Ultraviolet Properties of Perturbative Supergravity

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Based on work with Bern, Davies, Huang, Nohle, Smirnov, Smirnov

Outline

- Ultimate question:
  - Is a profoundly new framework needed for quantum gravity, or is there more to be learned from conventional approaches?
- Motivations – Why study UV divergences in N=4 supergravity?
- Methods – Lightning review of BCJ & Double copy
- Explicit results
  - Highlight unexplained cancellations
  - Role of anomalies
- Punch line: "Enhanced cancellations" and limitations of standard symmetry considerations

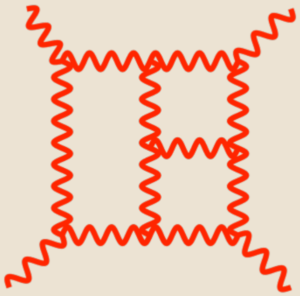
$$\mathcal{A}_n^{\text{loop}} = i^L g^{n-2+2L} \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j c_j}{D_j}$$

$$\mathcal{M}_n^{\text{loop}} = i^{L+1} \left(\frac{\kappa}{2}\right)^{n-2+2L} \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{D_j}$$

ULTRAVIOLET DIVERGENCES AND THE DOUBLE COPY METHOD

UV Divergences in Supergravity

- Supergravity is a nice test-bed for expanding our understanding of gravitational theories
  - Supersymmetry & duality symmetry can help tame the naïve power-counting from gravity's two-derivative coupling



$$\text{gauge theory:} \quad \int \prod \frac{d^D p_i}{(2\pi)^D} \frac{(g p_i^\mu) \cdots}{\text{propagators}}$$

$$\text{gravity:} \quad \int \prod \frac{d^D p_i}{(2\pi)^D} \frac{(\kappa p_i^\mu p_j^\nu) \cdots}{\text{propagators}}$$

  - Simultaneously, powerful new (~2008) methods of calculation allow access to high loop orders
- Allows both theoretical and practical access deep into the perturbative series
  - Verify our understanding, or indicate something new

UV Divergences in Supergravity

- Quantum gravity needs good control over ultraviolet behaviour
- Renormalisation group flow dictated by scale dependence
  - i.e. log(μ^2) terms
- Typically, L-loop Feynman integrals in dim reg have a global factor out front, and look like

$$\left(\frac{-s}{\mu^2}\right)^{-L\epsilon}\left(\frac{A}{\epsilon^L}+\ldots\right)$$

- This pegs log(μ^2) to singularities in ε.
- Therefore, we use divergence as proxy for scale dependence
  - (somewhat easier to calculate)

UV Divergences in Supergravity

- Most of this talk is about half-maximal supergravity
  - *N* = 4 supergravity in D=4

helicity	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$+\frac{1}{2}$	$+1$	$+\frac{3}{2}$	$+2$
state count	1	4	6	4	2	4	6	4	1

Das (1977);
Cremmer, Scherk, Ferrara (1978)

  - Duality symmetry SU(4) x SU(1,1)
  - The two scalars parameterize the coset space SU(1,1)/U(1)
  - Anomaly in the U(1) subgroup means quantum inequivalence between different classical formulations

Marcus (1985)

    - We use the SU(4) formulation of Cremmer, Scherk, Ferrara

Expectations from Symmetry

- 1970's-1980's: Supersymmetry delays UV divergences until three loops in all 4D pure supergravity theories
  - Expected counterterm is R^4
- In N=8, SUSY and duality symmetry rule out couterterms until 7 loops
  - Expected counterterm is D^8R^4
- 7-loop counterterm has an analog in $N = 4$ supergravity at three loops
  - But the divergence is not present

Grisaru; Tomboulis; Deser, Kay, Stelle; Ferrara, Zumino; Green, Schwarz, Brink; Howe, Stelle; Marcus, Sagnotti; etc.

Bern, Dixon, Dunbar; Perelstein, Rozowsky (1998);
Howe and Stelle (2003, 2009);
Grisaru and Siegel (1982);
Howe, Stelle and Bossard (2009);
Vanhove; Bjornsson, Green (2010);
Kiermaier, Elvang, Freedman (2010);
Ramond, Kallosh (2010);
Kallosh; Howe and Lindström (1981);
Green, Russo, Vanhove (2006)
Bern, Carrasco, Dixon, Johansson, Roiban (2010)
Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger (2010)

Duality Symmetries

- Analogs of $E_{7(7)}$ for lower supersymmetry

N=8: $E_{7(7)}$	$E_{7(7)}/SU(8)$
N=6: SO*(12)	SO*(12)/U(6)
N=5: SU(5,1)	SU(5,1)/U(5)
N=4: SU(4) x SU(1,1)	SU(1,1)/U(1)

- Can help UV divergences in these theories
  - **Still have candidate counterterms at *L* = *N* - 1 (1/*N* BPS)**

Bossard, Howe, Stelle, Vanhove (2010)

- Nice analysis for *N* = 8 counterterms

Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger (2010)

Recent Field Theory Calculations

- How are the calculations done?

