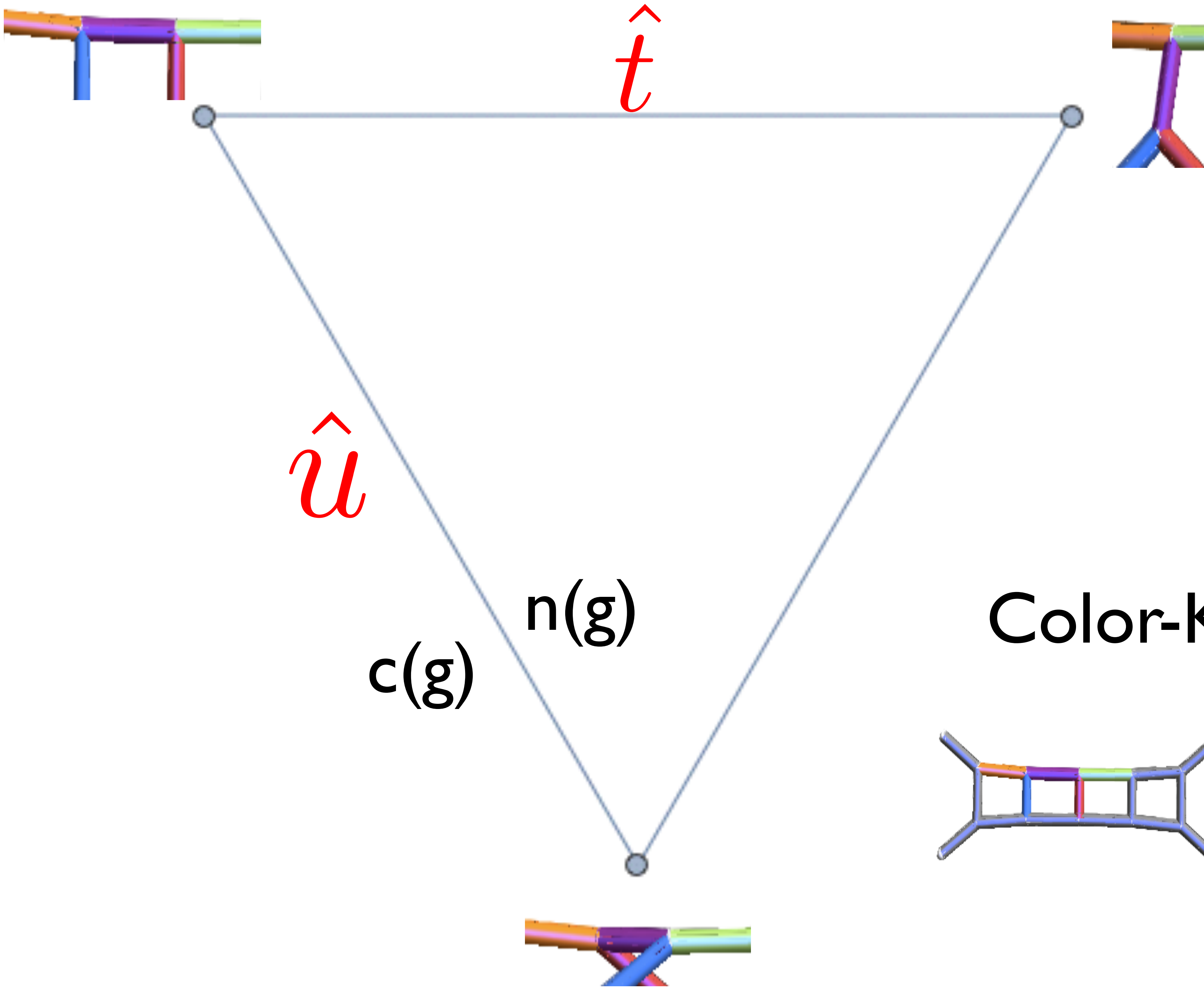


Solving Yang-Mills theories means solving Gravity theories.

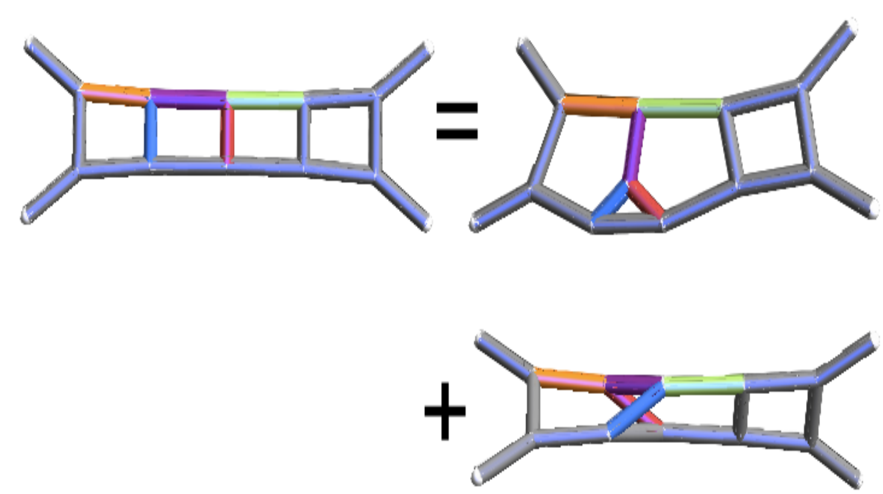


$$\mathcal{A}_m^{\text{tree}}(12\ldots m) = \sum_{g\in\Gamma_{m,\text{UO}}^{\text{tree}}} \frac{c(g)n(g)}{d(g)}$$

YM

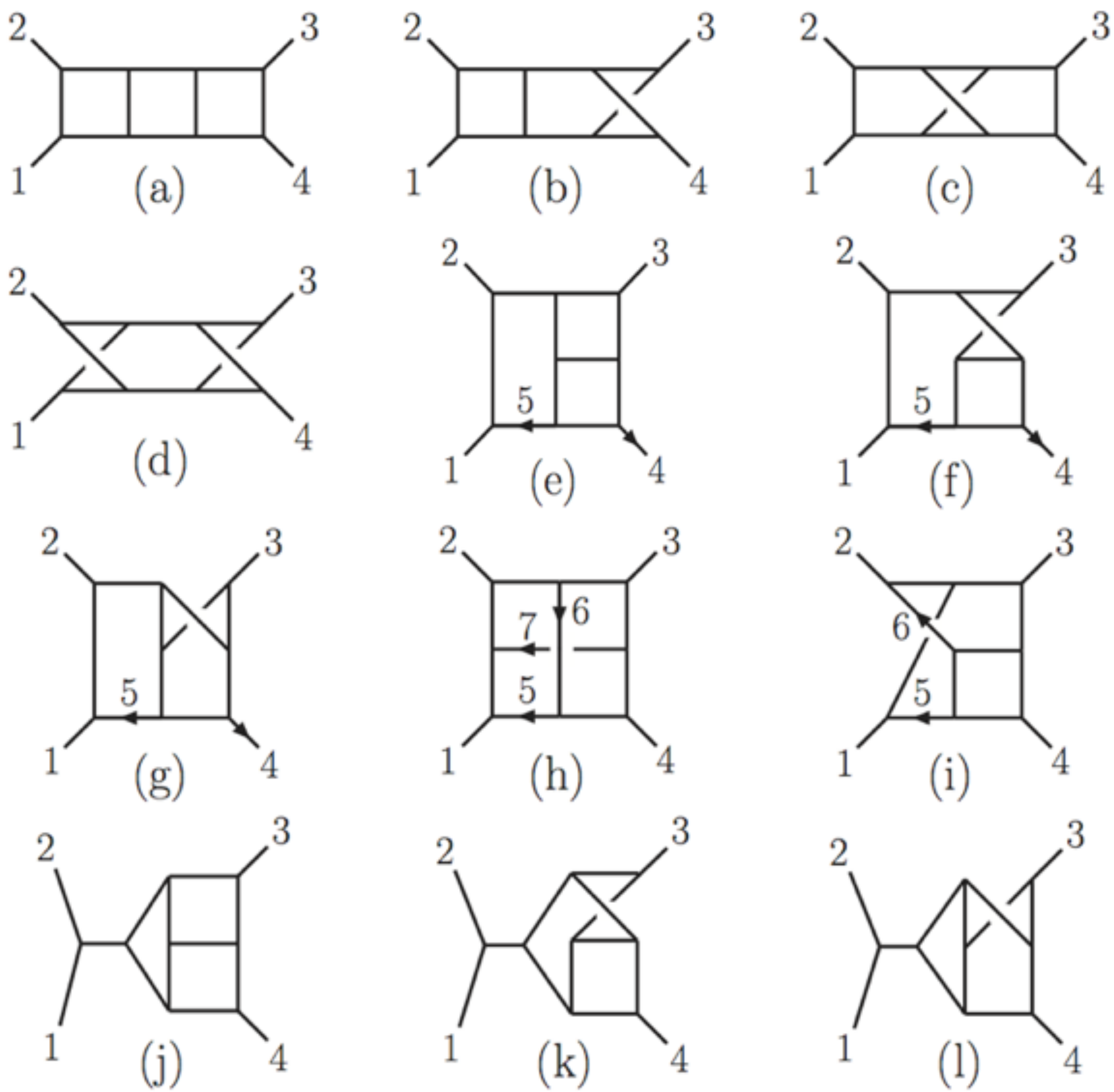



Color-Kinematics

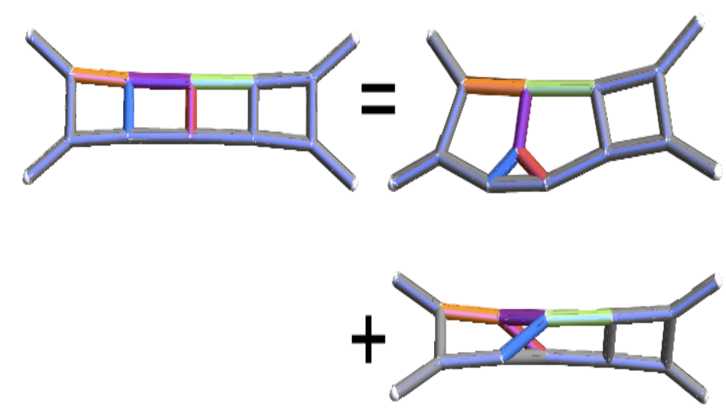


# Calculate by Exploiting Color-Kinematics Duality

Bern, JJMC, Johansson (`08, `10)




Leads to important constraints at tree & loop-level for gauge theories



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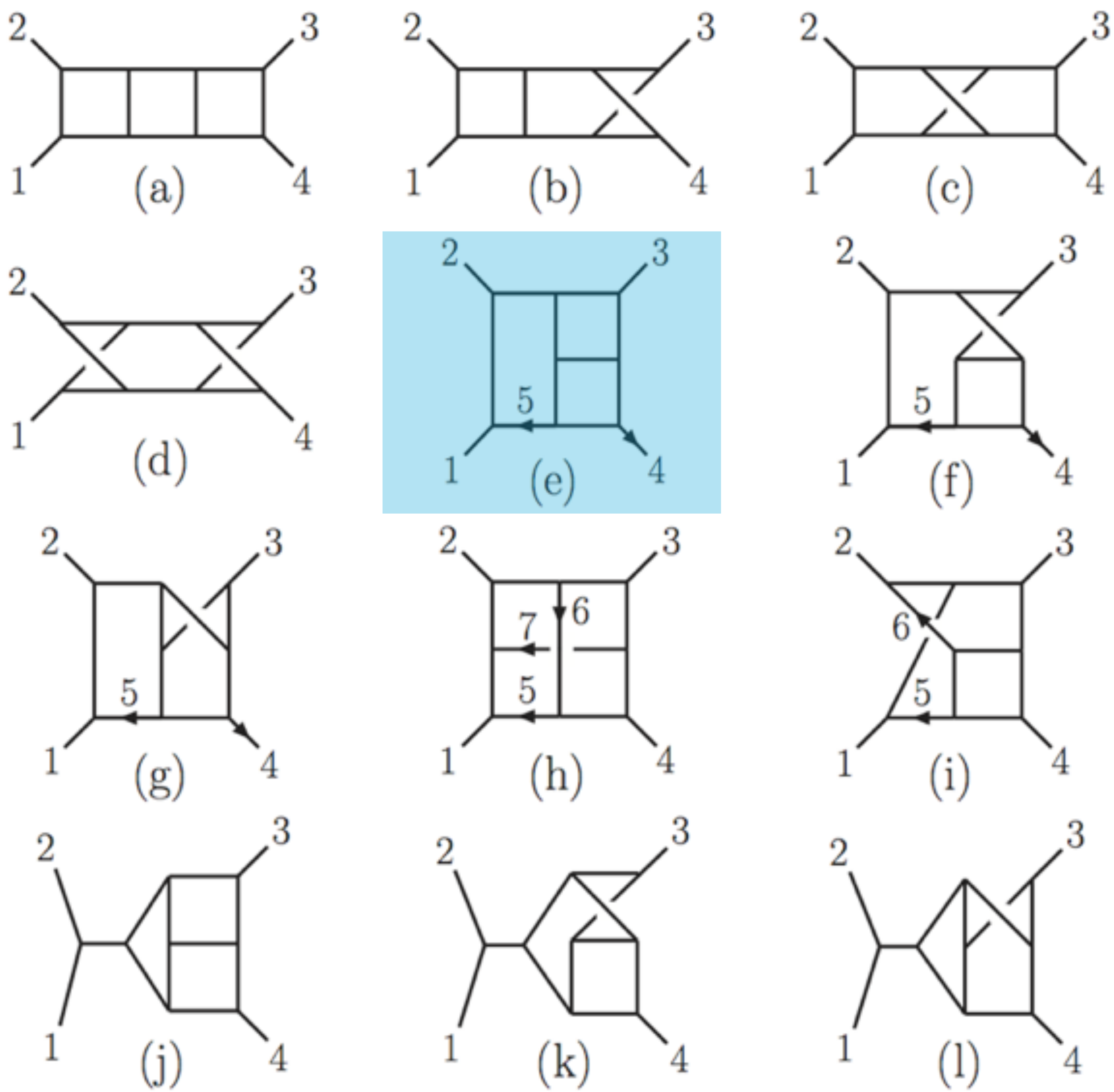


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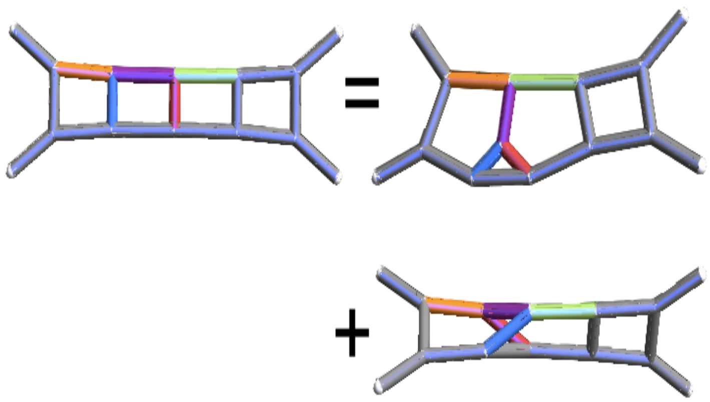

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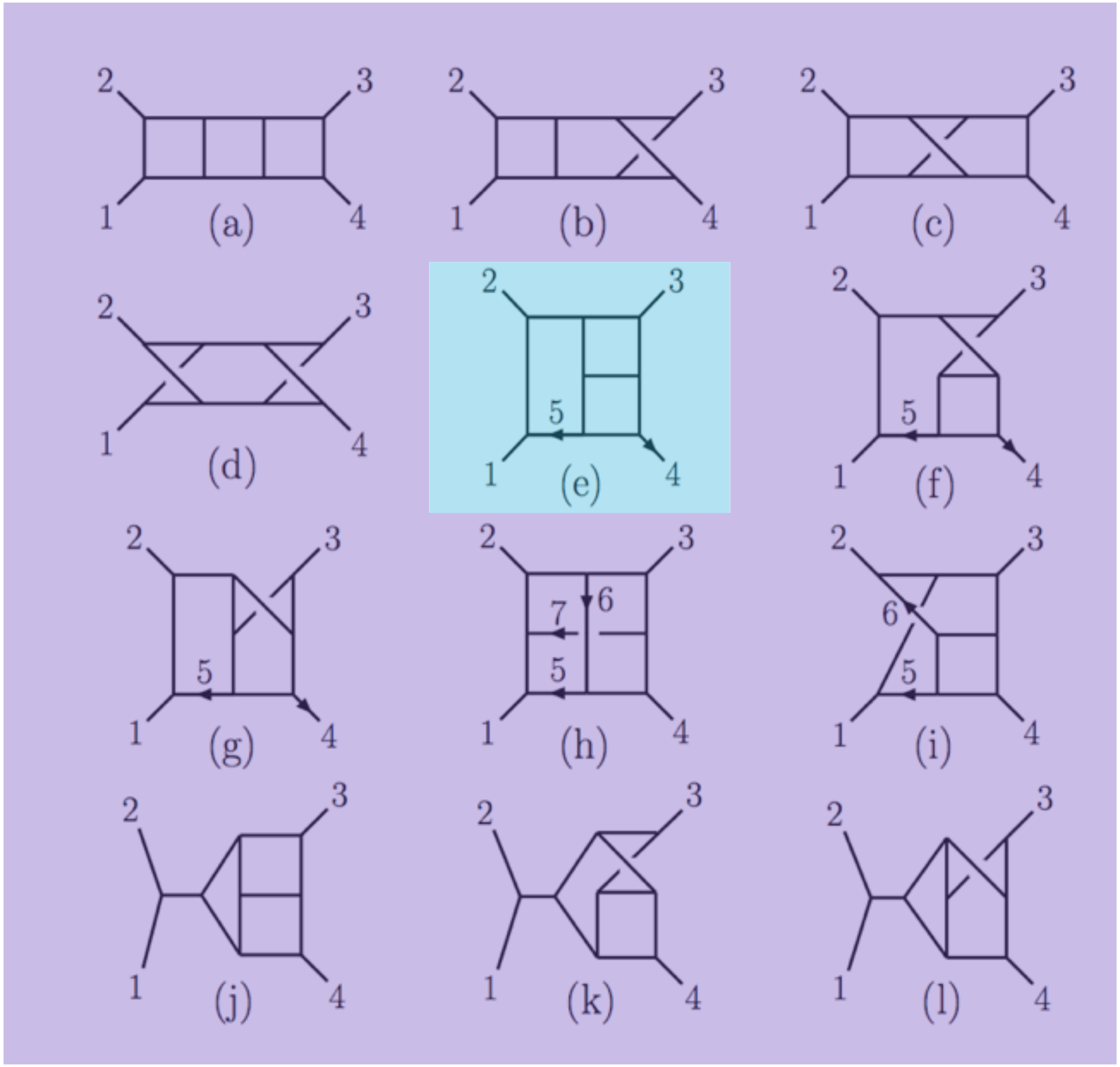


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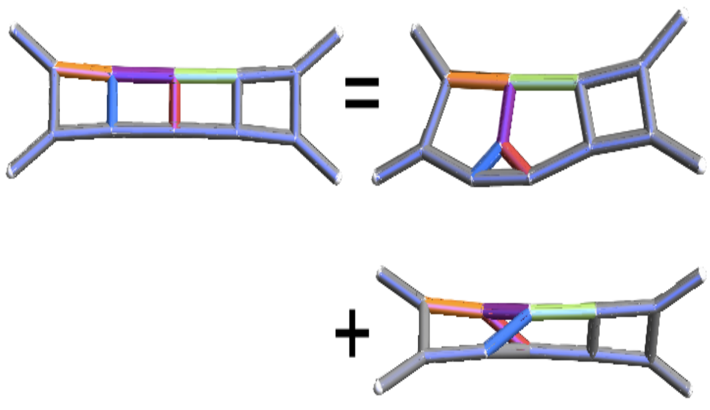
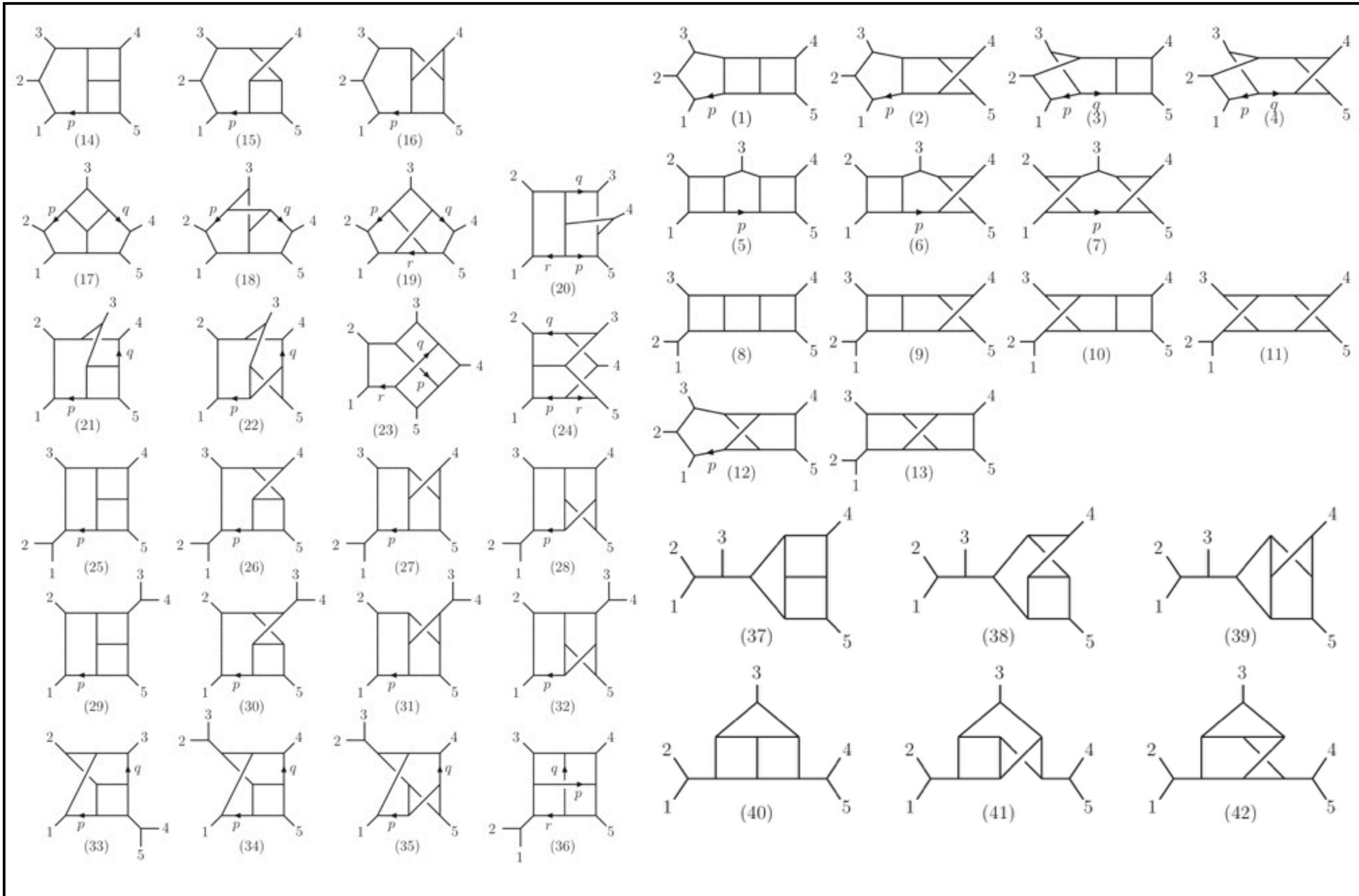