1. Find a representation of SYM that satisfies color-kinematics duality (hard)
2. Construct the integrand for a gravity amplitude using the double copy method (easy)
3. Extract the ultraviolet divergences from the integrals (straightforward, but a practical challenge)

Color-Kinematics Duality

- **Color-kinematics duality provides a construction of gravity amplitudes from knowledge of Yang-Mills amplitudes**

Bern, Carrasco, Johansson (2008)

- **In general, Yang-Mills amplitudes can be written as a sum over trivalent graphs**

$$\mathcal{A}_n = g^{n-2} \sum_i \frac{n_i c_i}{D_i}$$

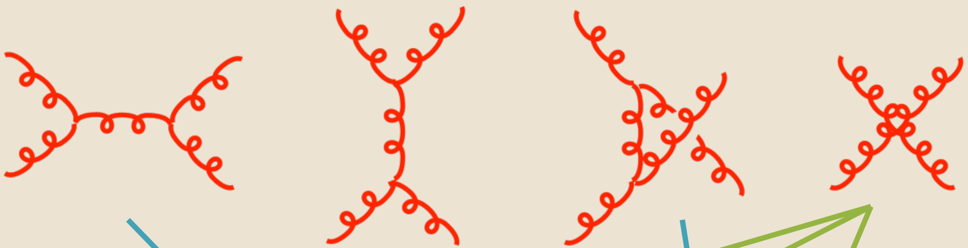
  - **Color factors** $c_i \sim f^{abc} f^{cde}$
  - **Kinematic factors** $n_i \sim (\epsilon_1 \cdot k_2)\,(\epsilon_2 \cdot k_3)\,(\epsilon_3 \cdot \epsilon_4) + \ldots$

- **Duality rearranges the amplitude so color and kinematics satisfy the same identities (Jacobi)**

$$c_i + c_j + c_k = 0 \leftrightarrow n_i + n_j + n_k = 0$$

Example: Four Gluons

- Four Feynman diagrams
- Color factors based on a Lie algebra



$$c_s = f^{a_1 a_2 b} f^{b a_3 a_4}$$

$$c_t = f^{a_1 a_4 b} f^{b a_2 a_3}$$

$$c_u = f^{a_1 a_3 b} f^{b a_4 a_2}$$

$$\mathcal{A}_4^{\text{tree}} = g^2 \left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u} \right)$$

$$n = \epsilon_1 \cdot k_2 \, \epsilon_2 \cdot \epsilon_3 \, \epsilon_4 \cdot k_1 + \ldots$$

- Color factors satisfy Jacobi identity: $c_s + c_t + c_u = 0$
- Numerator factors satisfy similar identity: $n_s + n_t + n_u = 0$
  - Color and kinematics satisfy the same identity!

Gravity from Double Copy

- **Once numerators are in color-dual form, "square" to construct a gravity amplitude**

Bern, Carrasco, Johansson (2008)

$$\mathcal{A}_n = g^{n-2} \sum_i \frac{n_i c_i}{D_i} \longrightarrow \mathcal{M}_n = i\left(\frac{\kappa}{2}\right)^{n-2} \sum_i \frac{n_i \tilde{n}_i}{D_i}$$

  - **Gravity numerators are a double copy of gauge theory ones!**

- **Proved using BCFW on-shell recursion** Bern, Dennen, Huang, Kiermaier (2010)

- **The two copies of gauge theory don't have to be the same theory.**

Gravity from Double Copy

- The two copies of gauge theory don't have to be the same theory.
  - Spectrum controlled by tensor product of Yang-Mills theories

$\mathcal{N}$ = 8 sugra: ($\mathcal{N}$ = 4 SYM) x ($\mathcal{N}$ = 4 SYM)
$\mathcal{N}$ = 5 sugra: ($\mathcal{N}$ = 4 SYM) x ($\mathcal{N}$ = 1 SYM)
$\mathcal{N}$ = 4 sugra: ($\mathcal{N}$ = 4 SYM) x ($\mathcal{N}$ = 0 SYM)
$\mathcal{N}$ = 0 sugra: ($\mathcal{N}$ = 0 SYM) x ($\mathcal{N}$ = 0 SYM)

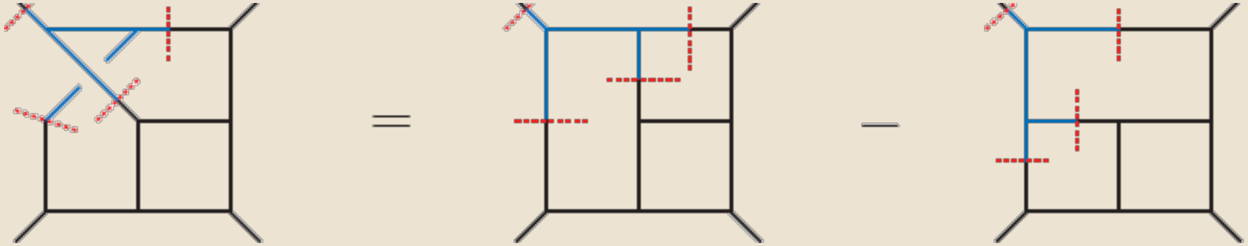
- More-sophisticated supergravity theories

Damgaard, Huang, Sondergaard, Zhang; Anastasiou, Borsten, Duff; Duff, Hughes, Nagy; Johansson, Ochirov; Carrasco, Chiodaroli, Gunaydin, Roiban; permutations…

- Relatively compact expressions for gravity amplitudes

Loop Level

- What we really need are *multiloop* gravity amplitudes
- Color-kinematics duality at loop level
  - Consistent loop labeling between three diagrams
  - Non-trivial to find duality-satisfying sets of numerators



$$c_i = c_j - c_k$$
$$n_i = n_j - n_k$$

Bern, Carrasco, Johansson (2010)

- Double copy gives gravity

$$\mathcal{A}_n^{\text{loop}} = i^L g^{n-2+2L} \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j c_j}{D_j}$$