**Gluons for (almost) nothing... gravitons for free!**

# Five point 3-loop N=4 SYM & N=8 SUGRA

JJMC, Johansson (to appear)



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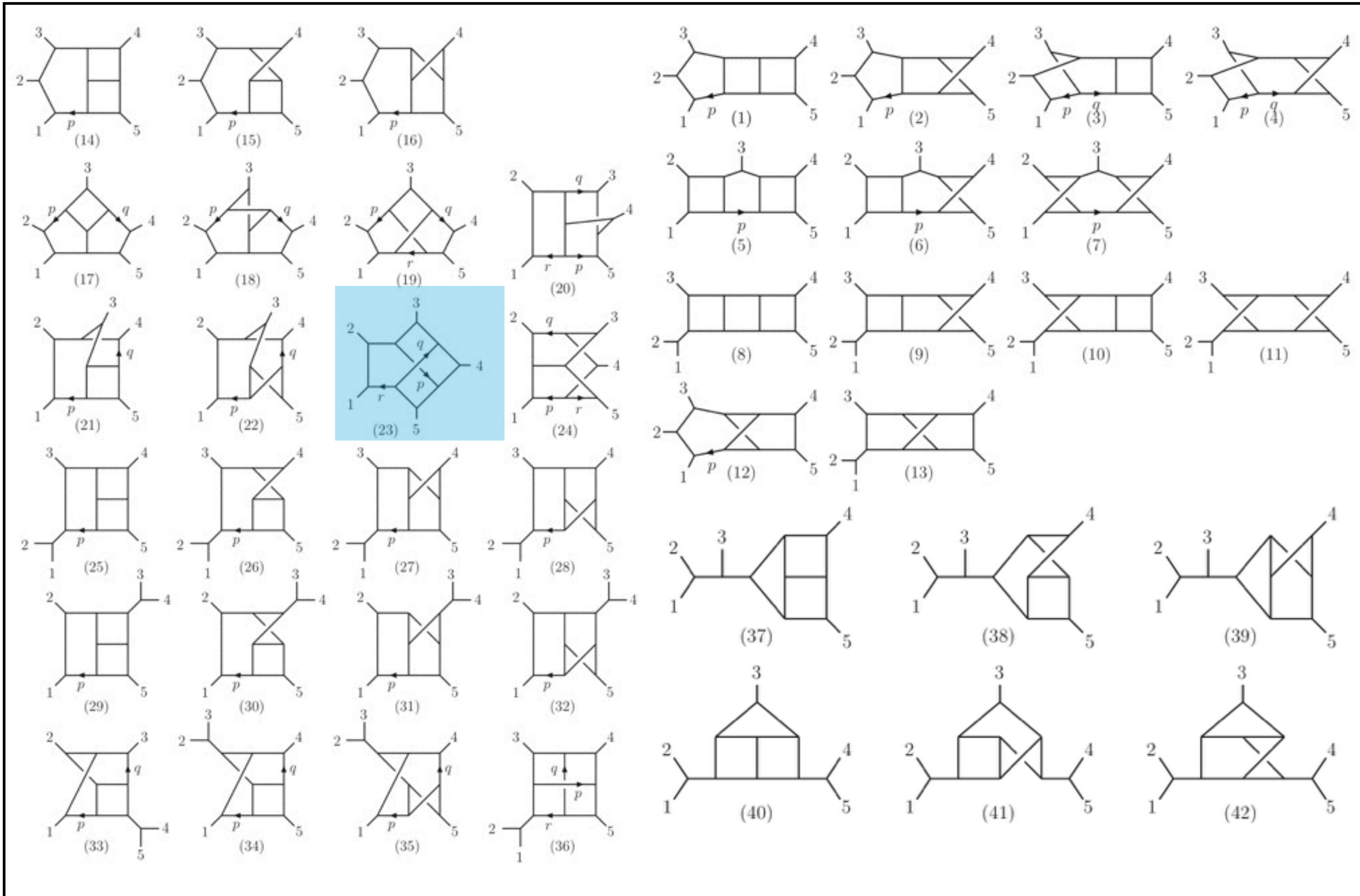

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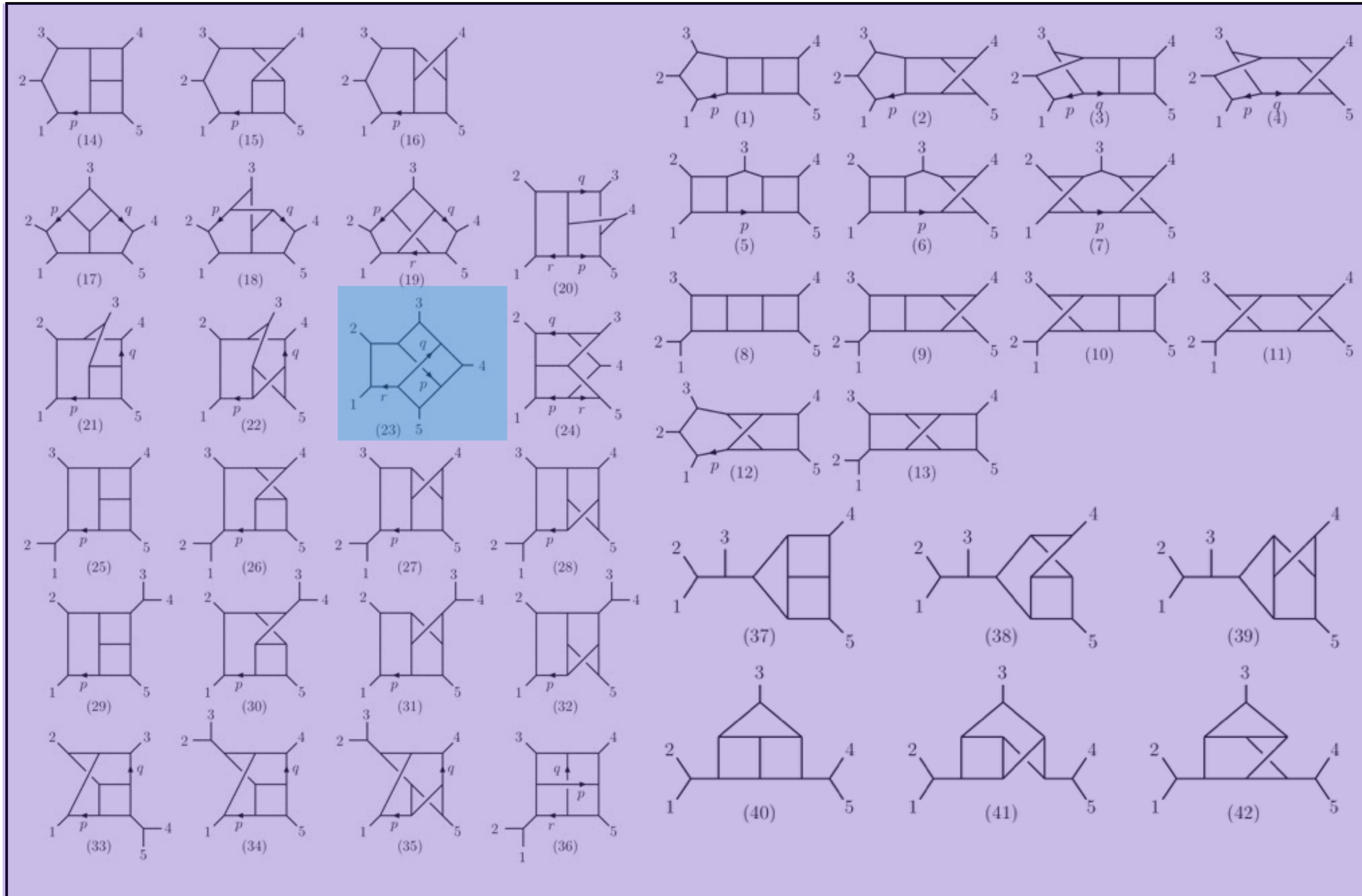
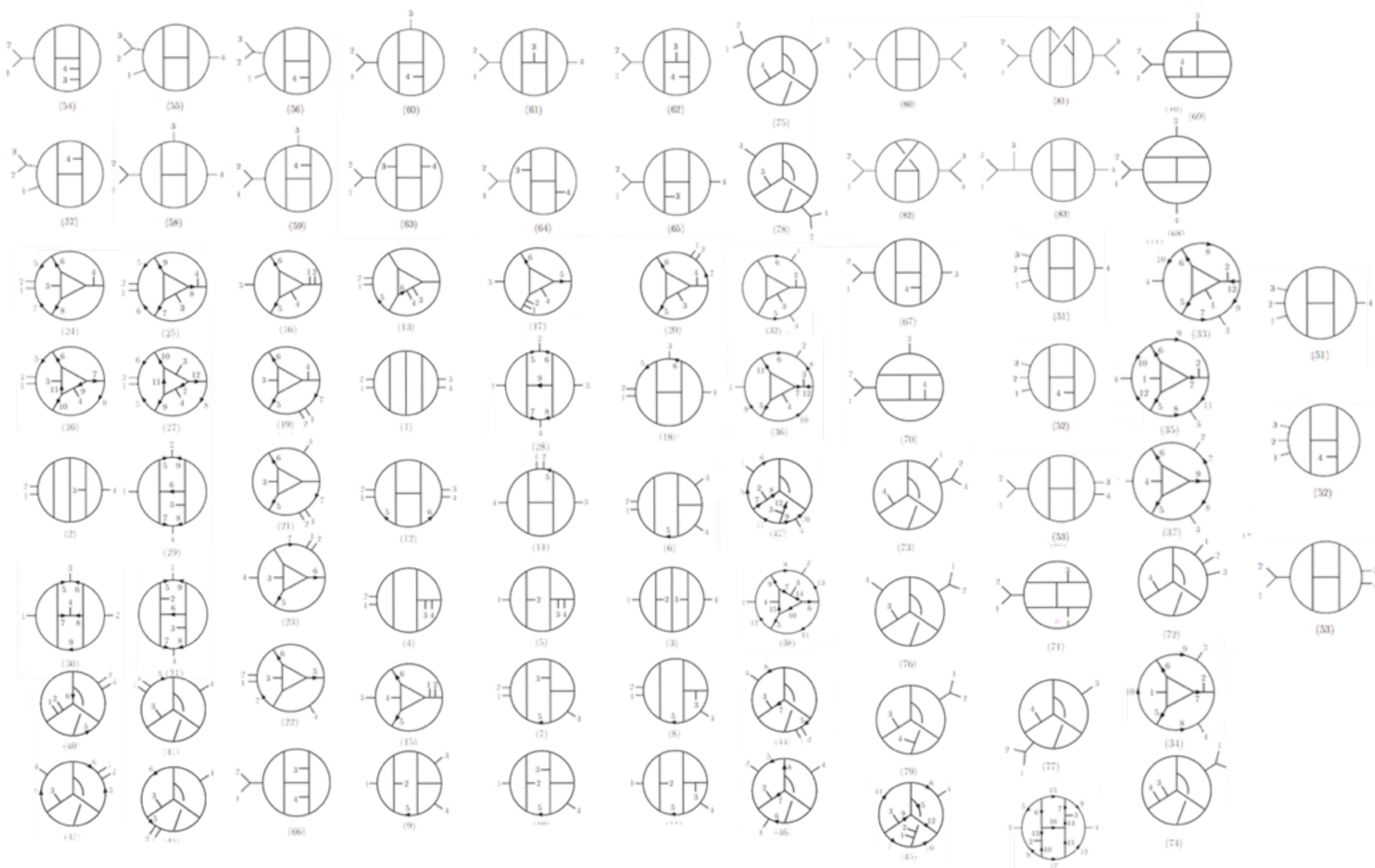


# Full four loop N=4 SYM & N=8 SUGRA

Bern, JJMC, Dixon, Johansson, Roiban (2012)



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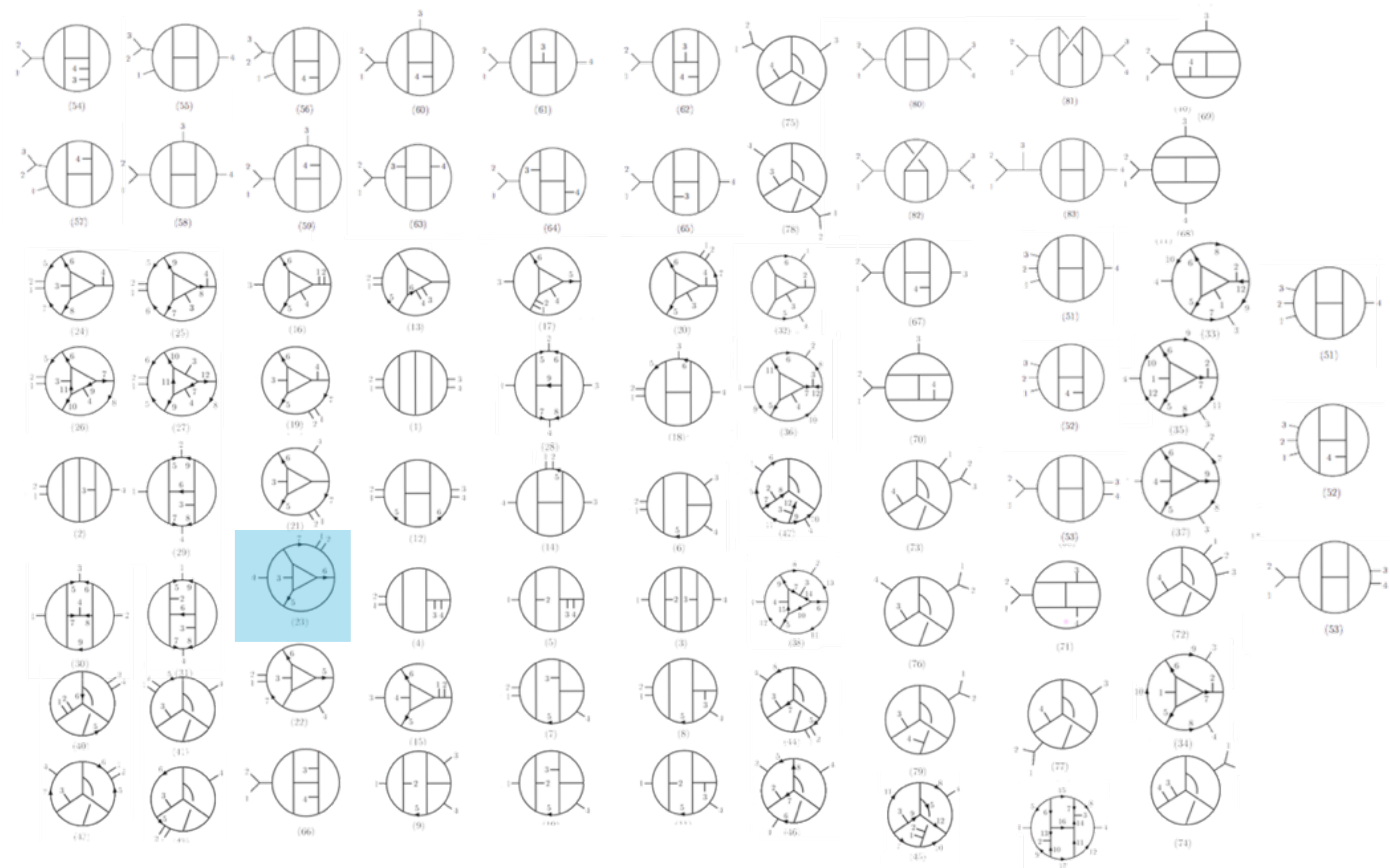

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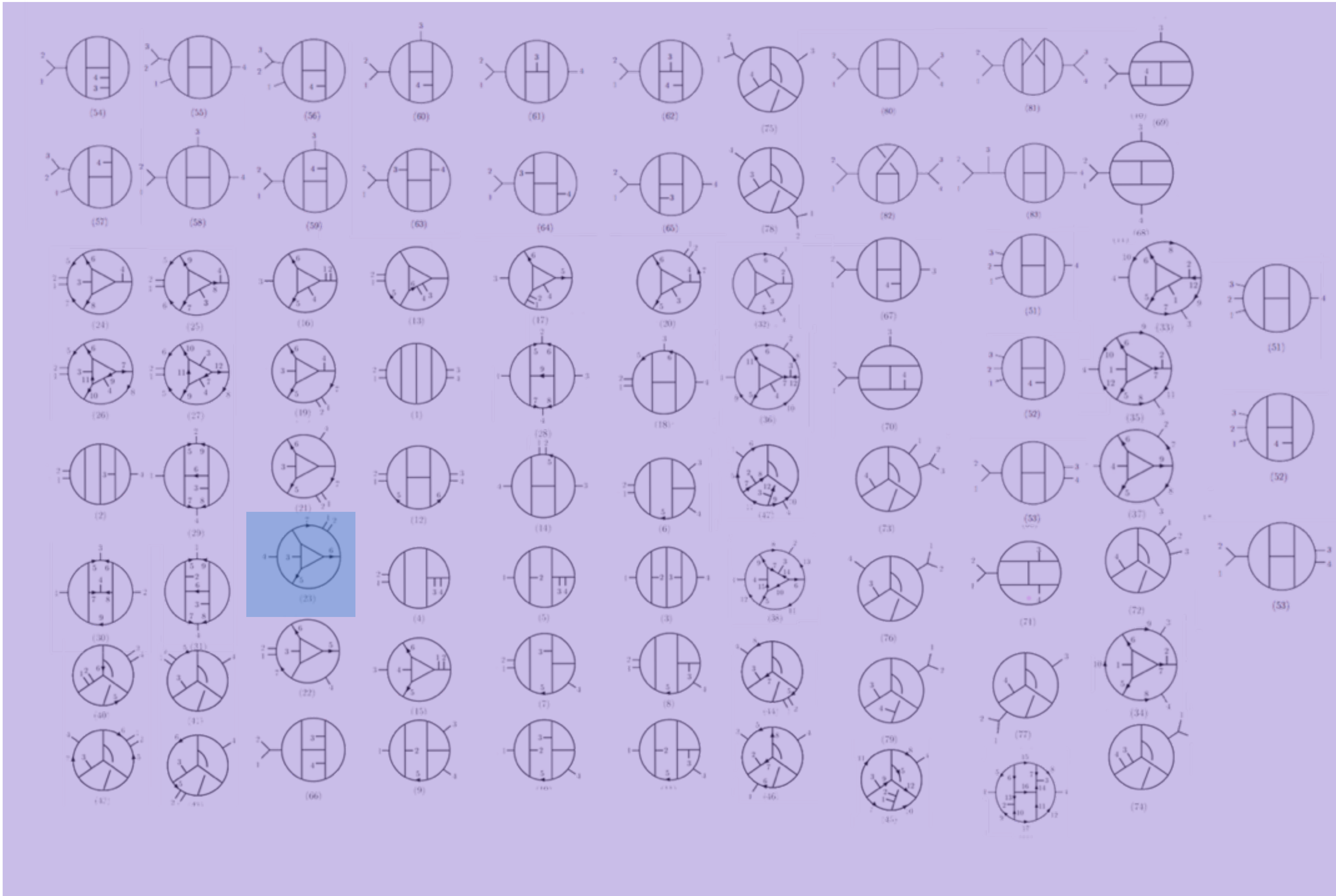
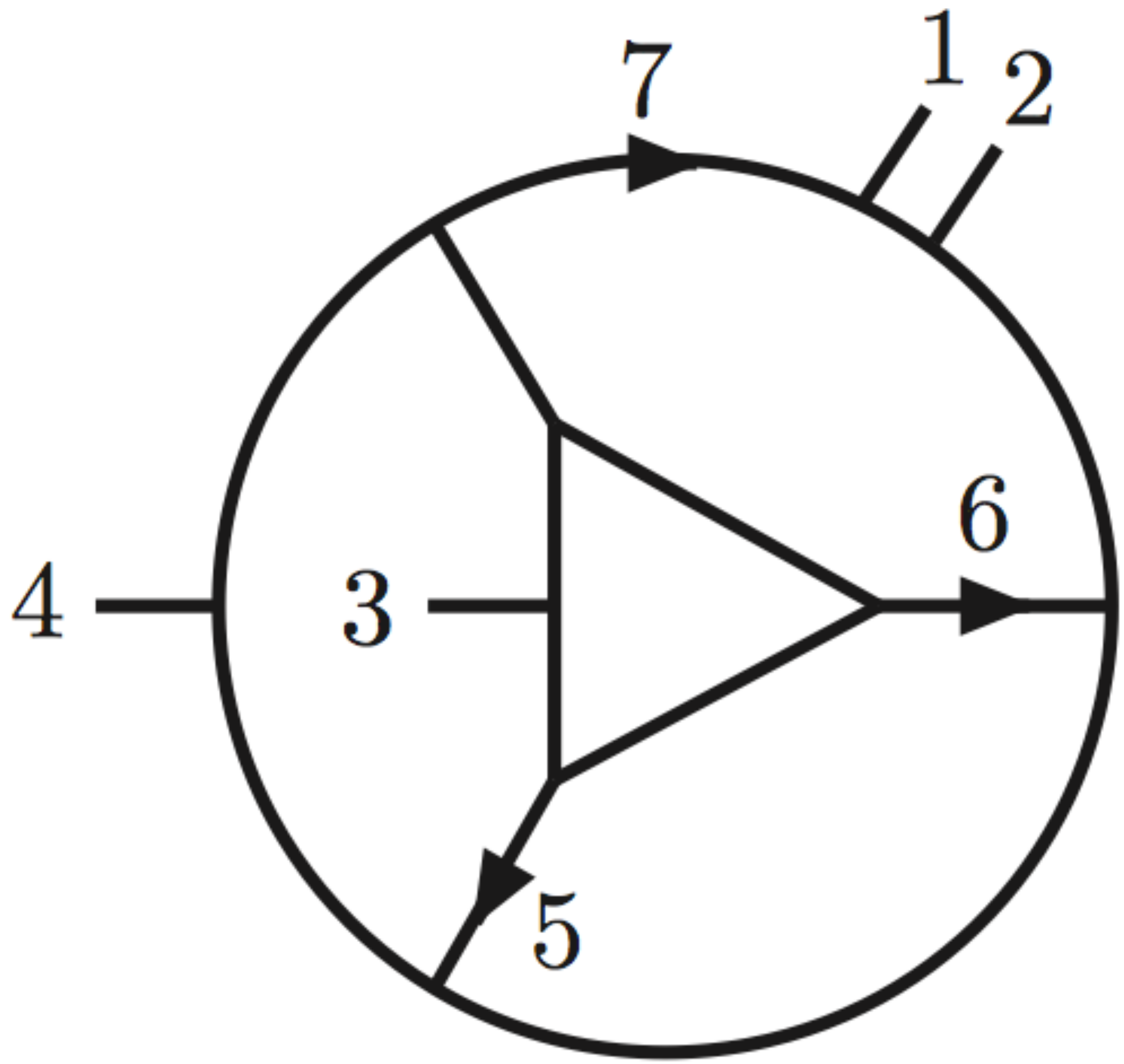