Just replace c with n

$$\mathcal{M}_n^{\text{loop}} = i^{L+1} \left(\frac{\kappa}{2}\right)^{n-2+2L} \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{D_j}$$

Known Color-Dual Numerators

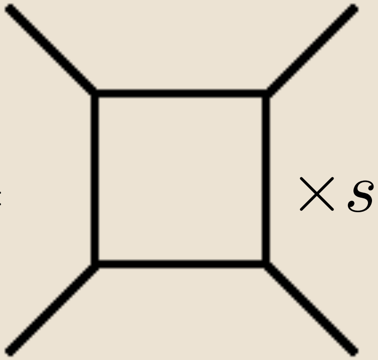
N = 4 SYM	1 Loop	2 Loops	3 Loops	4 Loops	*L* Loops
4 point	trivial	trivial	ansatz	ansatz	
5 point	construction	ansatz	ansatz		
6 point	construction				
7 point	construction				
n point	construction				

Bern, Carrasco, Johansson (2010)
Carrasco, Johansson (2011)
Bern, Carrasco, Dixon, Johansson, Roiban (2012)
Yuan (2012)
Bjerrum-Bohr, Dennen, Monteiro, O'Connell (2013)

L = 1

Colour-dual representation

- N = 4 SYM BCJ-satisfying numerators are trivially obtained
- Numerator is totally symmetric
  - Triangle numerators vanish through Jacobi identities (No triangle property of N=4 SYM)
- Numerator is independent of loop momentum
  - Can factor it out of integrals

$A^{(1)}_{\mathcal{N}=4} =$  $\times st A^{\text{tree}}_{\mathcal{N}=4}$

$$\mathcal{M}_n^{\text{loop}} = i^{L+1}\left(\frac{\kappa}{2}\right)^{n-2+2L}\sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D}\frac{1}{S_j}\frac{n_j \tilde{n}_j}{D_j}$$

Double copy at one loop

- Through the double copy, the gravity amplitude for Q+16 supercharge supergravity is

$$\mathcal{M}^{(1)}_{Q+16} = i\left(\frac{\kappa}{2}\right)^4 \boxed{st A^{\text{tree}}_{\mathcal{N}=4}(1,2,3,4)} \left[\boxed{A^{(1)}_Q(1,2,3,4) + A^{(1)}_Q(1,3,4,2) + A^{(1)}_Q(1,4,2,3)}\right]$$

N=4 box numerator

Linear Combination of one-loop YM amplitudes

- The N=4 numerator factors out of all of the integrals
  - Second copy of YM reorganises into a particular linear combination of colour-ordered amplitudes
- Valid in any number of dimensions
  - Q=16 gives N=8 supergravity
  - Q=0 gives N=4 supergravity

YM and Gravity Linked

$$\mathcal{M}^{(1)}_{Q+16} = i\left(\frac{\kappa}{2}\right)^4 st A^{\rm tree}_{\mathcal{N}=4}(1,2,3,4)\left[A^{(1)}_Q(1,2,3,4) + A^{(1)}_Q(1,3,4,2) + A^{(1)}_Q(1,4,2,3)\right]$$

N=4 box numerator

Linear Combination of one-loop YM amplitudes

- There is the possibility that supergravity amplitudes can vanish in the UV even when Yang-Mills does not.

$$\mathcal{A}^{(1)}(1,2,3,4) = g^4\left[c^{(1)}_{1234}A^{(1)}(1,2,3,4) + c^{(1)}_{1342}A^{(1)}(1,3,4,2) + c^{(1)}_{1423}A^{(1)}(1,4,2,3)\right]$$

- Renormalisability of Yang-Mills theory imposes a relationship between UV of different colour-ordered amplitudes
  - In precisely the combination that appears in the gravity amplitude

$$A^{(1)}_Q(1,2,3,4) + A^{(1)}_Q(1,3,4,2) + A^{(1)}_Q(1,4,2,3)\Big|_{\rm UV\ pole} = 0$$

N=4 supergravity UV

$$\mathcal{M}^{(1)}(1_{\rm H},2_{\rm H},3_{\rm H},4_{\rm H})\big|_{D=4\,\rm div.}=0\,,$$
$$\mathcal{M}^{(1)}(1_{\rm H},2_{\rm H},3_{\rm V},4_{\rm V})\big|_{D=4\,\rm div.}=0\,,$$
$$\mathcal{M}^{(1)}(1_{\rm V},2_{\rm V},3_{\rm V},4_{\rm V})\big|_{D=4\,\rm div.}=-\frac{1}{\epsilon}\frac{1}{(4\pi)^2}\left(\frac{\kappa}{2}\right)^4 stA^{\rm tree}_{Q=16}\frac{3(D_s-2)}{2}\,,$$
$$\mathcal{M}^{(1)}(1_{\rm V_1},2_{\rm V_1},3_{\rm V_2},4_{\rm V_2})\big|_{D=4\,\rm div.}=-\frac{1}{\epsilon}\frac{1}{(4\pi)^2}\left(\frac{\kappa}{2}\right)^4 stA^{\rm tree}_{Q=16}\frac{D_s-2}{2}\,,$$