# 4-loops Maximal SUSY




Many things to be learned, not the least, the existence of integral relations between gauge and gravity theories

Bern, JJMC, Dixon, Johansson, Roiban (2012)

# Problem Solved?

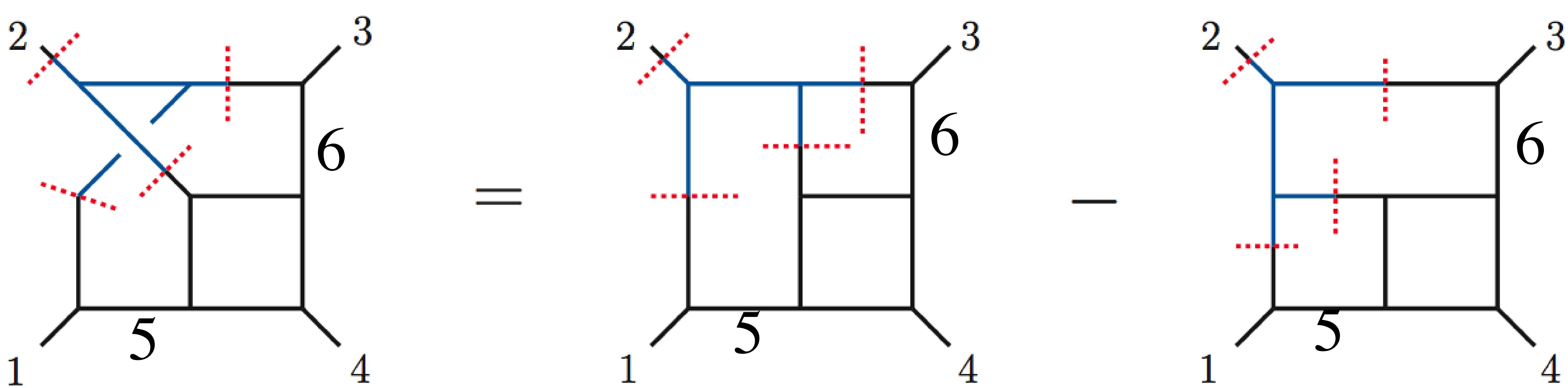
No.

We want all-order understanding!

# What's the barrier?

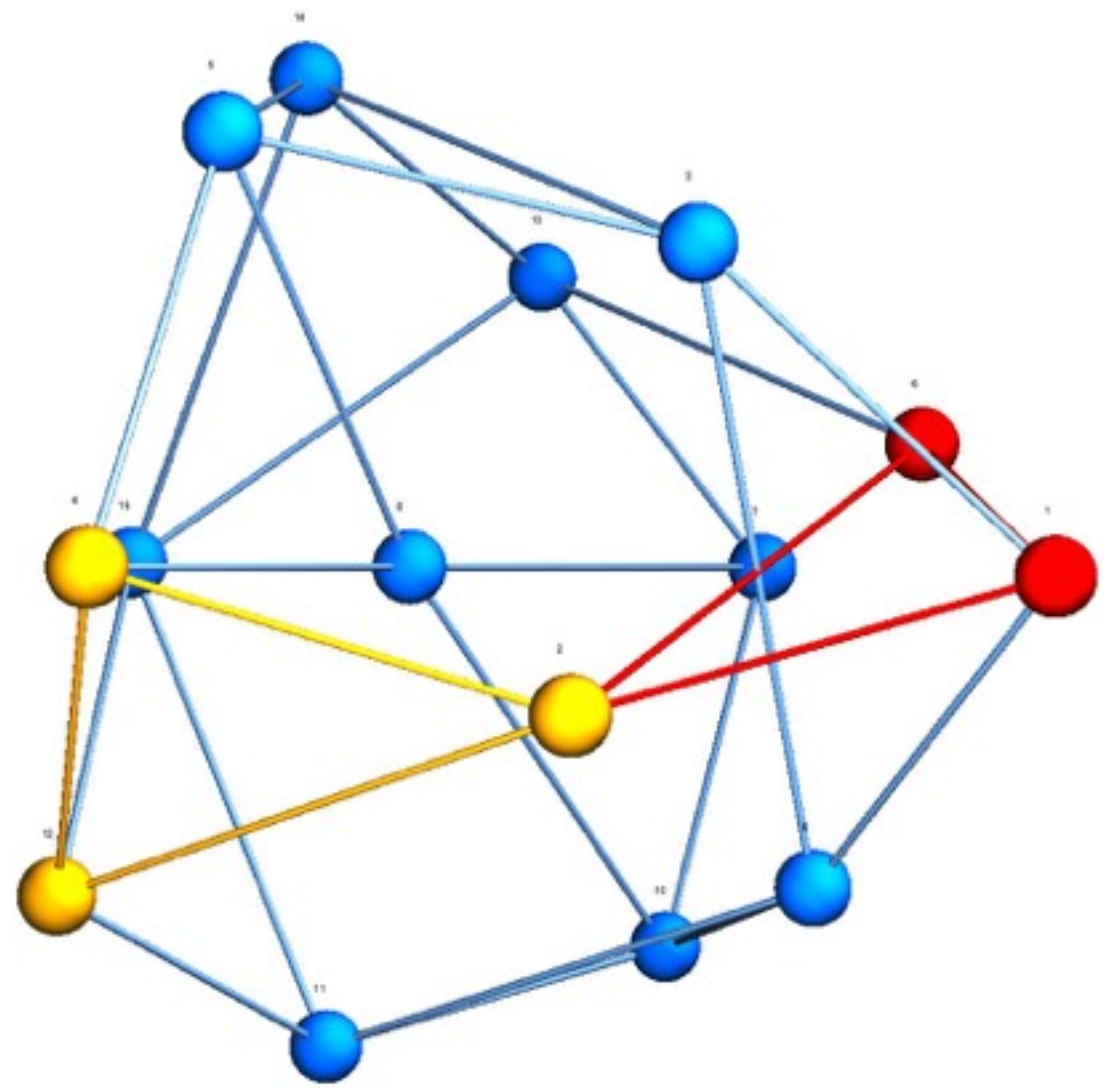
Frustrating Problem:

- Exploiting Color-Kinematics duality at loop-level means solving functional equations: number of master graphs controlled, but now need an ansatz.



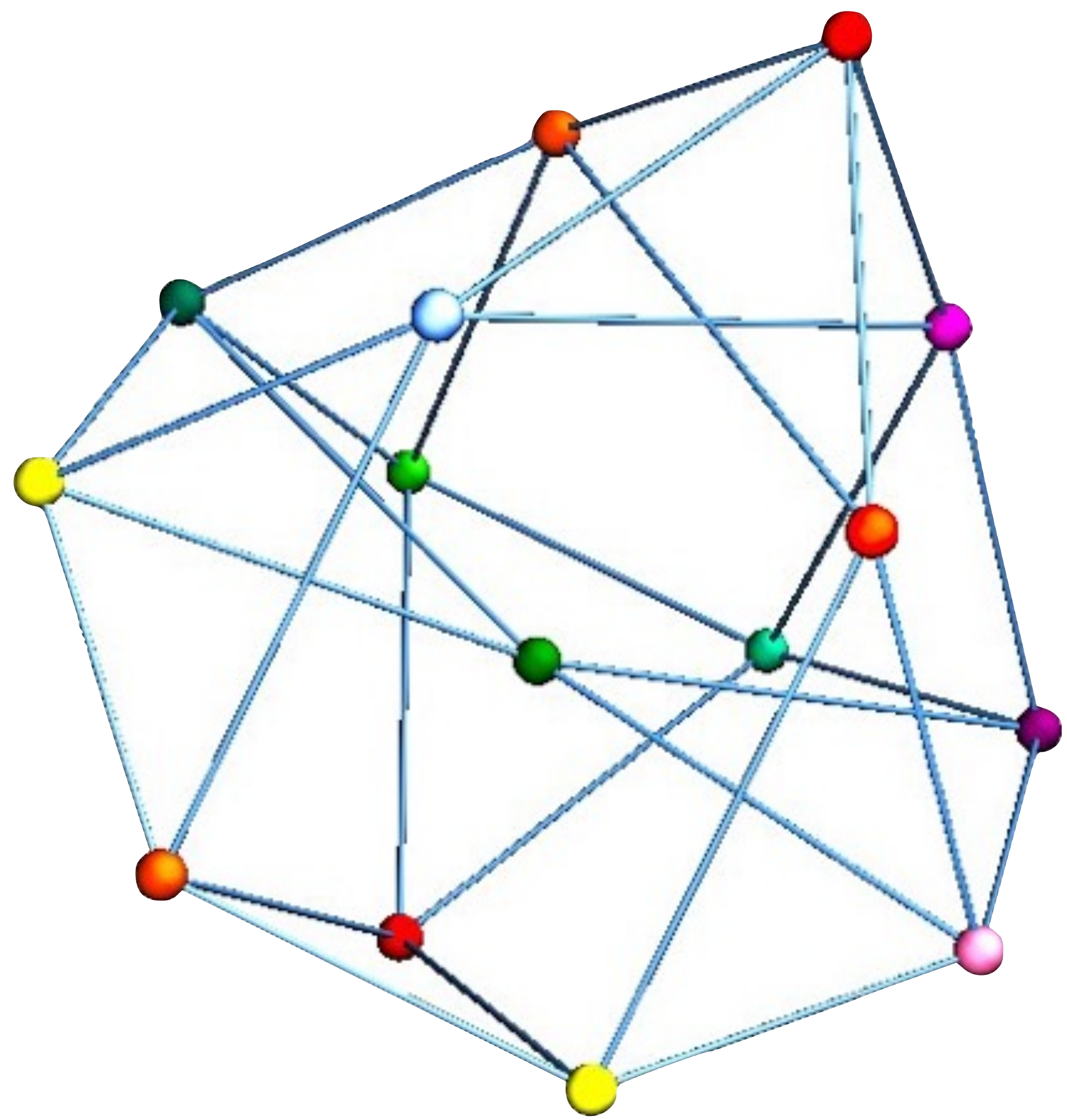

The set of multi loop Jacobi equations will relate the ***same*** numerator functions with permuted arguments.

tree-level, no problem



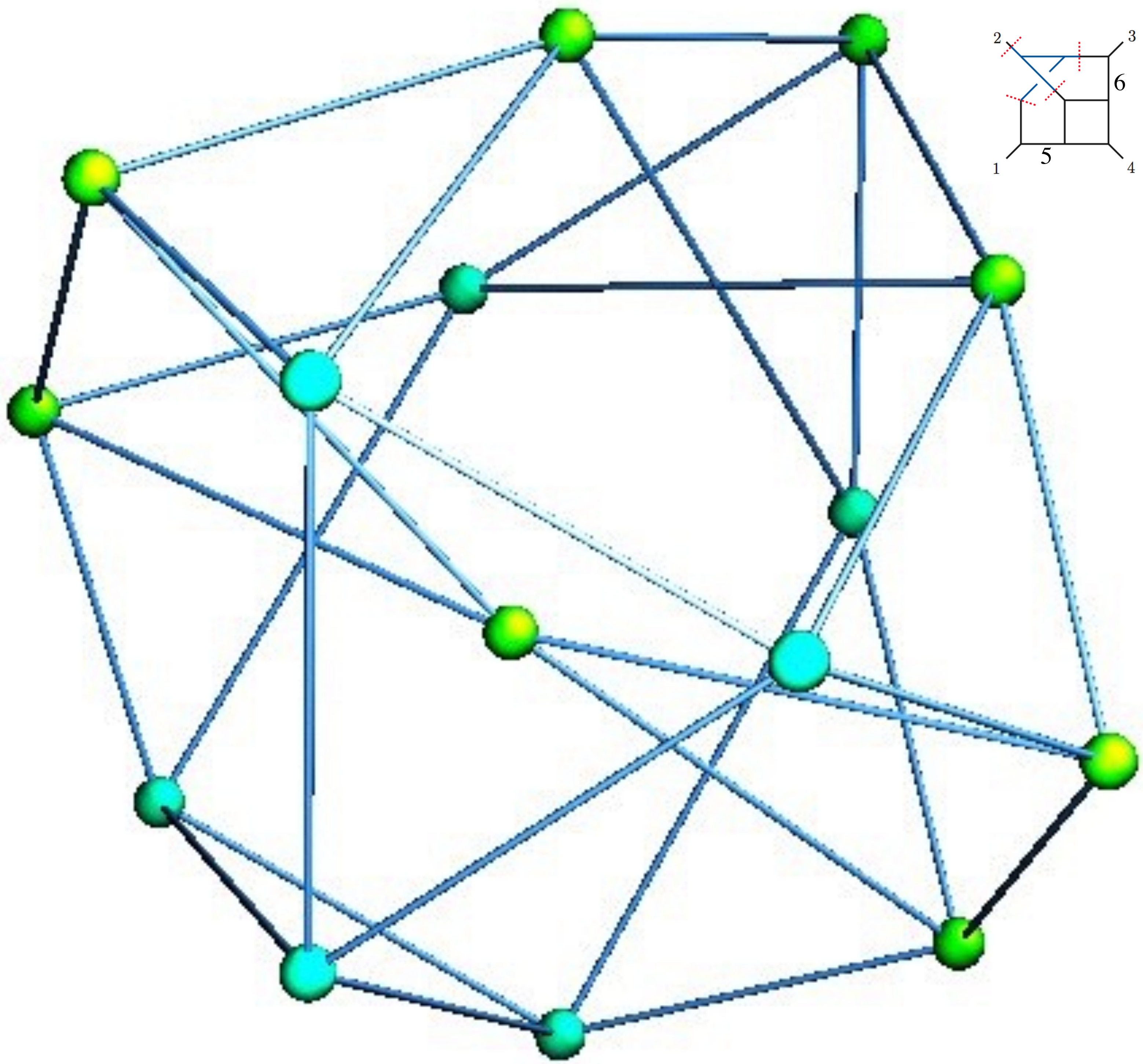
each vertex is a graph

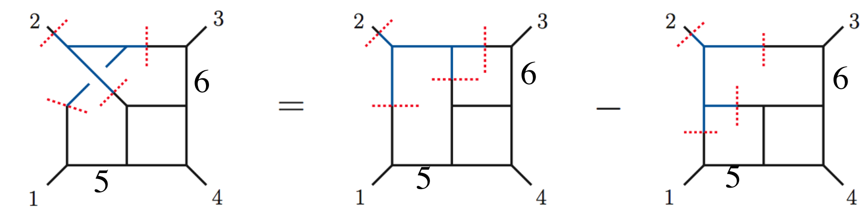
each triangle represents a Jacobi identity between graphs tree-level, no problem



as each node represents a separate graph, Jacobi eqns impose linear relations between numerators

# loop level, functional constraints






LABEL-SHIFTING PROBLEM

nodes can be the same graph with permuted labels!

# Let's get specific...

- It would be great to understand the origins of divergences in QFT of gravity.
- Evidence that they're intimately related to anomalies (some of which, at least, can be appreciated in non-supersymmetric YM.)

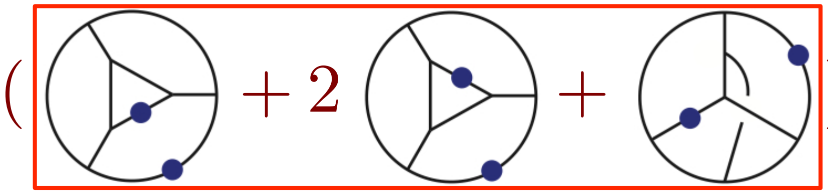
*JJMC, Kallosh, Roiban, Tseytlin '13*

*Bern, Davies, Dennen, Smirnov, Smirnov '13*

*[see Tristan's talk]*

Is N=8 SG perturbatively finite?

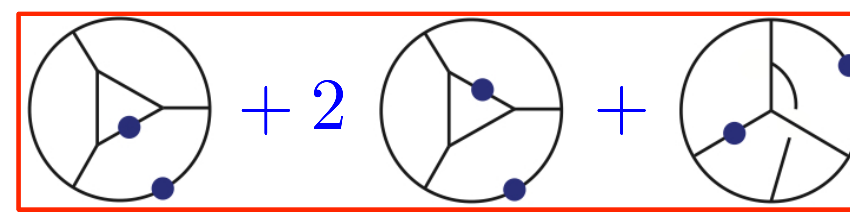
# Integral Relations at 4-loops

$\mathcal{M}_4^{(4)}\Big|_{\text{pole}} = -\frac{23}{8}\left(\frac{\kappa}{2}\right)^{10} stu(s^2+t^2+u^2)^2\, M_4^{\text{tree}}\,($  $)$

$-256 + \frac{2025}{8}$ ← 12- and 13-propagator integrals

11-propagator integrals; same as in sYM

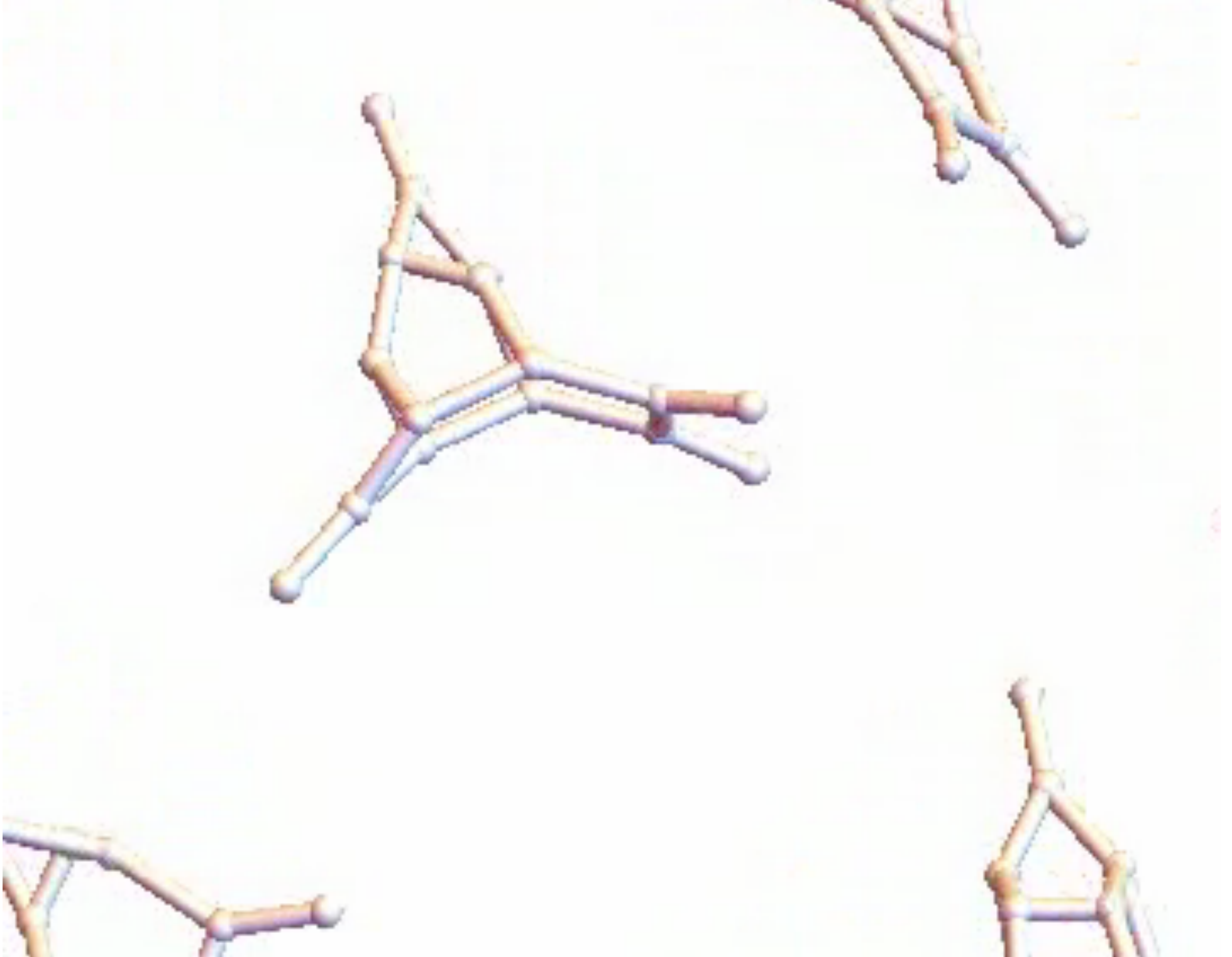
D=11/2

$\mathcal{A}_4^{(4)}\Big|_{\text{pole}}^{SU(N_c)} = -6\, g^{10}\, \mathcal{K}\, N_c^2 \Big(N_c^2$  $+ 12\, ($  $)\Big)$

$\times \Big(s\,(\text{Tr}_{1324}+\text{Tr}_{1423}) + t\,(\text{Tr}_{1243}+\text{Tr}_{1342}) + u\,(\text{Tr}_{1234}+\text{Tr}_{1432})\Big)$

D=11/2

5-loops? Need the SG integrand first!