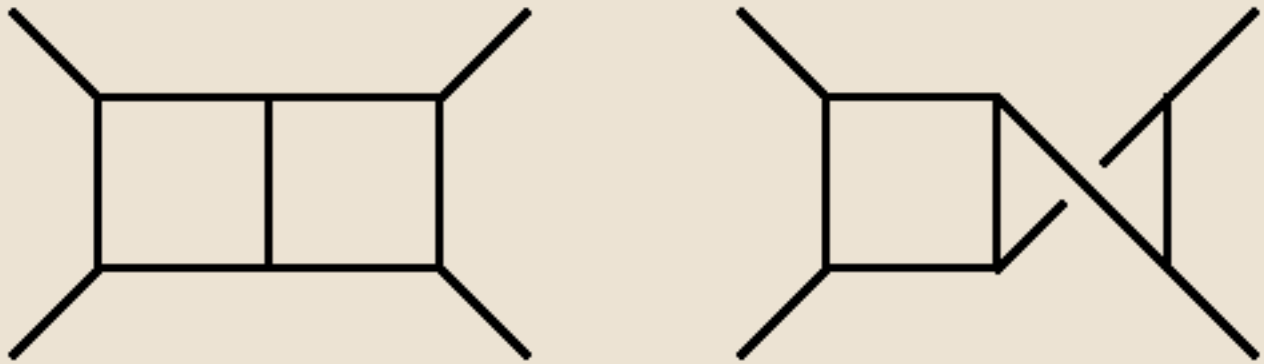
- **In 4D, factor of D_s-2 is due to U(1) duality symmetry anomaly**
  - **(more later)**

$$\mathcal{M}^{(1)}(1_{\rm H},2_{\rm H},3_{\rm H},4_{\rm H})\big|_{D=6\,\rm div.}=0\,,$$
$$\mathcal{M}^{(1)}(1_{\rm V},2_{\rm V},3_{\rm V},4_{\rm V})\big|_{D=6\,\rm div.}=0\,,$$
$$\mathcal{M}^{(1)}(1_{\rm V_1},2_{\rm V_1},3_{\rm V_2},4_{\rm V_2})\big|_{D=6\,\rm div.}=\frac{1}{\epsilon}\frac{1}{(4\pi)^3}\left(\frac{\kappa}{2}\right)^4 stA^{\rm tree}_{Q=16}\frac{26-D_s}{12}s\,, \qquad (4.10)$$
$$\mathcal{M}^{(1)}(1_{\rm H},2_{\rm H},3_{\rm V},4_{\rm V})\big|_{D=6\,\rm div.}=-\frac{1}{\epsilon}\frac{1}{(4\pi)^3}\left(\frac{\kappa}{2}\right)^4 stA^{\rm tree}_{Q=16}\frac{26-D_s}{24}\left(\varepsilon_1\cdot\varepsilon_2\, s-2k_1\cdot\varepsilon_2\, k_2\cdot\varepsilon_1\right)\,.$$

- **In 6D, factor of 26-D_s suggestive of bosonic string**
  - **But no real surprises here**

L = 2

Colour-dual representation



- Only two graphs (and permutations) contribute to the N=4 SYM amplitude
  - Numerators of the two graphs are the same
    - Implies vanishing of triangle-graph numerators

$$n^x_{1234} = sK\,, \quad n^x_{3421} = sK\,, \quad n^x_{1423} = tK\,,$$
$$n^x_{2341} = tK\,, \quad n^x_{1342} = uK\,, \quad n^x_{4231} = uK\,,$$
$$K = st A^{\text{tree}}_{Q=16}(1,2,3,4)$$

- Numerators are all independent of loop momentum

N=4 SYM amplitude at two loops

$$\mathcal{A}^{(2)}(1,2,3,4) = g^6 \sum_{x\in\{\mathrm{P},\mathrm{NP}\}} \Big[c^x_{1234} A^x(1,2,3,4) + c^x_{3421} A^x(3,4,2,1) + c^x_{1423} A^x(1,4,2,3) + c^x_{2341} A^x(2,3,4,1) + c^x_{1342} A^x(1,3,4,2) + c^x_{4231} A^x(4,2,3,1) \Big],$$

- **Again, renormalisability gives constraints on the 4D UV behaviour of different colour-ordered amplitudes**
- **Recall instructions to replace colour factors with kinematic numerators to get Gravity amplitudes:**

$$\mathcal{M}_n^{\mathrm{loop}} = i^{L+1} \left(\frac{\kappa}{2}\right)^{n-2+2L} \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{D_j}$$

Double copy at two loops

- Through the double copy, the gravity amplitude for Q+16 supercharge supergravity is

$$\mathcal{M}^{(2)}_{Q=16}(1,2,3,4) \;=\; i\left(\frac{\kappa}{2}\right)^6 st A^{\text{tree}}_{Q=16}(1,2,3,4) \sum_{x\in\{\text{P,NP}\}} \Big[s\big(A^x(1,2,3,4) + A^x(3,4,2,1)\big)$$
$$+t\big(A^x(1,4,2,3) + A^x(2,3,4,1)\big) + u\big(A^x(1,3,4,2) + A^x(4,2,3,1)\big)\Big]$$

- Again, the N=4 SYM copy factorises out of the integrals
  - Second copy of YM reorganises into a particular linear combination of colour-ordered amplitudes
  - In 4D, precisely the combination(s) that are UV finite due to renormalisability of YM

N=4 Supergravity in D=4

$$\mathcal{M}^{(2)}(1_{\mathrm{V}_1}, 2_{\mathrm{V}_1}, 3_{\mathrm{V}_2}, 4_{\mathrm{V}_2})\big|_{D=4\,\mathrm{div.}} = -\frac{1}{\epsilon^2}\frac{1}{(4\pi)^4}\left(\frac{\kappa}{2}\right)^6 s^2 t A^{\mathrm{tree}}_{Q=16}\frac{(D_s-2)^2}{4},$$

- The only UV divergent 4pt amplitude is with four external vector multiplets
- D_s-2 indicates connection to U(1) anomaly
- No real surprises

N=4 Supergravity in D=5

$$\mathcal{M}^{(2)}(1_H,2_H,3_H,4_H)\Big|_{D=5\text{ div.}} = 0$$

$$\mathcal{M}^{(2)}(1_V,2_V,3_V,4_V)\Big|_{D=5\text{ div.}} = \frac{1}{\epsilon}\frac{1}{(4\pi)^5}\left(\frac{\kappa}{2}\right)^5 stA^{\text{tree}}_{Q=16}\frac{(10-D_s)\pi}{3}(s^2+t^2+u^2)$$

$$\mathcal{M}^{(2)}(1_{V_1},2_{V_1},3_{V_2},4_{V_2})\Big|_{D=5\text{ div.}} = \frac{1}{\epsilon}\frac{1}{(4\pi)^5}\left(\frac{\kappa}{2}\right)^5 stA^{\text{tree}}_{Q=16}\frac{(10-D_s)\pi}{6}(3s^2+2tu)$$

$$\mathcal{M}^{(2)}(1_H,2_H,3_V,4_V)\Big|_{D=5\text{ div.}} = -\frac{1}{\epsilon}\frac{1}{(4\pi)^5}\left(\frac{\kappa}{2}\right)^5 s^2tA^{\text{tree}}_{Q=16}\frac{(10-D_s)\pi}{6} \times(\varepsilon_1\cdot\varepsilon_2 s-2k_1\cdot\varepsilon_2 k_2\cdot\varepsilon_1)$$

- **Pure graviton amplitudes are UV finite at two loops in D=5**
  - **No known symmetry explanation**
  - **But follows from YM renormalisability arguments similar to 4D**
    - i.e. Follows because double copy directly links SUGRA UV to YM UV
- **Also see suggestive 10-D_s factors**
  - **Just numerology?**

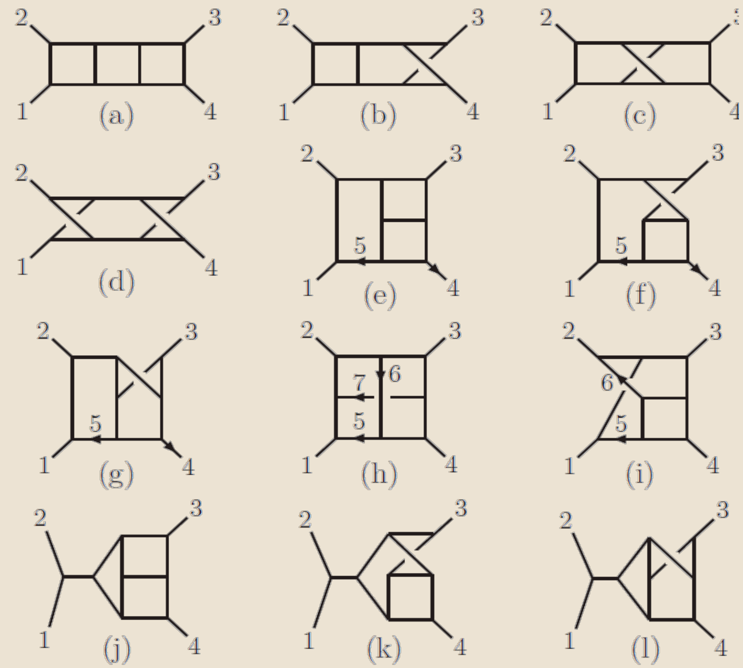
L = 3

Colour-dual numerators

- **Numerators satisfy BCJ duality**

  Bern, Carrasco, Johansson (2010)

- **Factor of stA_4^{tree} pulls out of every graph, but loop momentum dependence remains**
- **Graphs with triangle subdiagrams have vanishing numerators**



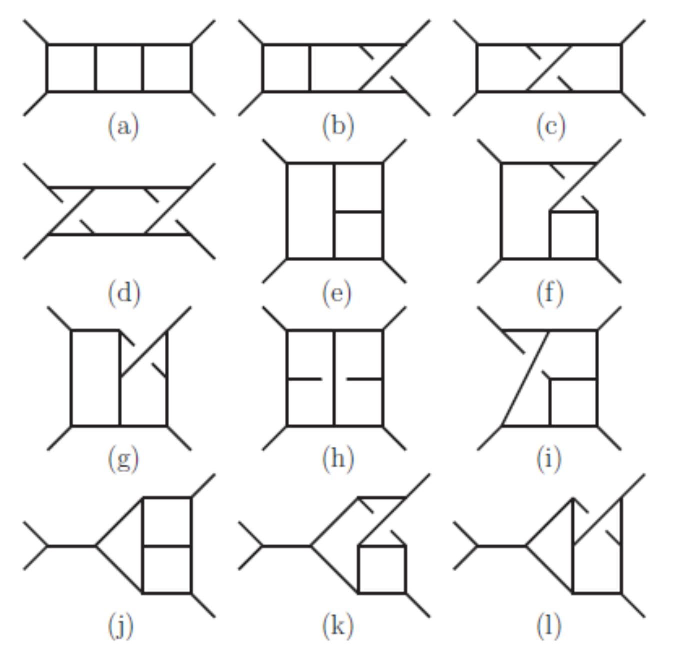
Integral $I^{(x)}$	$\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8}$ supergravity) numerator
(a)–(d)	s^2
(e)–(g)	$\left(s\,(-\tau_{35}+\tau_{45}+t) - t\,(\tau_{25}+\tau_{45}) + u\,(\tau_{25}+\tau_{35}) - s^2 \right)/3$
(h)	$\left(s\,(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u) \right.$ $\left. +t\,(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17}) + s^2 \right)/3$
(i)	$\left(s\,(-\tau_{25}-\tau_{26}-\tau_{35}+\tau_{36}+\tau_{45}+2t) \right.$ $\left. +t\,(\tau_{26}+\tau_{35}+2\tau_{36}+2\tau_{45}+3\tau_{46}) + u\,\tau_{25} + s^2 \right)/3$
(j)-(l)	$s(t-u)/3$