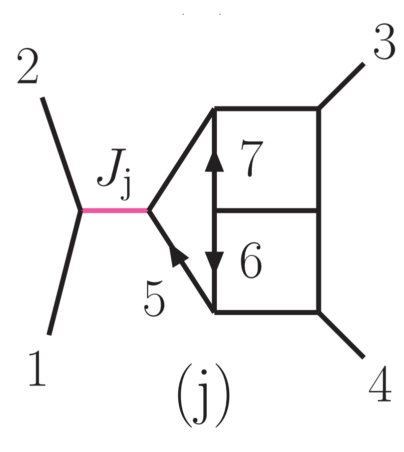


# 5-loops N=4 sYM:

- ~900 cubic graphs with no bubbles, and no triangles
- Jacobi's fix to a set of 2 non-planar masters!
- can impose a consistent minimal power-counting.
- can impose all symmetries

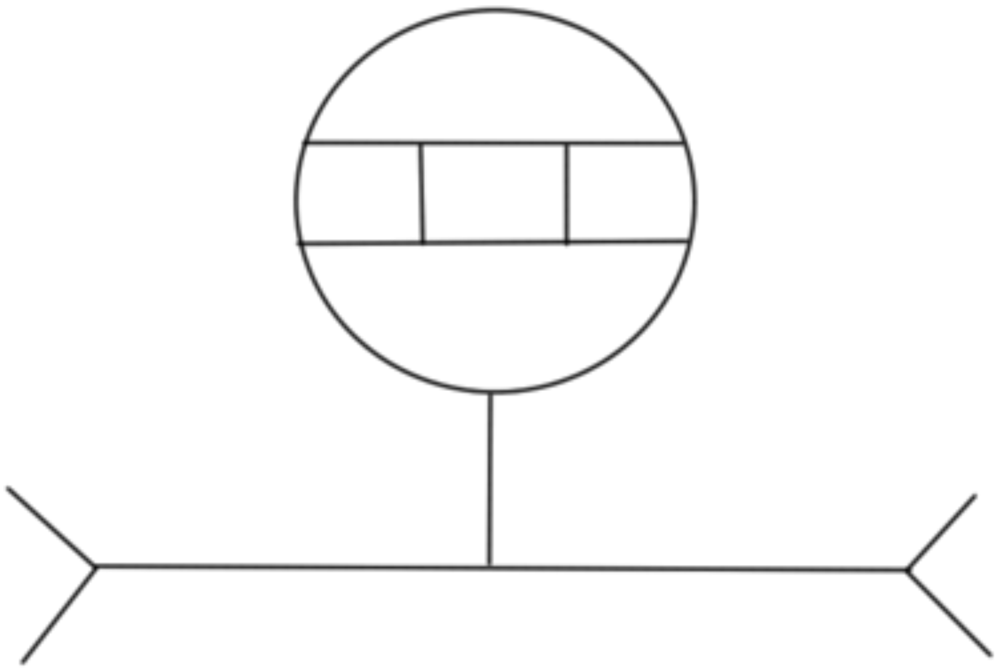
Maximal cuts break almost immediately!

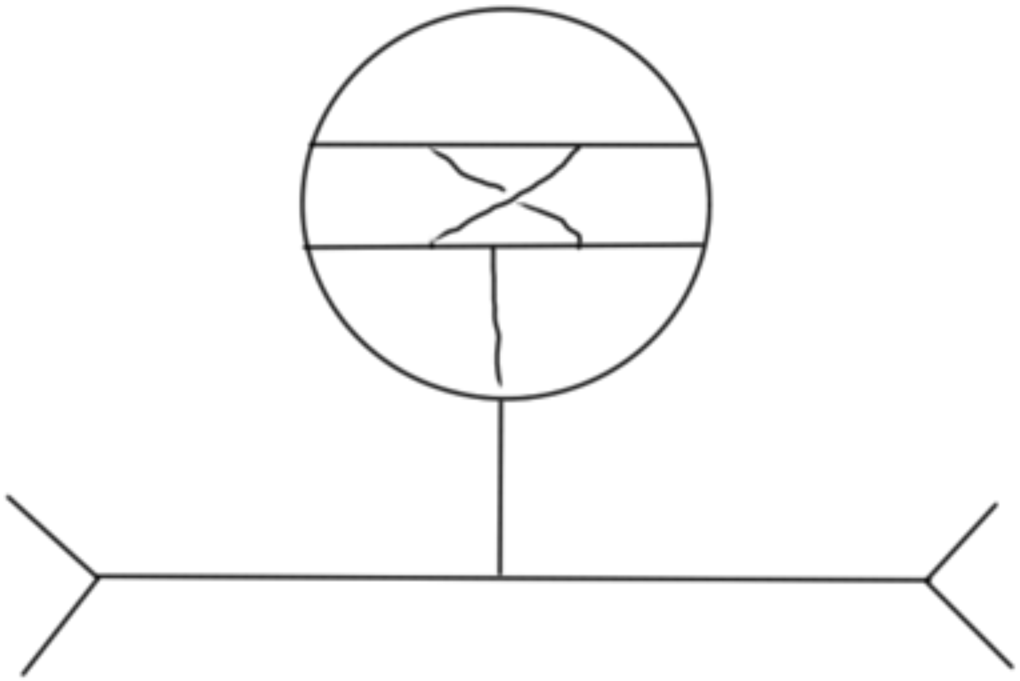
Just like 3-loops before including:



# Relaxing ansatz:

- Relax power counting
- Allow for add'l graphs (e.g non-planar triangles)
- Generalize prescription to handle **unusual** graphs
- Nonlocal numerators? (unbounded complexity)





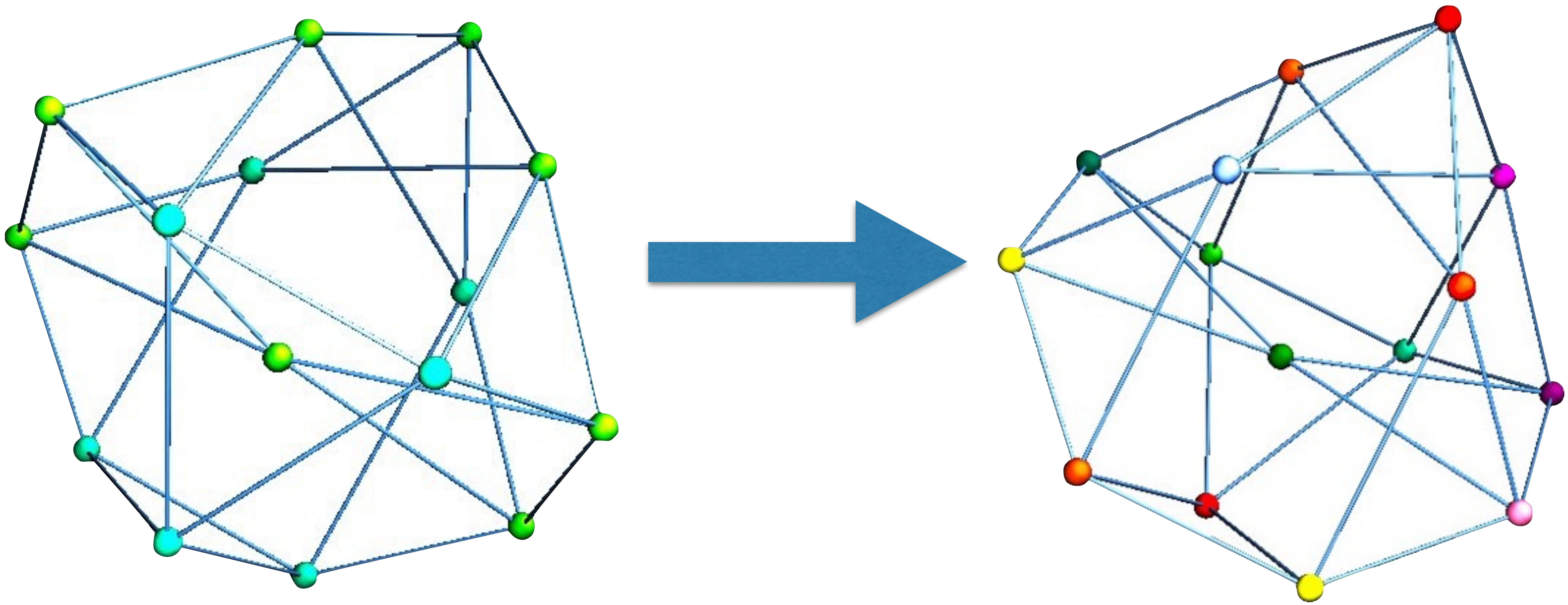
The sea of space to explore is unfortunately vast and expensive at 5 loops

Is there a non-ansatz path forward?

$$\left\langle \text{Is there a non-ansatz path forward?} \right\rangle$$

= **YES!**

## a solution to label-shifting



introduce a distinct graph for every possible labeling of m-point L-loop graph topologies

this will be isomorphic to a subset of (2L+m)-point tree graphs, with 2L “ext” labels: $\{l_1, -l_1, \ldots, -l_L, l_L\}$

Brief interlude, a comment on my title:

nice

*adjective:*

1. pleasing; agreeable; delightful.
2. characterized by great accuracy, precision, or delicacy.

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*caution*

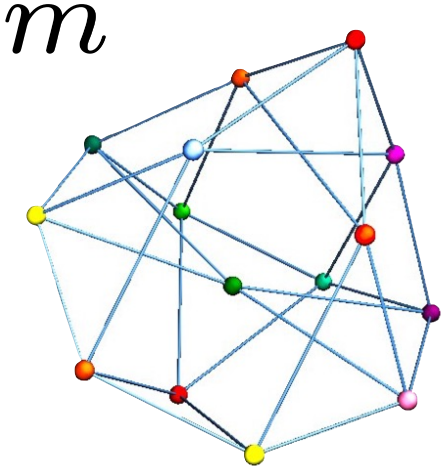
origin:
1250-1300; Middle English: foolish, stupid < Old French: silly

there will be a lot of graphs.

# An algebraic loop-level approach

JJMC

- Introduce multi-loop objects: pre-Integrands $\mathcal{I}_m^L =$



  - will contain all cut information manifestly, not functionally!
  - can decompose into color-stripped polytopes just like at tree-level
  - introduce enough graphs to cover all labelings
  - each graph appears with fixed labels so can solve Jacobi's linearly

$$\{n_a + n_b + n_c = 0\} \to n_j = J_{jk} m_k$$

# Functional Integrands

- Functional integrands are the set of maps from labeled graphs to:
  - n (numerators)
  - d (propagators as in a scalar theory)
  - c (color factors)
  - S (symmetry factors: # of automorphisms)
- cut satisfaction means: Amplitude = $\int \prod_{j=1}^{L} \frac{d^D l_j}{(2\pi)^D} \sum_{i \in \text{cubic}} \frac{1}{S_i} \frac{n_i c_i}{d_i}.$
- Two integrands that satisfy all cuts differ at most by something that integrates to zero.

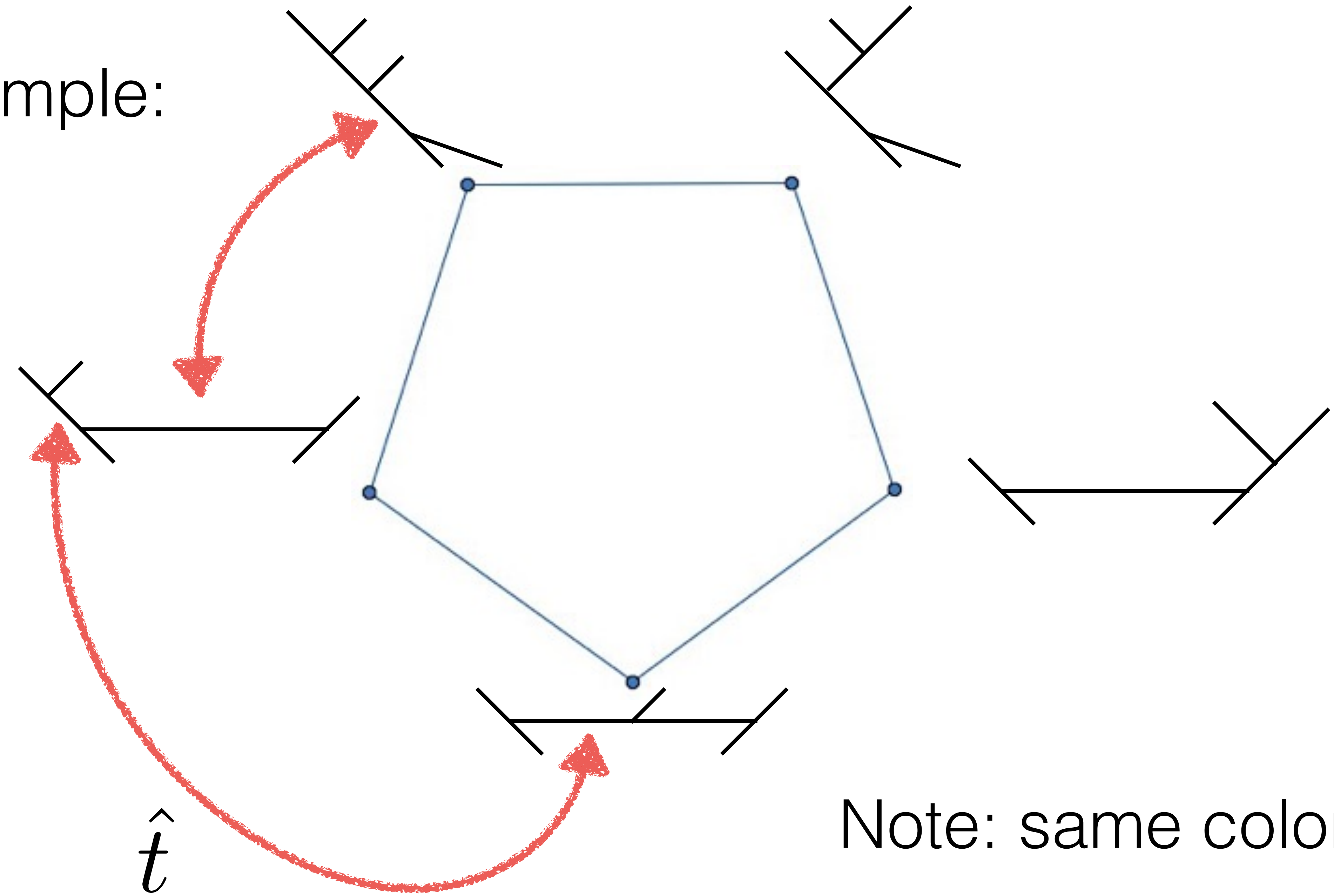
Pre-Integrand: $\mathcal{I}_m^L$ from integrand $\{n, d, c, S\}$

- take (2L +M)-point color-dressed tree graphs
- identify 2L ext legs with +/- indep loop momental labels $\{k_1, \ldots, k_m, -l_1, l_1, \ldots, -l_L, l_L\}$
- label channels momentum can run through at L-loops M-point.

$$\mathcal{I}_m^L = \sum_{j \in \text{assym}} \frac{n_j c_j}{d_j} \qquad \mathrm{I}_i = \sum_{j \in \text{perm}_i} \frac{n_j}{d_j}$$

Graphs contributing to a color-ordered tree, generate the 1-skeleton of Stasheff polytopes joined only by $\hat{t}$

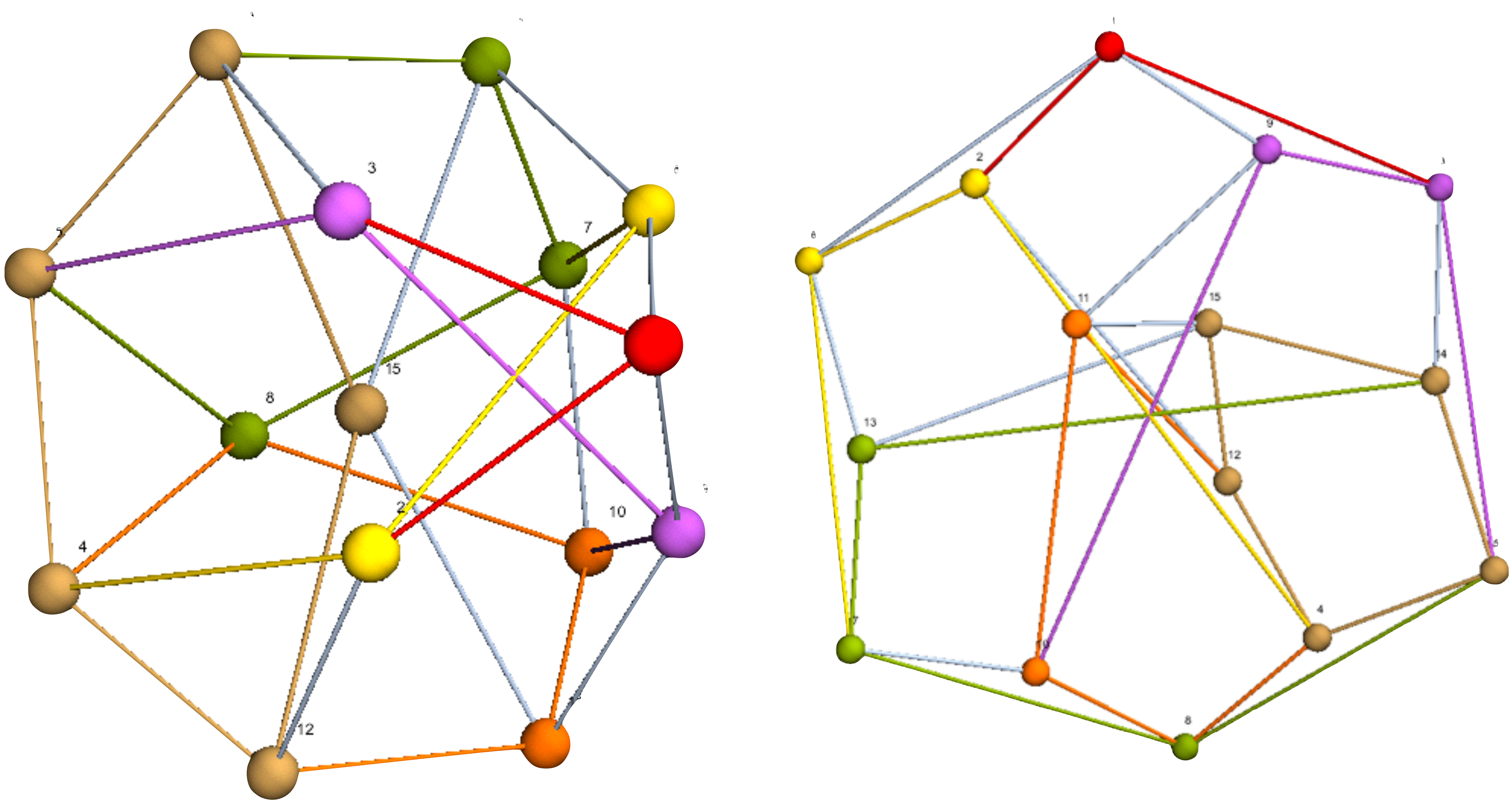
5pt example:




Note: same color-order!