$$\tau_{ij} = 2k_i \cdot l_j$$

N=4 Supergravity in D=4

- ✓ **Series expand the integrand and select the logarithmic terms**
- ✓ **Reduce all the tensors in the integrand**
- ✓ **Regulate infrared divergences**
- ✓ **Subtract subdivergences**
- ✓ **Evaluate vacuum integrals**

Graph	$(\text{divergence})/(\langle 12\rangle^2[34]^2 st A^{\text{tree}}(\frac{\kappa}{2})^8)$
(a)-(d)	0
(e)	$\frac{263}{768}\frac{1}{\epsilon^3}+\frac{205}{27648}\frac{1}{\epsilon^2}+\left(-\frac{5551}{768}\zeta_3+\frac{326317}{110592}\right)\frac{1}{\epsilon}$
(f)	$-\frac{175}{2304}\frac{1}{\epsilon^3}-\frac{1}{4}\frac{1}{\epsilon^2}+\left(\frac{593}{288}\zeta_3-\frac{217571}{165888}\right)\frac{1}{\epsilon}$
(g)	$-\frac{11}{36}\frac{1}{\epsilon^3}+\frac{2057}{6912}\frac{1}{\epsilon^2}+\left(\frac{10769}{2304}\zeta_3-\frac{226201}{165888}\right)\frac{1}{\epsilon}$
(h)	$-\frac{3}{32}\frac{1}{\epsilon^3}-\frac{41}{1536}\frac{1}{\epsilon^2}+\left(\frac{3227}{2304}\zeta_3-\frac{3329}{18432}\right)\frac{1}{\epsilon}$
(i)	$\frac{17}{128}\frac{1}{\epsilon^3}-\frac{29}{1024}\frac{1}{\epsilon^2}+\left(-\frac{2087}{2304}\zeta_3-\frac{10495}{110592}\right)\frac{1}{\epsilon}$
(j)	$-\frac{15}{32}\frac{1}{\epsilon^3}+\frac{9}{64}\frac{1}{\epsilon^2}+\left(\frac{101}{12}\zeta_3-\frac{3227}{1152}\right)\frac{1}{\epsilon}$
(k)	$\frac{5}{64}\frac{1}{\epsilon^3}+\frac{89}{1152}\frac{1}{\epsilon^2}+\left(-\frac{377}{144}\zeta_3+\frac{287}{432}\right)\frac{1}{\epsilon}$
(l)	$\frac{25}{64}\frac{1}{\epsilon^3}-\frac{251}{1152}\frac{1}{\epsilon^2}+\left(-\frac{835}{144}\zeta_3+\frac{7385}{3456}\right)\frac{1}{\epsilon}$



- ✓ **The sum of all 12 graphs is finite!**

Bern, Davies, Dennen, Huang (2012)

N=4 Supergravity in D=4

$$\mathcal{M}^{(3)}(1_H, 2_H, 3_H, 4_H)\Big|_{D=4\text{ div.}} = 0$$

$$\mathcal{M}^{(3)}(1_H, 2_H, 3_V, 4_V)\Big|_{D=4\text{ div.}} = 0$$

$$\mathcal{M}^{(3)}(1_V, 2_V, 3_V, 4_V)\Big|_{D=4\text{ div.}} = -\frac{1}{(4\pi)^6}\left(\frac{\kappa}{2}\right)^6 (s^2+t^2+u^2)\, st A^{\text{tree}}_{Q=16}\frac{(D_s-2)^2}{4} \times\left(\frac{D_s-2}{2\epsilon^3} - \frac{1}{\epsilon^2} + \frac{1}{\epsilon}\right)$$

$$\mathcal{M}^{(3)}(1_{V_1}, 2_{V_1}, 3_{V_2}, 4_{V_2})\Big|_{D=4\text{ div.}} = (\text{ugly and not very interesting})$$

- **The vanishing of the pure supergravity divergence remains unexplained through symmetry understandings**
  - **Although, there is an understanding from string theory**

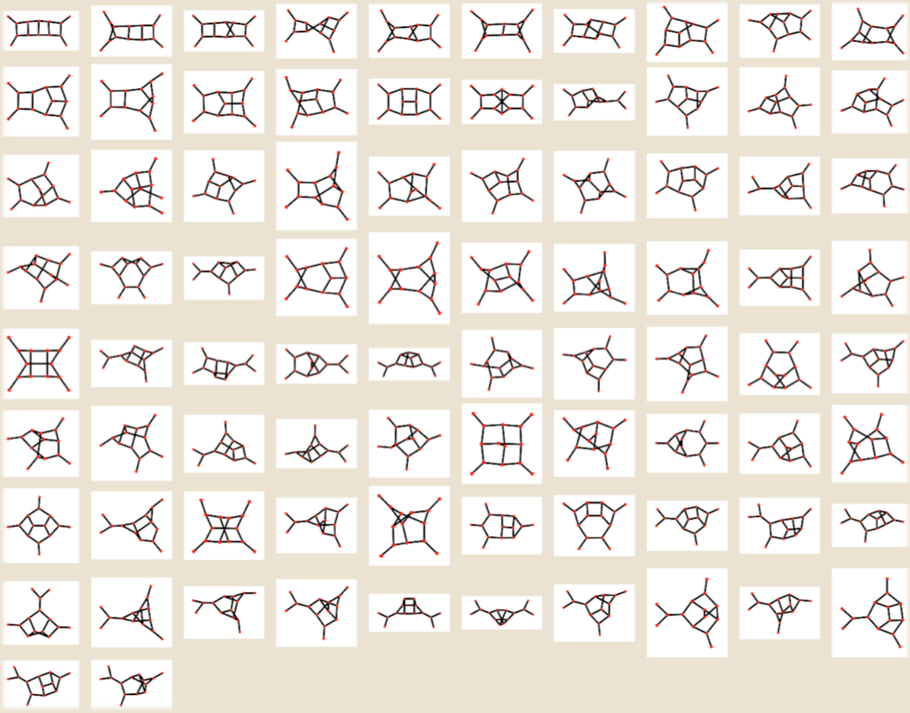
Tourkine, Vanhove (2012)

- **Another puzzle: R^4 counterterm is allowed by known symmetries**
  - **This is one of the critical L=N-1 cases.**
  - **What about N=5 and L=4?**

L = 4

Colour-dual numerators

- 82 colour-dual numerators in N=4 SYM theory at four loops
  - Too lengthy to put here
  - Factor of stA_4^{tree} pulls out of every numerator
  - Up to two powers of loop momentum in numerators



Bern, Carrasco, Dixon, Johansson, Roiban (2012)

N=5 Supergravity in D=4

graphs	(divergence) $\times\, u(4\pi)^8/(-\langle 12\rangle^2[34]^2 st A^{\mathrm{tree}}(\frac{\kappa}{2})^{10})$
1–30	$-\frac{1}{\epsilon^4}\left[\frac{297863}{3981312}s^2+\frac{7115179}{7962624}st+\frac{1230523}{2654208}t^2\right]+\frac{1}{\epsilon^3}\left[\frac{183507269}{318504960}s^2-\frac{121097629}{106168320}st-\frac{125340203}{159252480}t^2\right]+\frac{1}{\epsilon^2}\Big[\zeta_3\left(-\frac{54780317}{3686400}s^2-\frac{364821169}{22118400}st-\frac{19297919}{7372800}t^2\right)-\zeta_2\left(\frac{297863}{1990656}s^2+\frac{7115179}{3981312}st+\frac{1230523}{1327104}t^2\right)-\mathrm{S2}\left(\frac{1602535}{73728}s^2+\frac{10330175}{442368}st-\frac{14079343}{442368}t^2\right)-\frac{80222068879}{28665446400}s^2-\frac{949461174731}{57330892800}st-\frac{17877740021}{19110297600}t^2\Big]+\frac{1}{\epsilon}\Big[\zeta_5\left(\frac{42165713}{92160}s^2+\frac{12876011}{9216}st+\frac{10040753}{46080}t^2\right)+\zeta_4\left(\frac{1162609}{7372800}s^2+\frac{183267071}{14745600}st+\frac{110749763}{14745600}t^2\right)-\zeta_3\left(\frac{10506518408983}{71663616000}s^2+\frac{30289233413171}{71663616000}st-\frac{2013863213191}{35831808000}t^2\right)-\zeta_2\left(\frac{970317931}{159252480}s^2+\frac{59367181}{5898240}st-\frac{719420377}{79626240}t^2\right)-\mathrm{T1ep}\left(\frac{1602535}{995328}s^2+\frac{10330175}{5971968}st-\frac{14079343}{5971968}t^2\right)-\mathrm{S2}\left(\frac{33354691993}{53084160}s^2+\frac{19386147397}{10616832}st+\frac{9723954001}{8847360}t^2\right)-\mathrm{D6}\left(\frac{4137589}{552960}s^2+\frac{2283701}{184320}st+\frac{527011}{138240}t^2\right)-\frac{20252328329611}{143327232000}s^2-\frac{534679988685821}{1146617856000}st-\frac{8363829769903}{1146617856000}t^2\Big]$
31–60	$\frac{1}{\epsilon^4}\left[\frac{1788617}{3981312}s^2+\frac{20728021}{7962624}st+\frac{2452169}{2654208}t^2\right]+\frac{1}{\epsilon^3}\left[\frac{527762531}{318504960}s^2+\frac{1120727089}{106168320}st+\frac{122147731}{53084160}t^2\right]+\frac{1}{\epsilon^2}\Big[\zeta_3\left(\frac{6081287}{345600}s^2+\frac{13983243}{819200}st+\frac{98182043}{22118400}t^2\right)+\zeta_2\left(\frac{1788617}{1990656}s^2+\frac{20728021}{3981312}st+\frac{2452169}{1327104}t^2\right)+\mathrm{S2}\left(\frac{3516907}{73728}s^2+\frac{31188941}{442368}st-\frac{15998365}{442368}t^2\right)+\frac{545203990507}{28665446400}s^2+\frac{4109230335503}{57330892800}st+\frac{142686680113}{19110297600}t^2\Big]+\frac{1}{\epsilon}\Big[\zeta_5\left(-\frac{160438583}{245760}s^2-\frac{311758955}{147456}st-\frac{119748949}{368640}t^2\right)-\zeta_4\left(\frac{5925797}{921600}s^2+\frac{460780679}{14745600}st+\frac{126445477}{14745600}t^2\right)+\zeta_3\left(\frac{11662905491459}{53747712000}s^2+\frac{54035183618969}{71663616000}st+\frac{8467395805631}{214990848000}t^2\right)+\zeta_2\left(\frac{3059935571}{159252480}s^2+\frac{789428243}{17694720}st-\frac{197819569}{26542080}t^2\right)+\mathrm{T1ep}\left(\frac{3516907}{995328}s^2+\frac{31188941}{5971968}st-\frac{15998365}{5971968}t^2\right)+\mathrm{S2}\left(\frac{2658637313}{53084160}s^2+\frac{2611873009}{10616832}st+\frac{23301734753}{26542080}t^2\right)+\mathrm{D6}\left(\frac{6050189}{552960}s^2+\frac{10479103}{552960}st+\frac{233987}{46080}t^2\right)+\frac{455464156513}{1911029760}s^2+\frac{173334911330293}{229323571200}st+\frac{673760034799}{25480396800}t^2\Big]$
61–82	$-\frac{1}{\epsilon^4}\left[\frac{248459}{663552}s^2+\frac{756269}{442368}st+\frac{610823}{1327104}t^2\right]+\frac{1}{\epsilon^3}\left[-\frac{17781745}{7962624}s^2-\frac{5553497}{589824}st-\frac{24110299}{15925248}t^2\right]+\frac{1}{\epsilon^2}\Big[\zeta_3\left(-\frac{30260233}{11059200}s^2-\frac{1590799}{2764800}st-\frac{20144143}{11059200}t^2\right)-\zeta_2\left(\frac{248459}{331776}s^2+\frac{756269}{221184}st+\frac{610823}{663552}t^2\right)-\mathrm{S2}\left(\frac{53177}{2048}s^2+\frac{3476461}{73728}st-\frac{319837}{73728}t^2\right)-\frac{38748493469}{2388787200}s^2-\frac{9752373953}{176947200}st-\frac{31202235023}{4777574400}t^2\Big]+\frac{1}{\epsilon}\Big[\zeta_5\left(\frac{28798009}{147456}s^2+\frac{35247593}{49152}st+\frac{876065}{8192}t^2\right)+\zeta_4\left(\frac{15414589}{2457600}s^2+\frac{11563067}{614400}st+\frac{7847857}{7372800}t^2\right)-\zeta_3\left(\frac{15920366514887}{214990848000}s^2+\frac{4001452799633}{11943936000}st+\frac{20550575084777}{214990848000}t^2\right)-\zeta_2\left(\frac{52240441}{3981312}s^2+\frac{30566335}{884736}st+\frac{12596167}{7962624}t^2\right)-\mathrm{T1ep}\left(\frac{53177}{27648}s^2+\frac{3476461}{995328}st-\frac{319837}{995328}t^2\right)+\mathrm{S2}\left(\frac{767401367}{1327104}s^2+\frac{1397856199}{884736}st+\frac{587012725}{2654208}t^2\right)-\mathrm{D6}\left(\frac{47815}{13824}s^2+\frac{22675}{3456}st+\frac{17495}{13824}t^2\right)-\frac{434546648527}{4478976000}s^2-\frac{9221628964379}{31850496000}st-\frac{5488842949013}{286654464000}t^2\Big]$
sum	$\frac{1}{\epsilon}su\frac{1}{72}(264\,\zeta_3-1)$