(these polytopes are also called associahedra)

You might think you need (m-2)! of these color-ordered amplitudes to capture everything because this is what is required to touch every vertex at least once:



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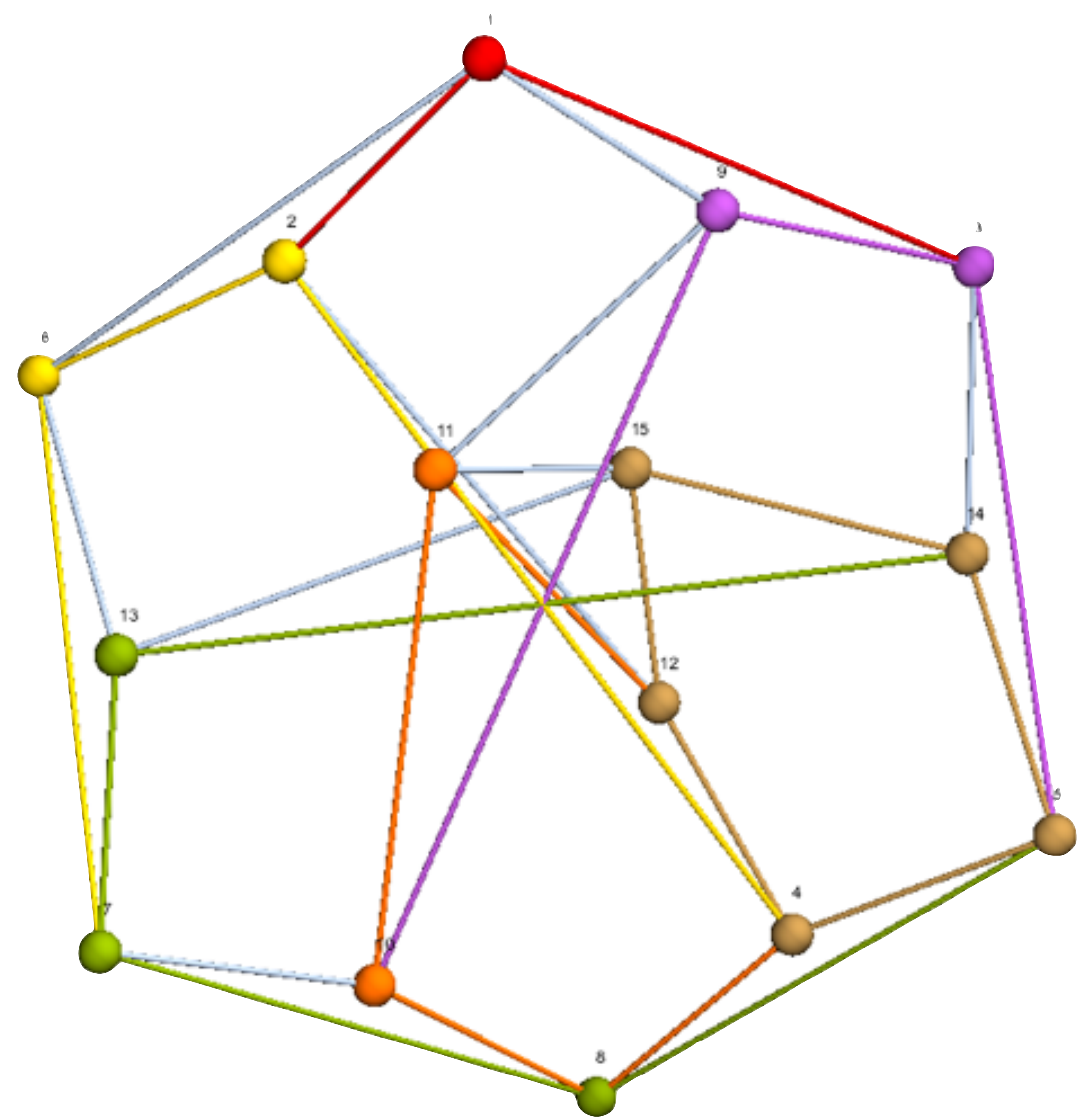
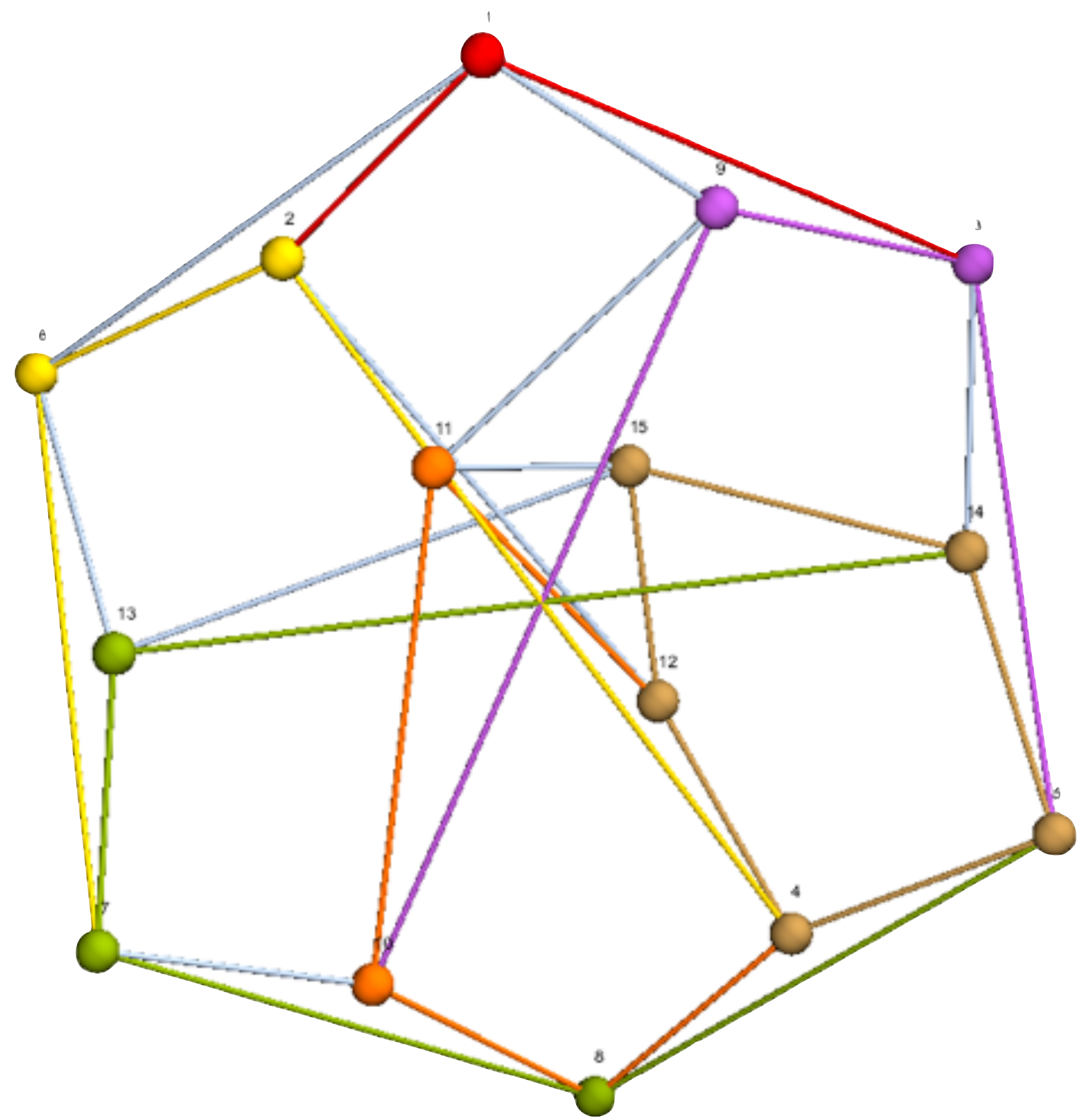


In fact, this is the KK basis, proven by Del Duca, Dixon, and Maltoni to be sufficient

But notice, only (m-2)! nodes are needed to specify both the color factors and numerator factors of everyone



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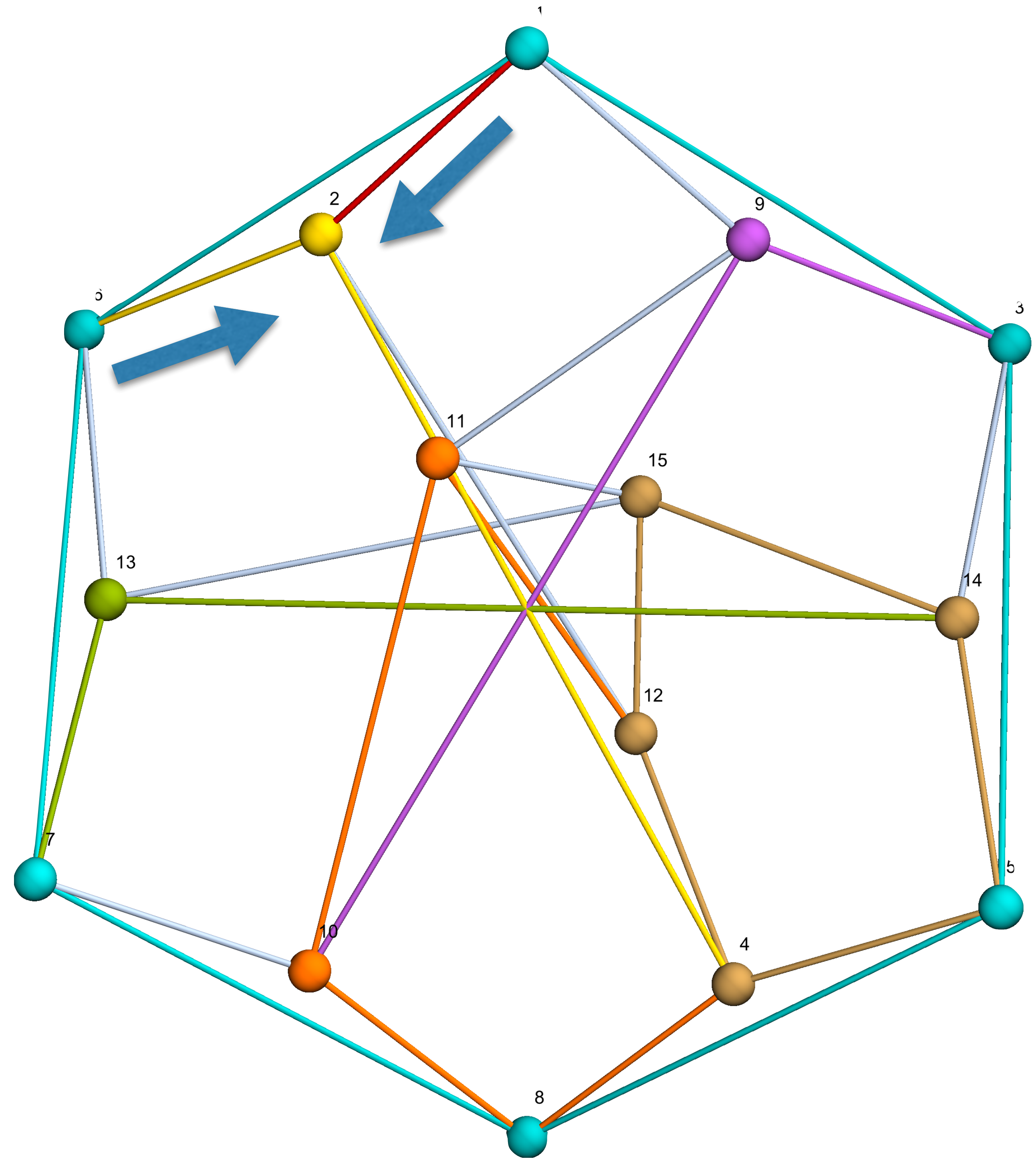

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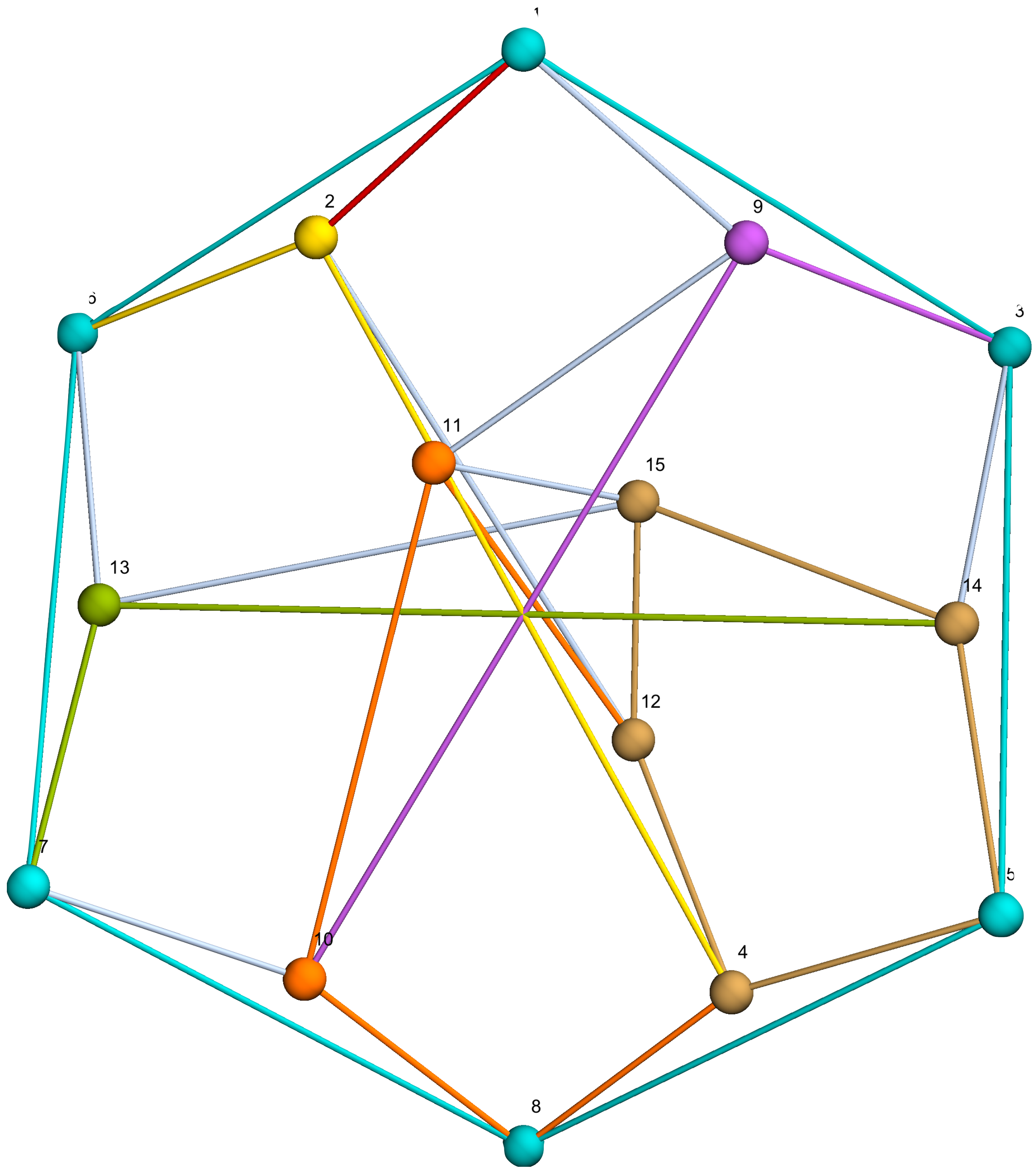
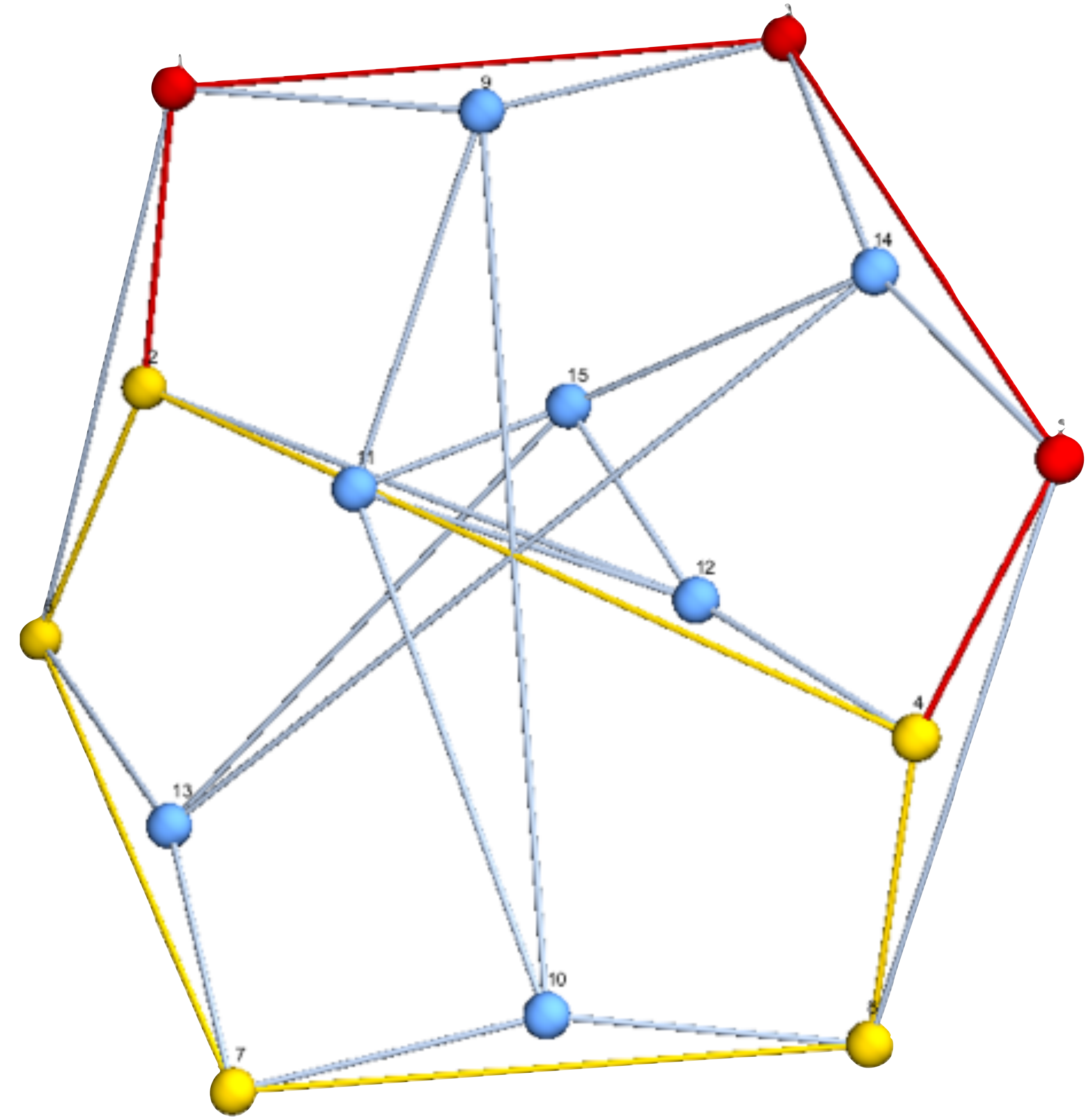
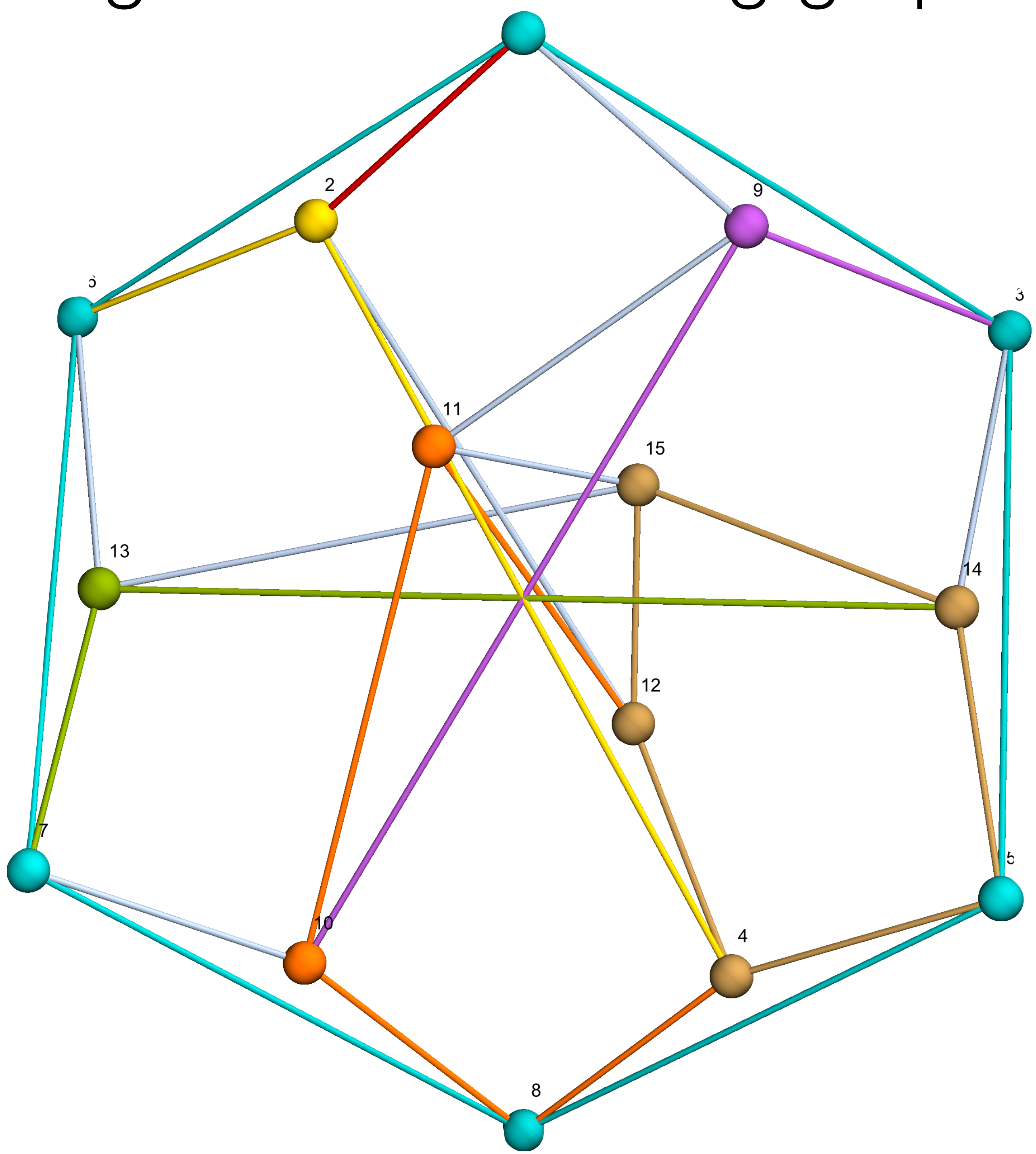
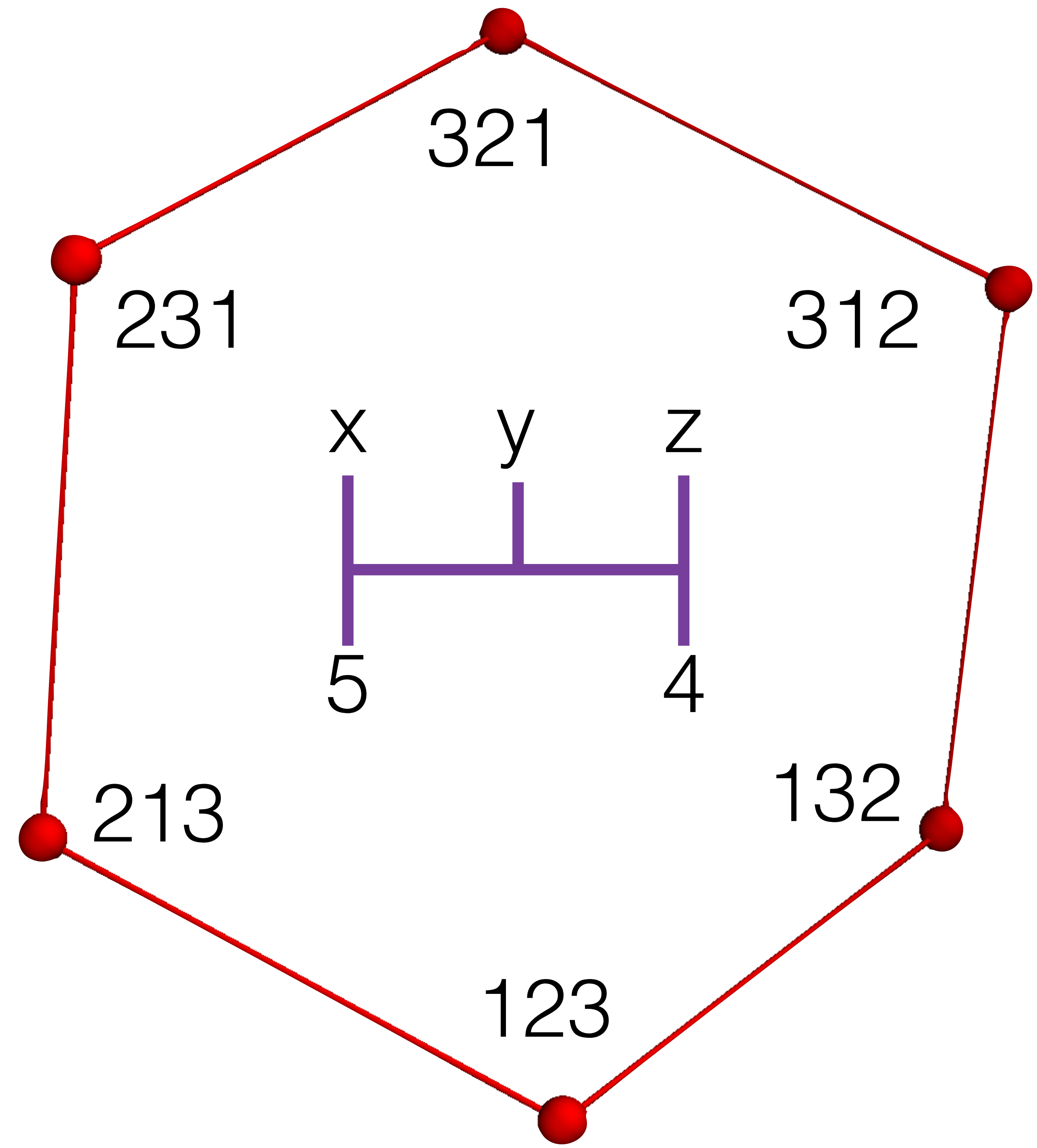




This reduces the set of necessary color-ordered amplitudes to (m-3!)

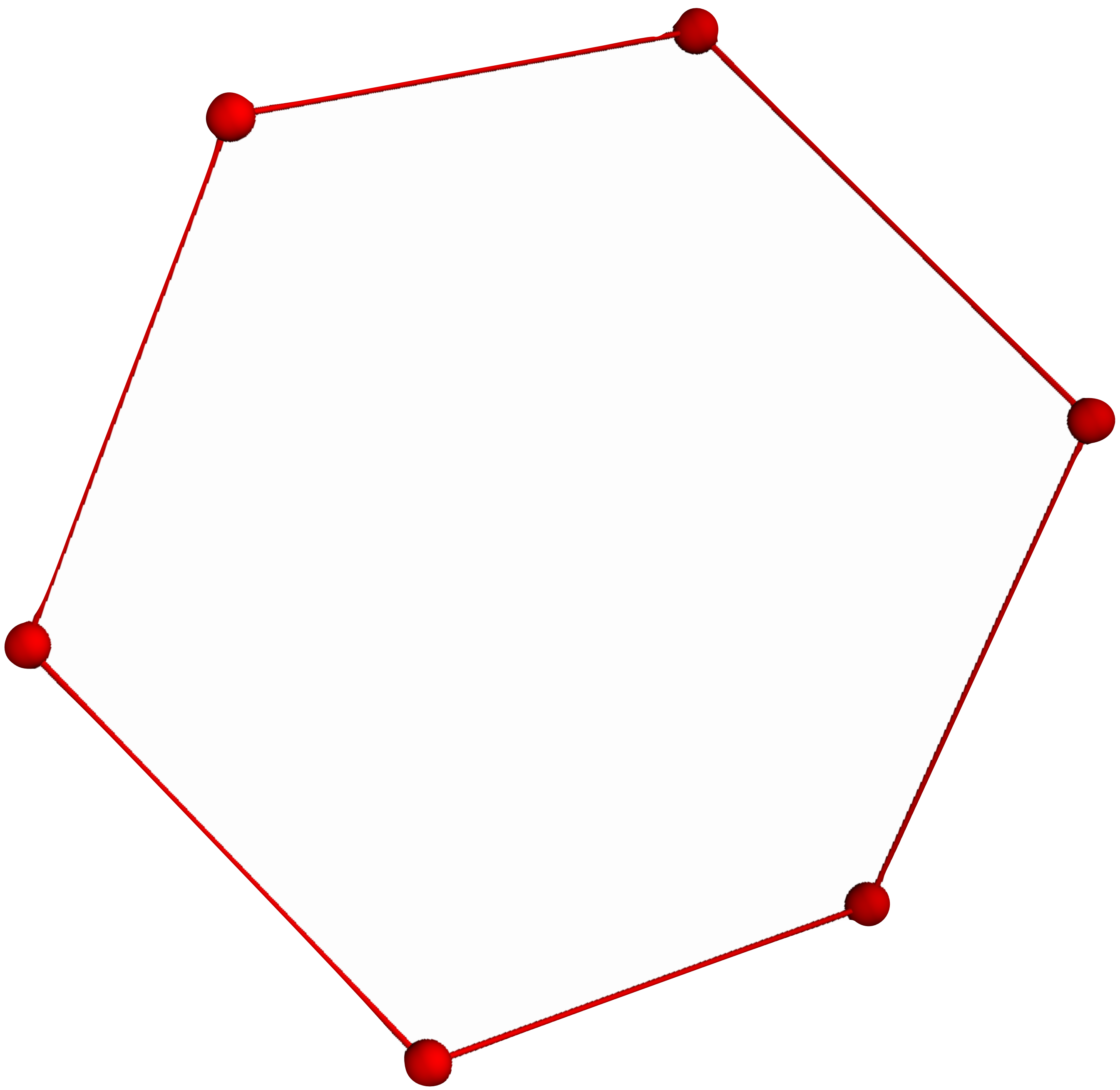
At every multiplicity the masters can be chosen to form the 1-skeleton of a polytope related by $\hat{u}$ on every internal edge of the scattering graphs

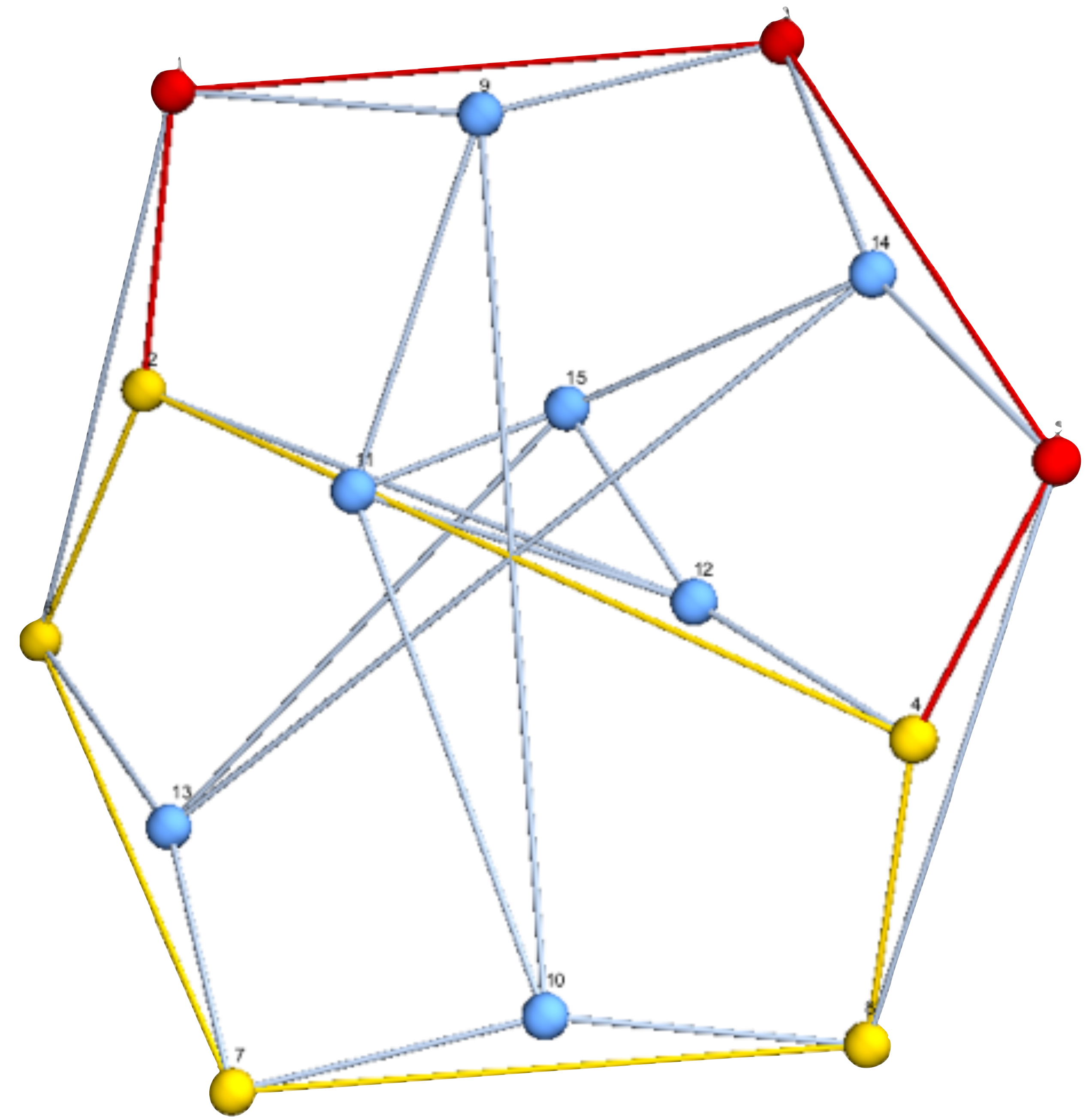



(these polytopes are called permutahedra)

Can linearly solve for the (m-2)! numerators of the masters in terms of the (m-3)! “BCJ” independent color-ordered amplitudes. In fact you get (m-3)! numerators in terms of the color-ordered amplitudes and (m-3)(m-3)! free functions.

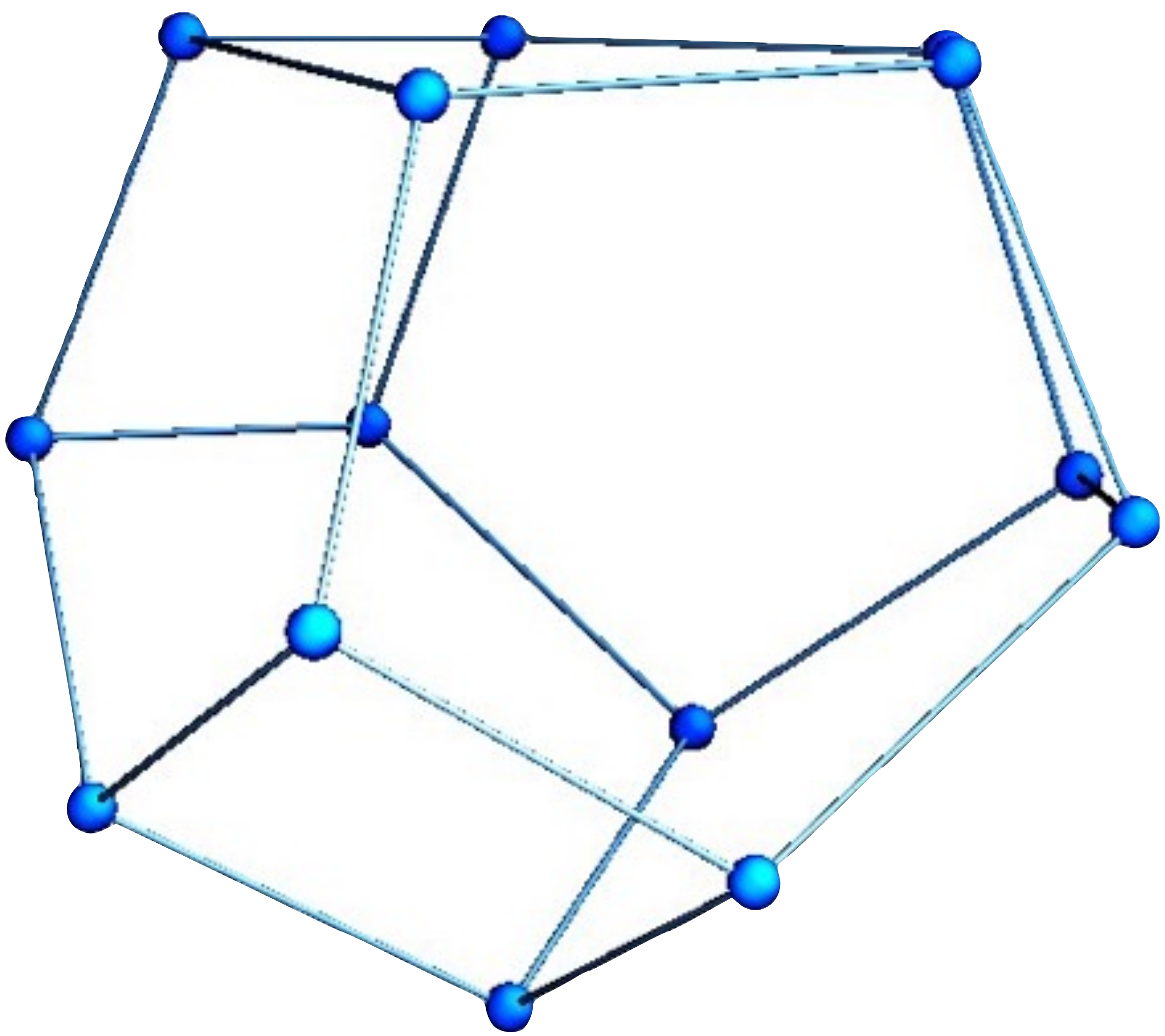





(generalized gauge freedom)

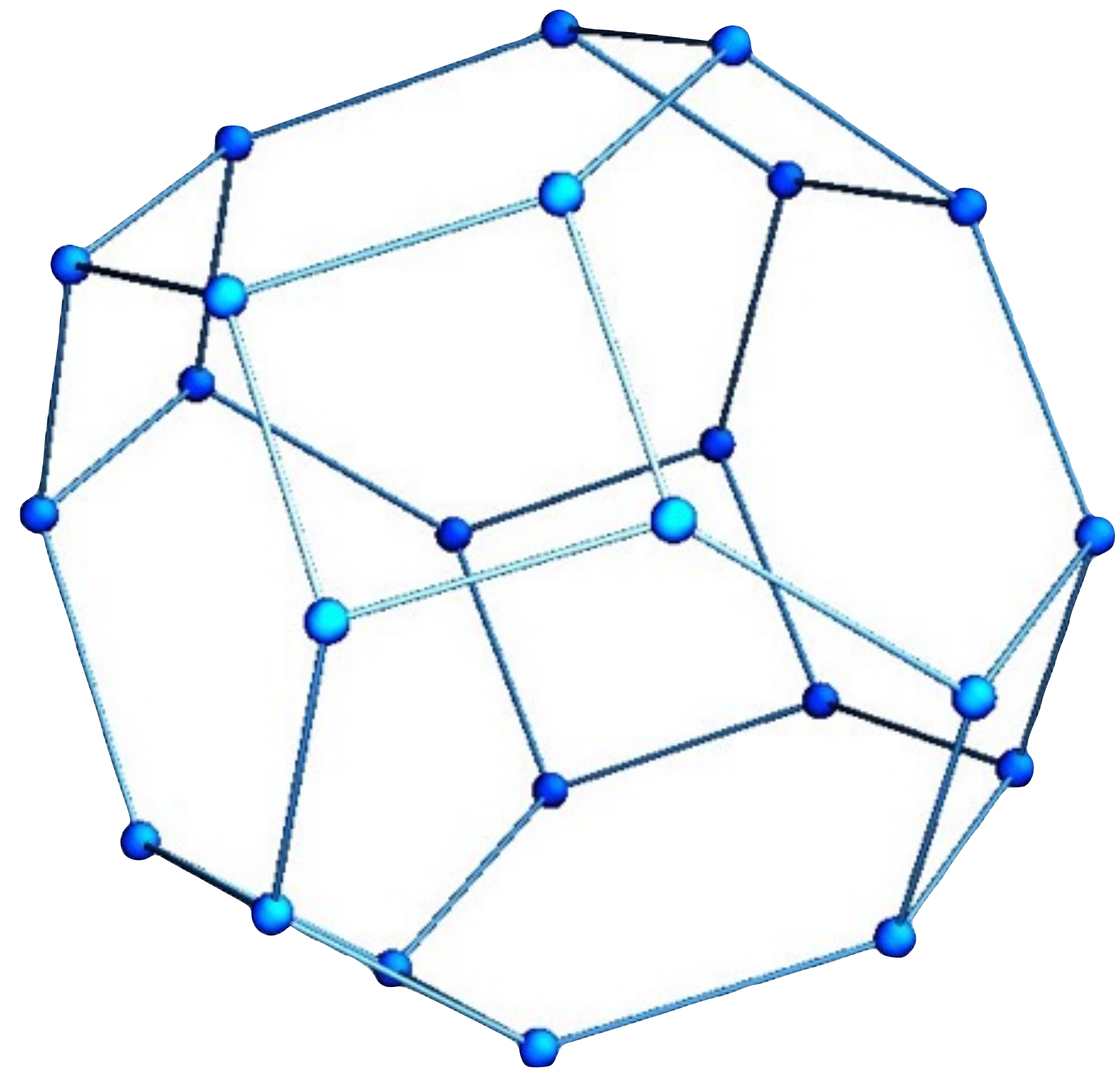
Building blocks at 6-points:

color-ordered amplitude



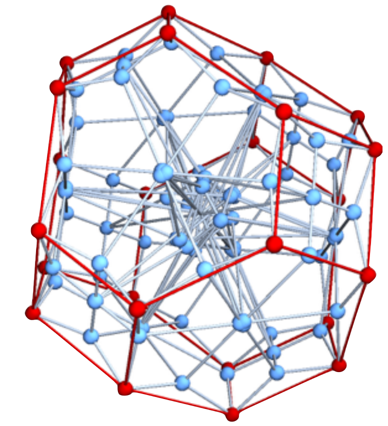
associahedron

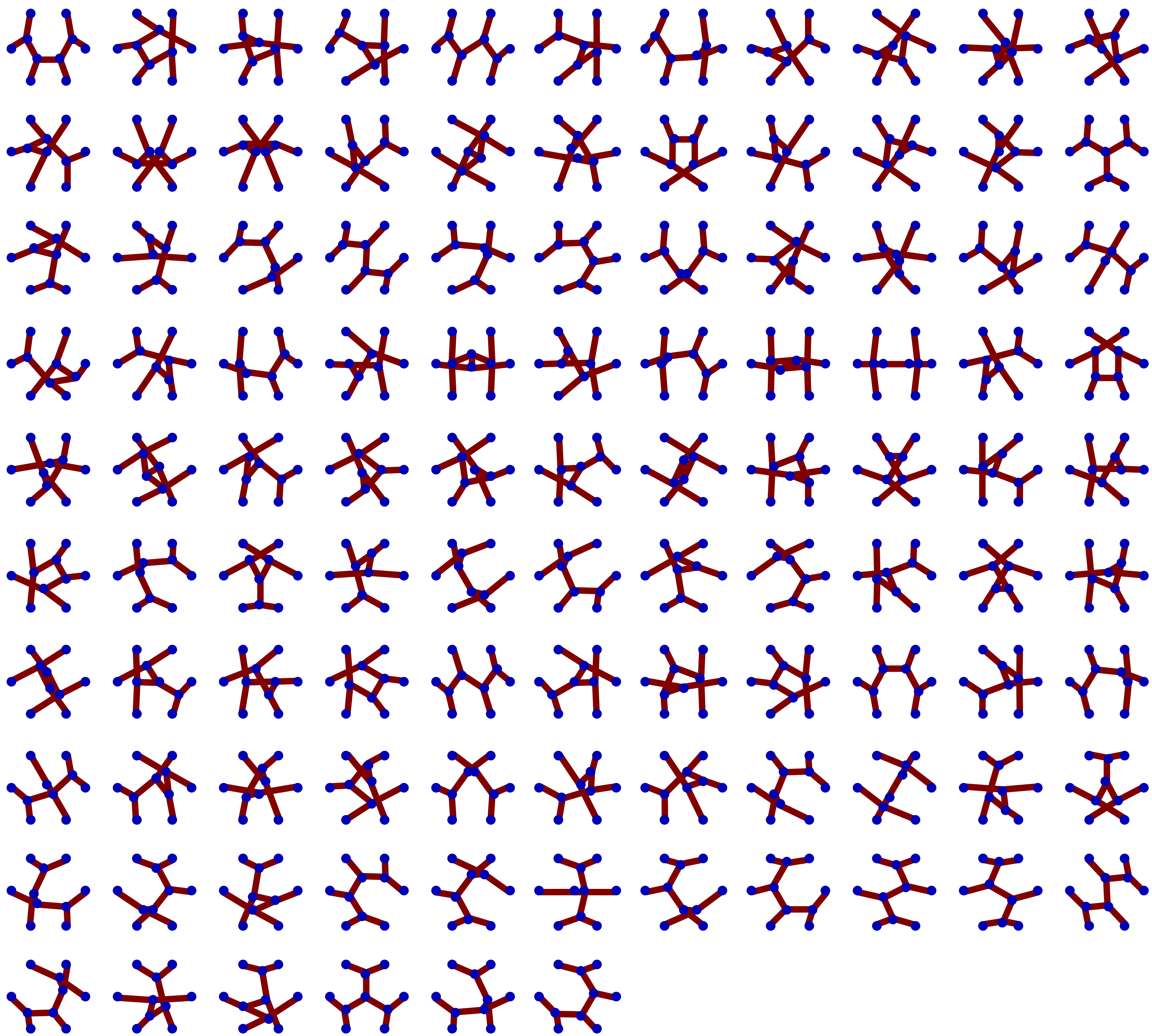
set of masters



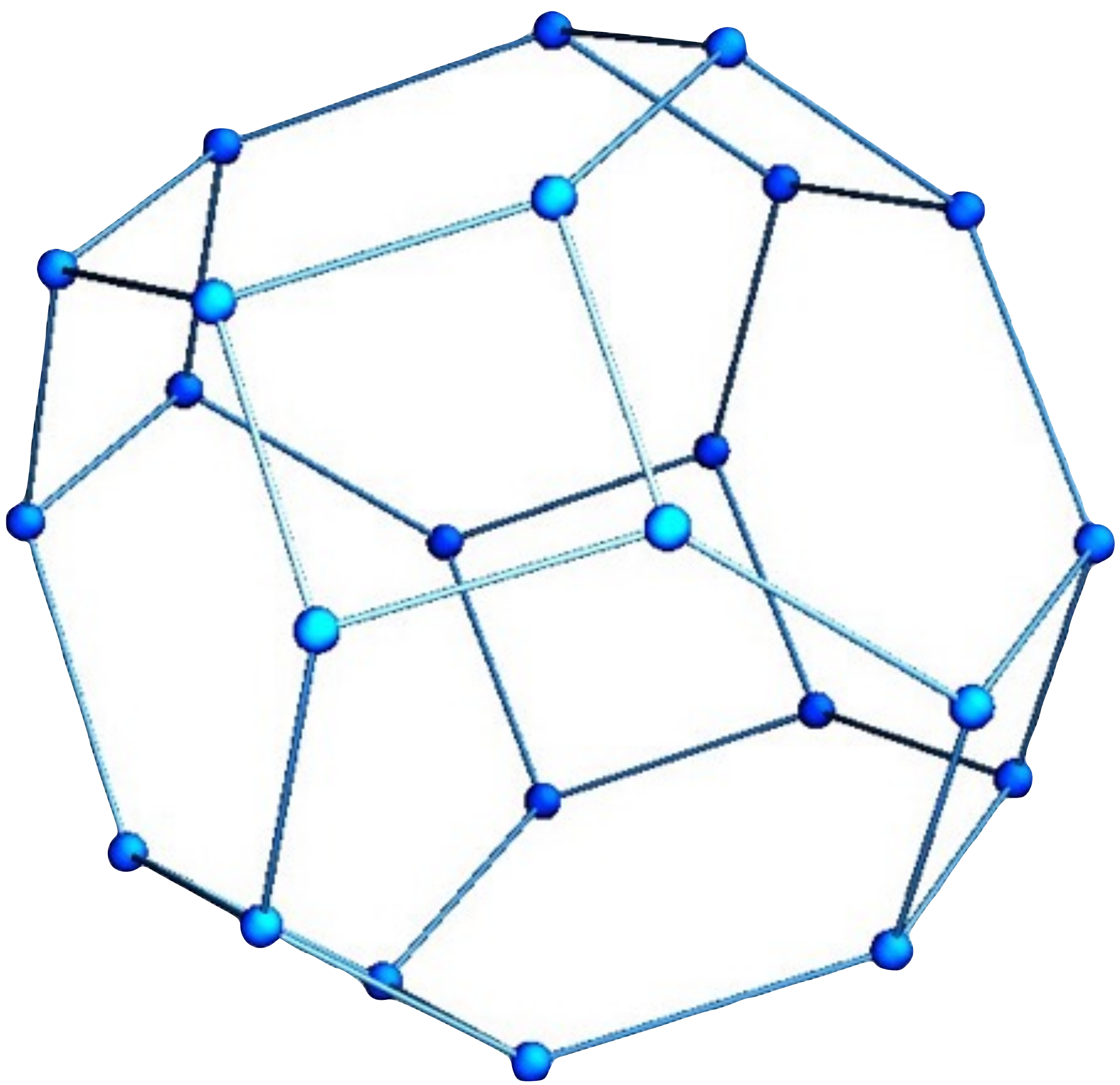
permutahedron

# 105
# cubic graphs at 6 pt

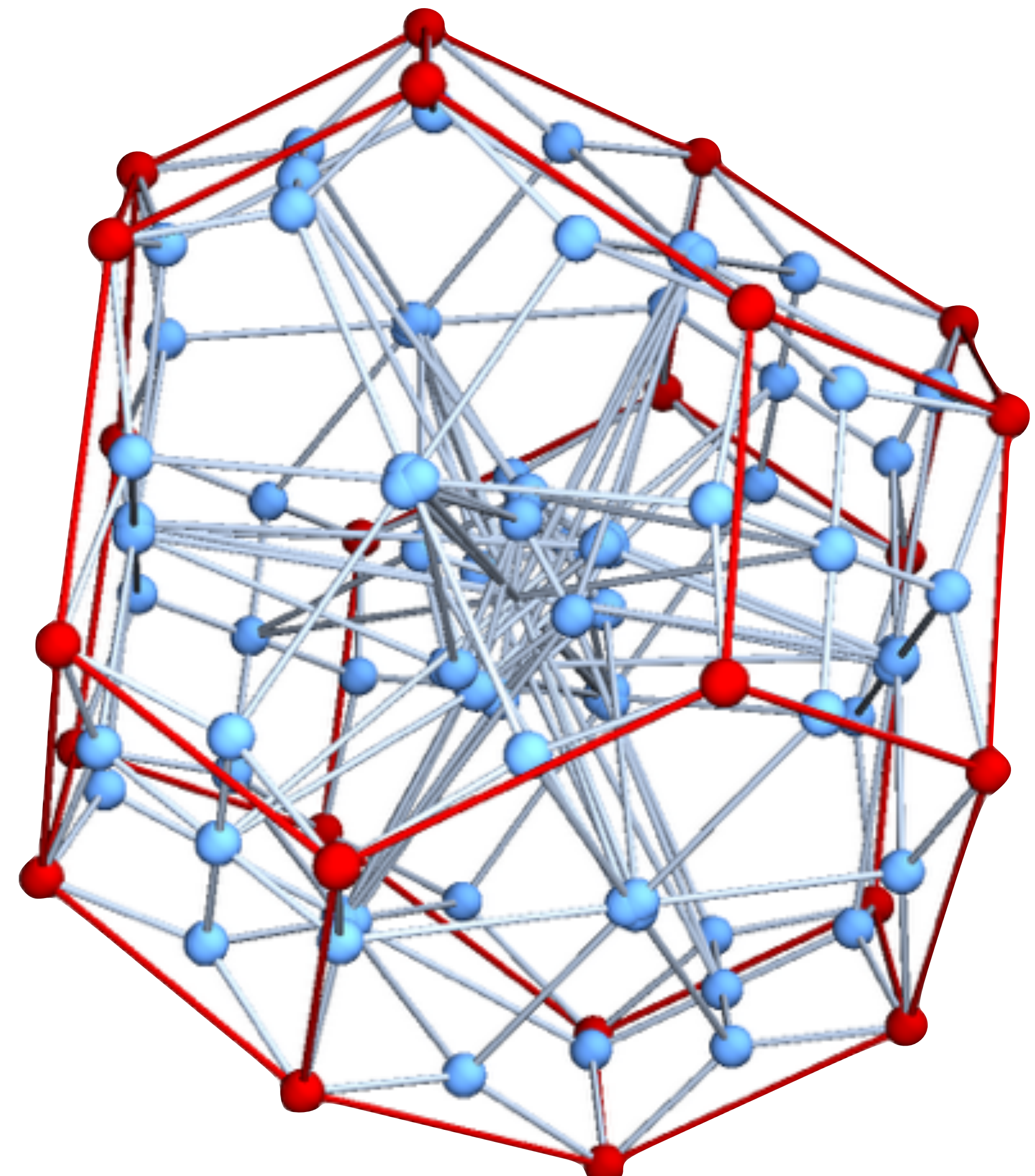


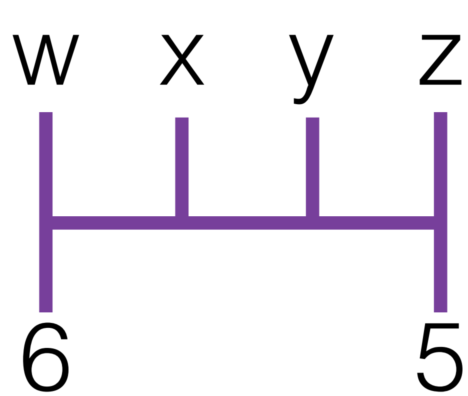


set of masters

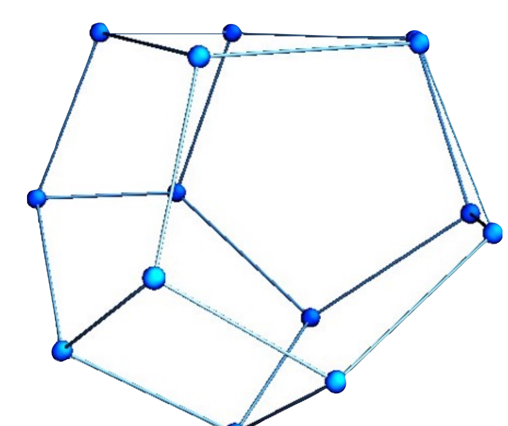


full amplitude



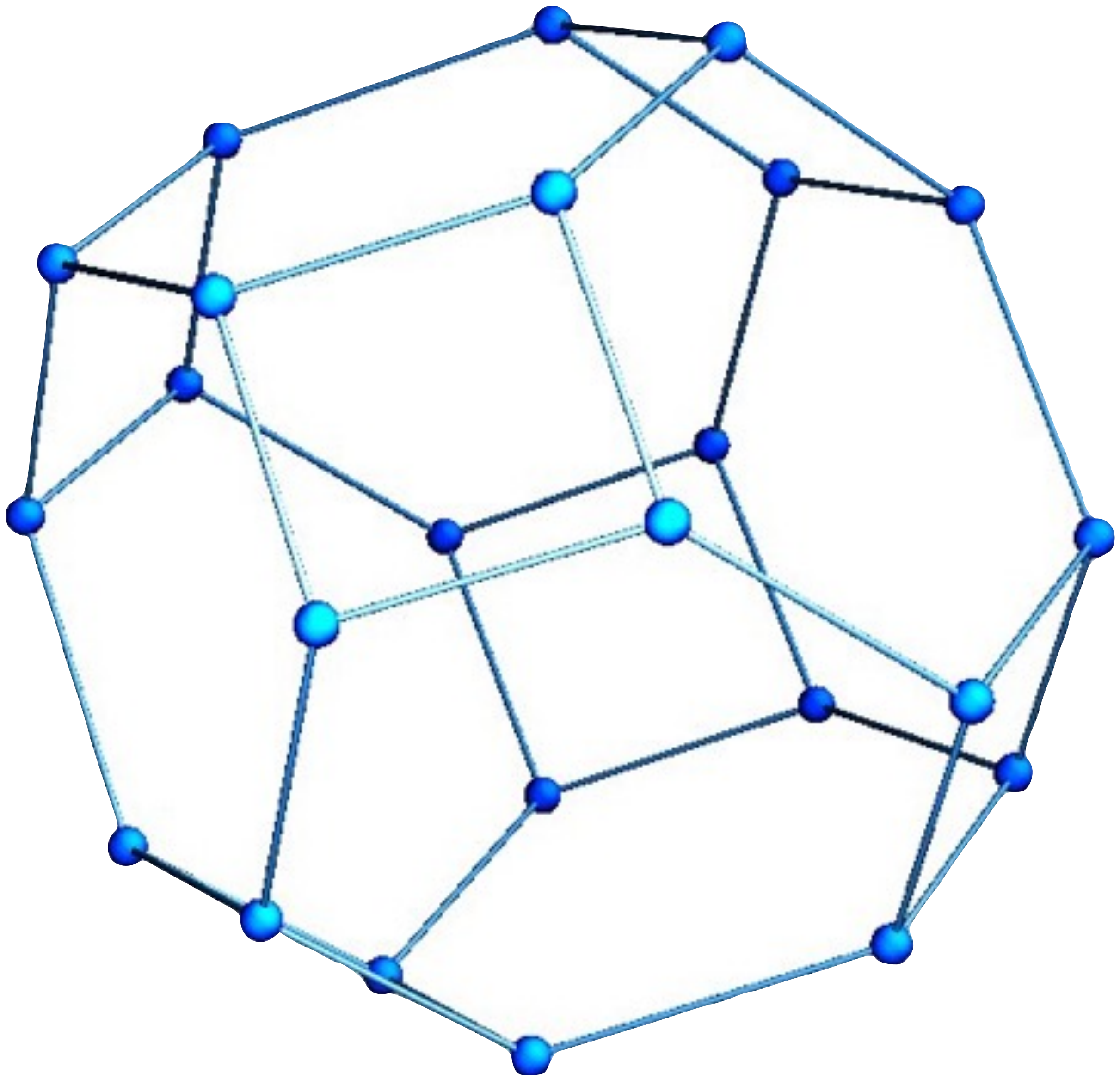


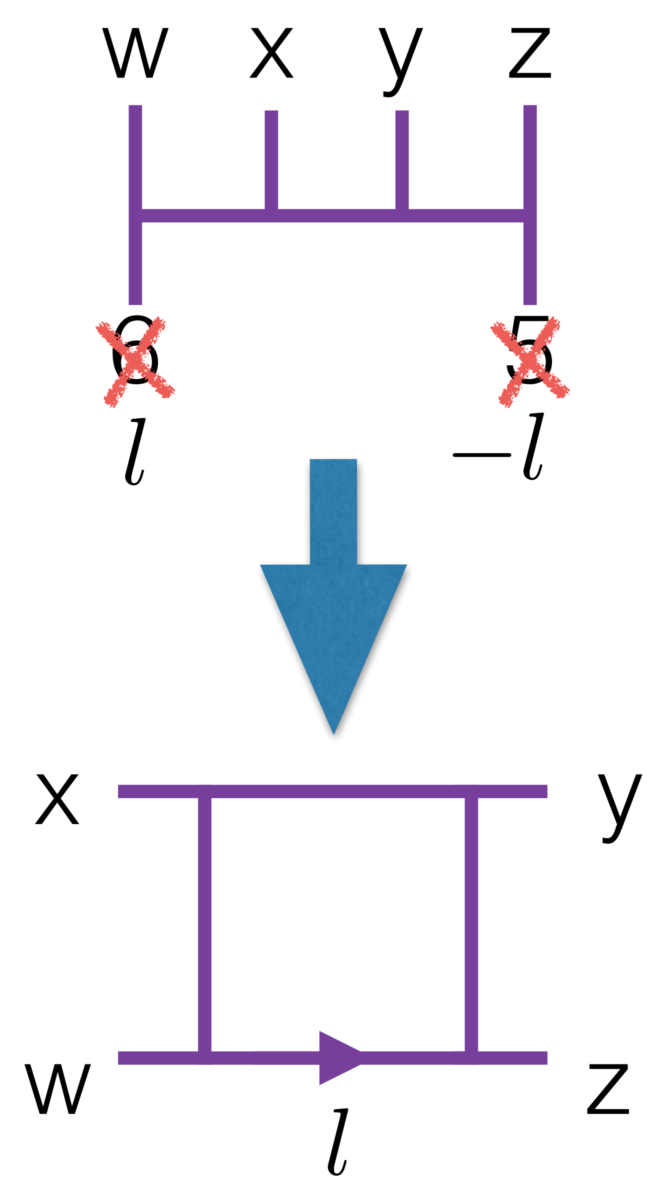

masters fixed by 6



# Now we can talk interestingly about pre-Integrands of loops

set of masters





# Now we can talk interestingly about pre-Integrands of loops

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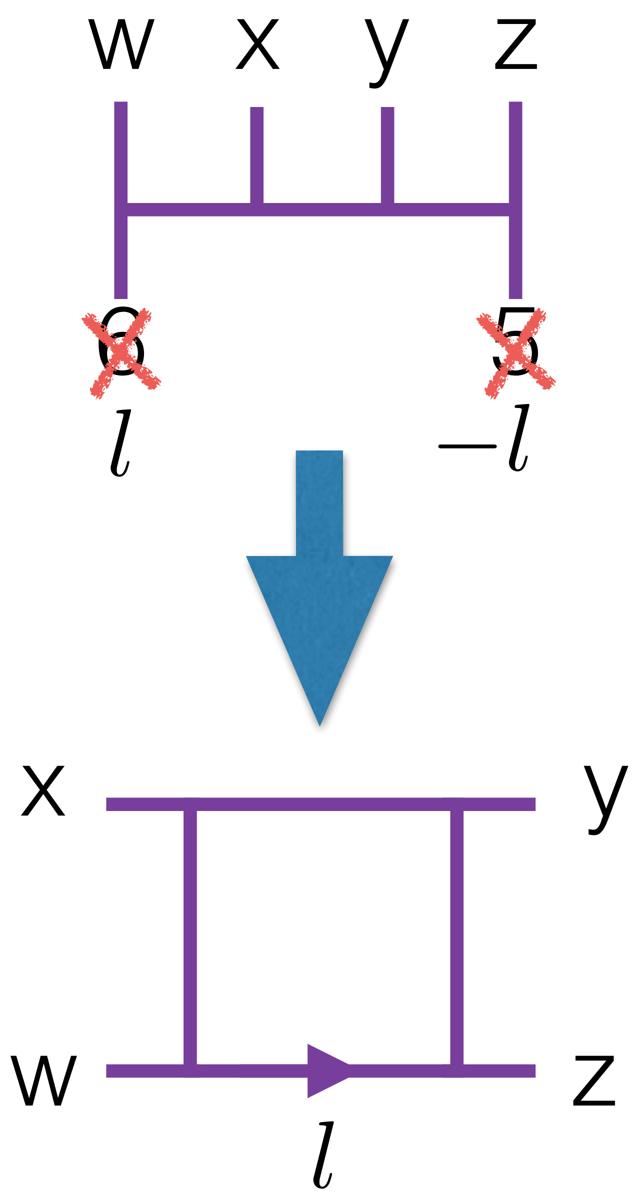
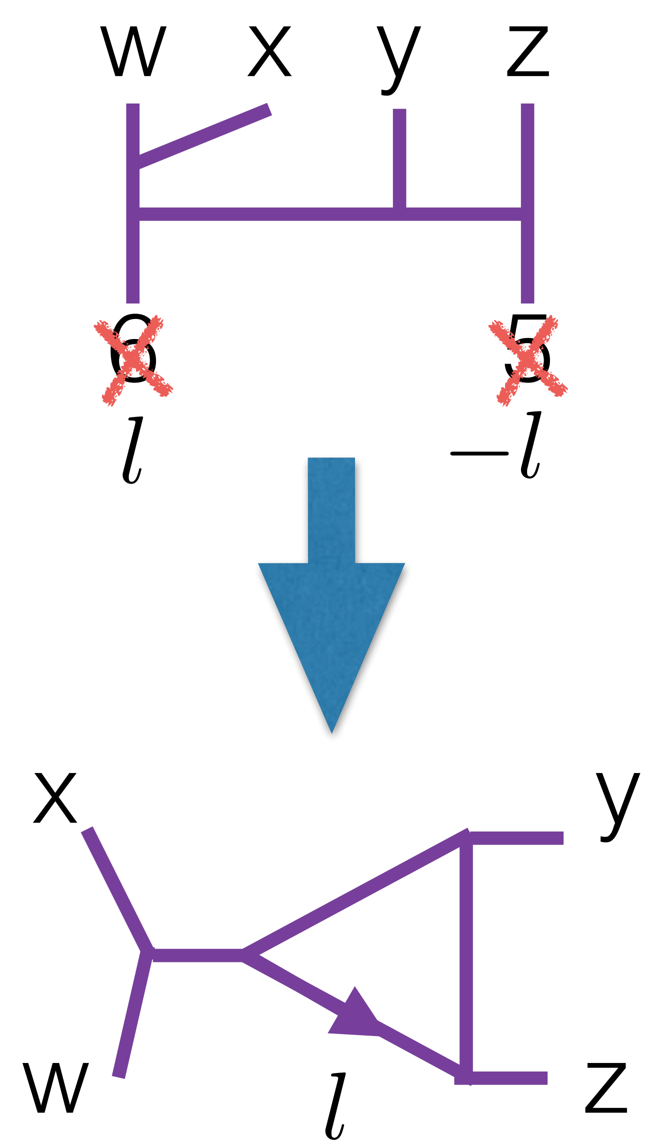
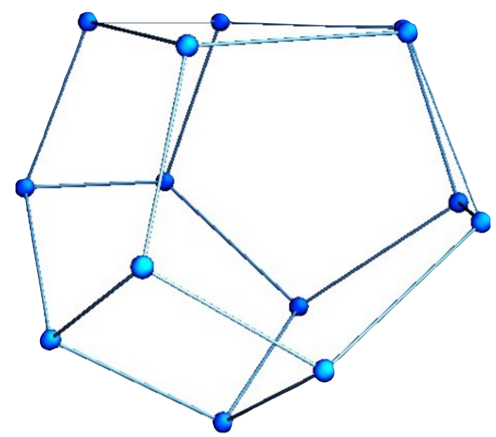


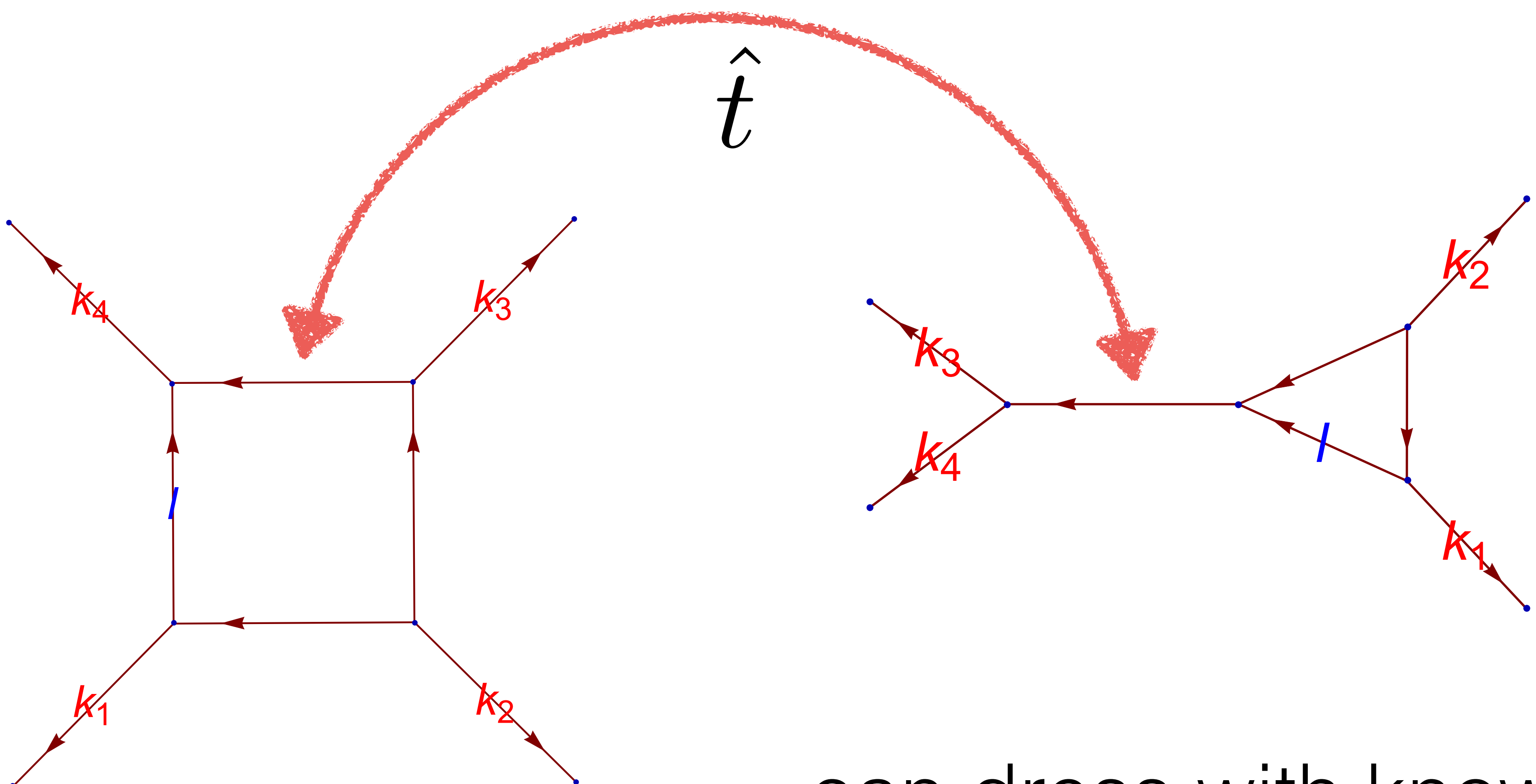

non-masters



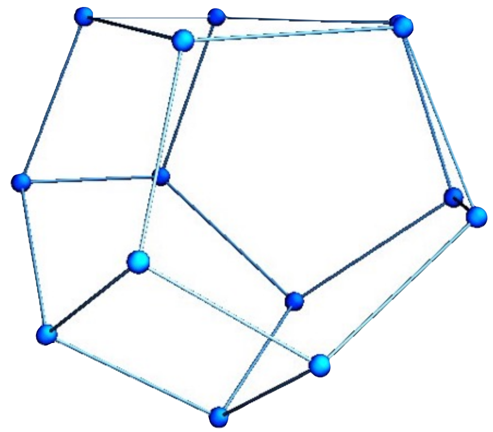
Any given



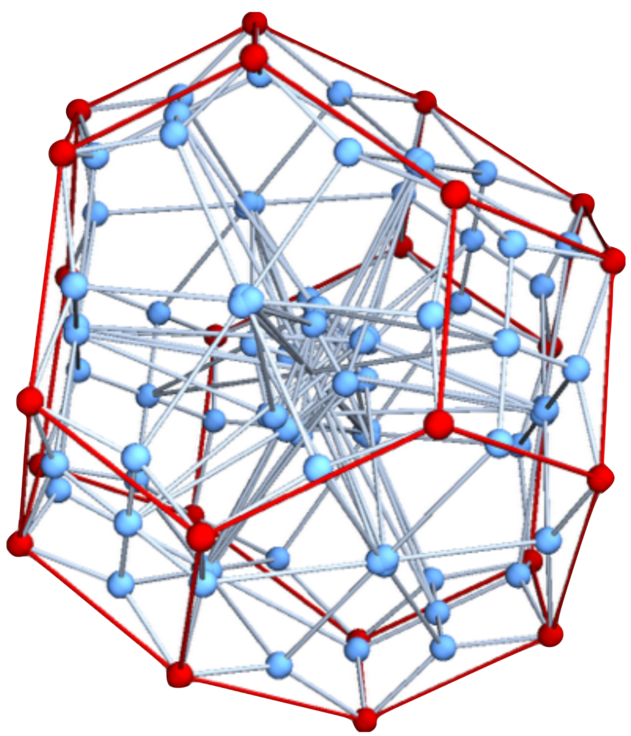
will be comprised of all the one-loop graphs labeled appropriately to the color-order:

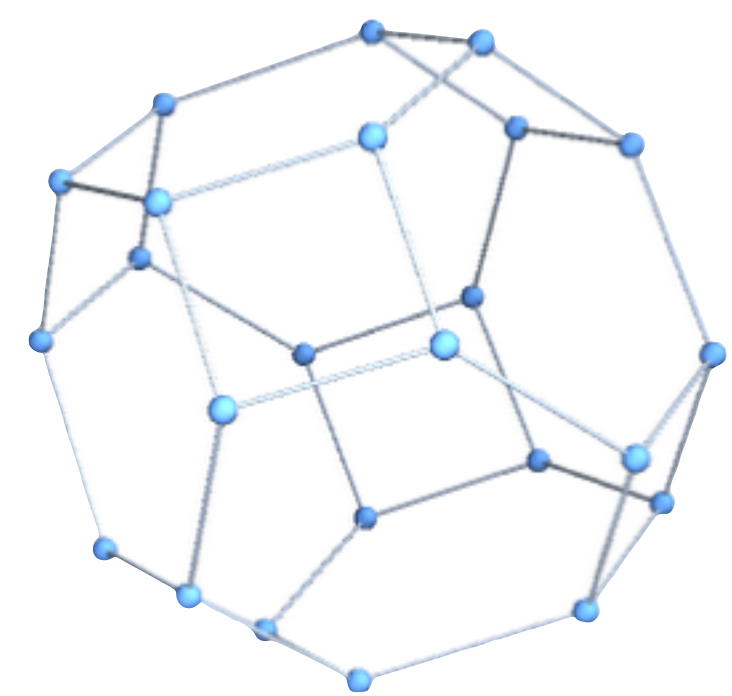



can dress with known off-shell information (unitarity, recursion, etc)

Any given  can dress with off-shell information (unitarity, recursion, etc)

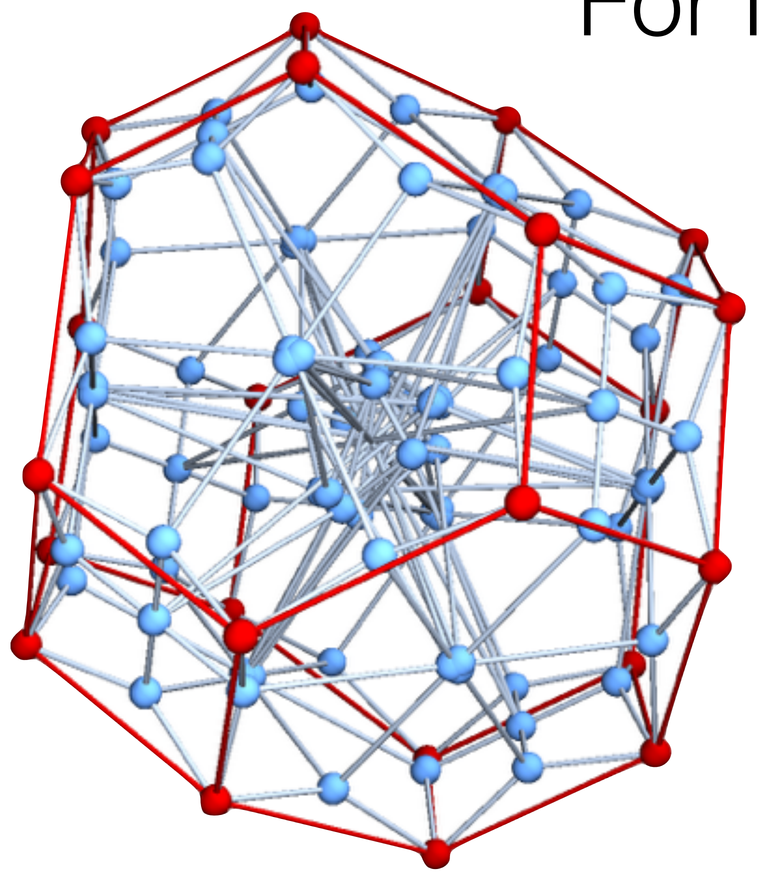
does not need to come from a Jacobi satisfying representation. This will be boundary data. It just has to be true and off-shell on internal legs.

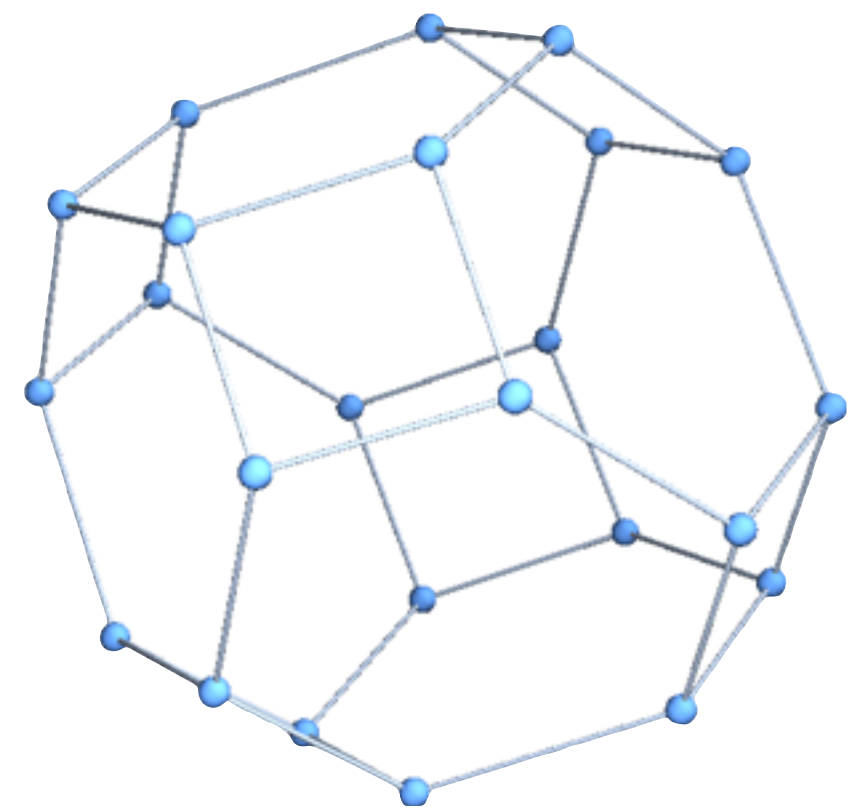
Then demand  satisfies Jacobi for a new rep.

and solve for new: 

This works!

For N=4 SYM at 4pt one-loop only need boxes

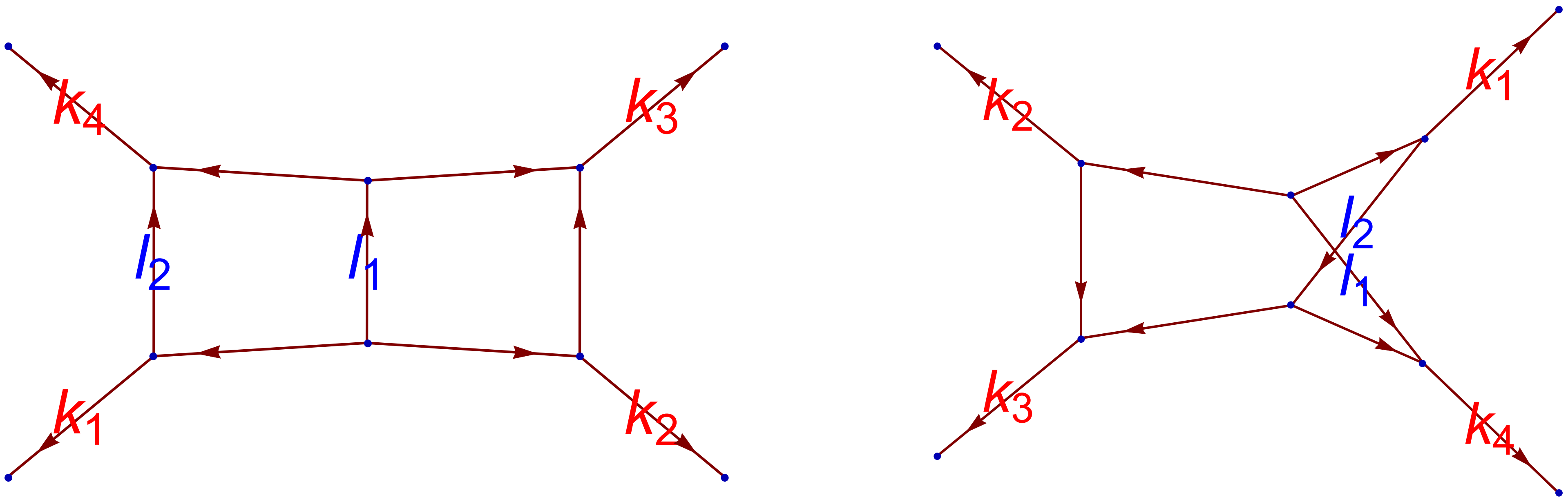




So Jacobi eqns reduce all numerators to the same function if you impose vanishing of all triangles.

$$s\ t\ A(1234)$$

For N=4 SYM at 4pt two-loop only need planar and non-planar boxes

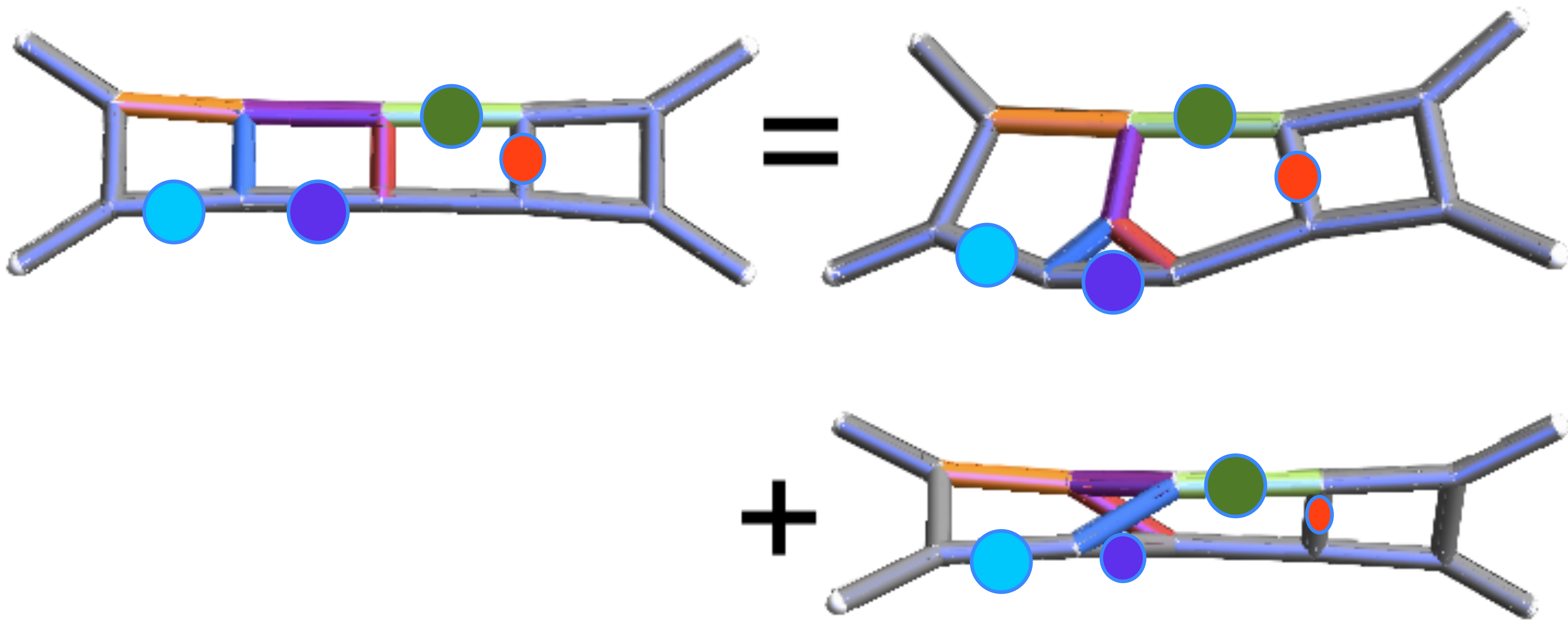



+ perms

Jacobi eqns reduce all numerators to linear combination of two functions

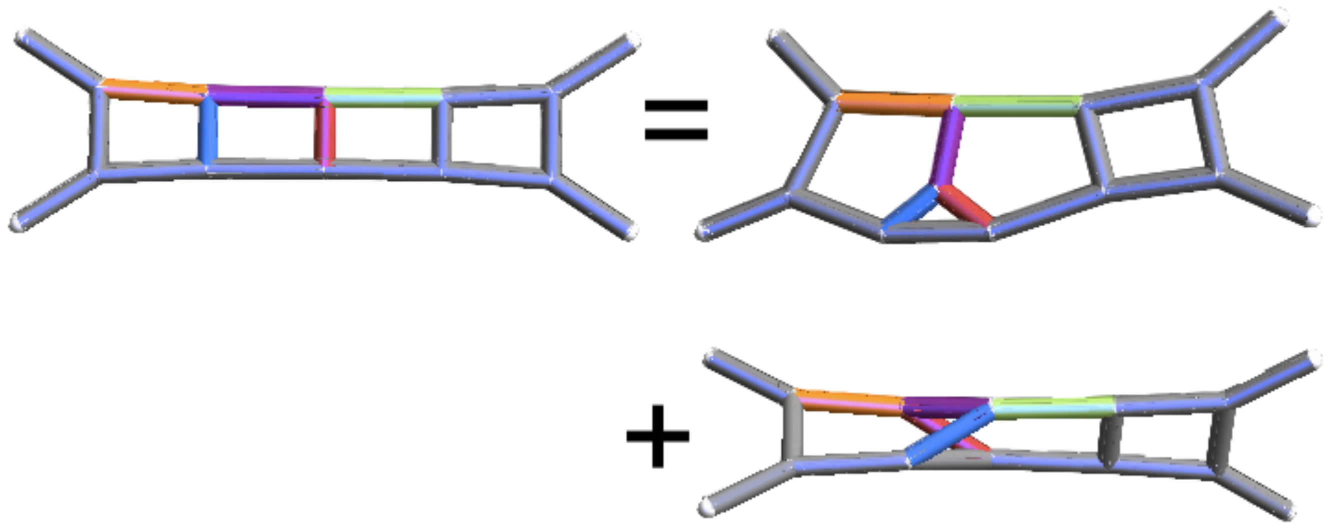
$$s\,(s\,t\,A(1234)) \qquad t\,(s\,t\,A(1234))$$

After Jacobi, now have a color-kinematic satisfying representation at loop level -- no ansatz.

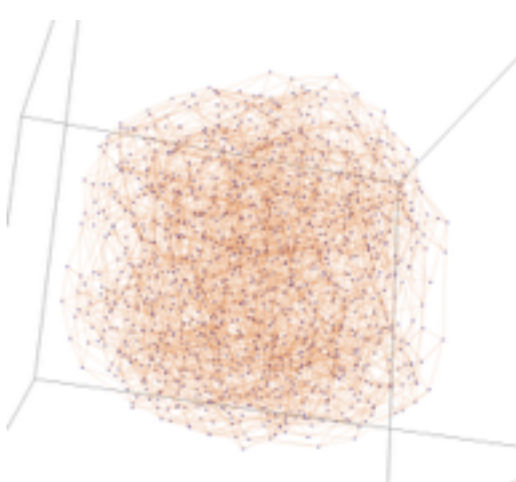


Asymmetric graphs can have Jacobi's imposed linearly on all edges but L

**Conjecture: this is sufficient for double-copy to hold**



Verification: Gravity amplitude must be checked on a spanning set of cuts by symmetrizing into symmetric functional representation.



Verified at 1 loop 4-pt for N<=4 SYM

Verified at 2 loop 4-pt for N=4 SYM

**Summary:** Presented path forward to find C/K satisfying representations without an ansatz.

There is a cautionary note, this way forward involves increasing the redundancy of graph descriptions — no free lunch, but at least a bounded complexity problem.

## The HOPE

Can be a spring board to a description that starts collapsing the redundancy.

May be an avenue to recycle formal all-multiplicity tree-level insight into all multiplicity loop-level insight

Should at least be a vehicle to get more c/k data at lower-loops in theories with less SUSY

Happy to help you play these games with your own non-planar integrands!