graphs	(divergence) $\times\, u(4\pi)^8/(-\langle 12\rangle^2[34]^2 st A^{\mathrm{tree}}(\frac{\kappa}{2})^{10})$
1–30	$\frac{1}{\epsilon^4}\left[\frac{607}{1990656}s^2-\frac{1323773}{1990656}st-\frac{14255}{41472}t^2\right]+\frac{1}{\epsilon^3}\left[\frac{4865671}{19906560}s^2+\frac{149977}{3317760}st-\frac{20170049}{19906560}t^2\right]+\frac{1}{\epsilon^2}\Big[\zeta_3\left(-\frac{3733153}{230400}s^2-\frac{5900609}{276480}st+\frac{3883097}{691200}t^2\right)+\zeta_2\left(\frac{607}{995328}s^2-\frac{1323773}{995328}st-\frac{14255}{20736}t^2\right)-\mathrm{S2}\left(\frac{625357}{36864}s^2+\frac{5161189}{110592}st-\frac{1428583}{55296}t^2\right)-\frac{7648139167}{3583180800}s^2-\frac{22568882383}{3583180800}st-\frac{55681241}{59719680}t^2\Big]+\frac{1}{\epsilon}\Big[\zeta_5\left(-\frac{225641}{1024}s^2-\frac{12931021}{18432}st-\frac{2378855}{18432}t^2\right)-\zeta_4\left(\frac{4044329}{460800}s^2+\frac{3646153}{921600}st-\frac{2056603}{153600}t^2\right)+\zeta_3\left(\frac{6076575618157}{17915904000}s^2+\frac{3396579085657}{3583180800}st+\frac{2089036585637}{8957952000}t^2\right)-\zeta_2\left(\frac{51416459}{9953280}s^2+\frac{801749}{51840}st-\frac{65544931}{9953280}t^2\right)-\mathrm{T1ep}\left(\frac{625357}{497664}s^2+\frac{5161189}{1492992}st-\frac{1428583}{746496}t^2\right)-\mathrm{S2}\left(\frac{8055438013}{16588800}s^2+\frac{555755309}{414720}st+\frac{555207793}{614400}t^2\right)-\mathrm{D6}\left(\frac{715513}{138240}s^2+\frac{718247}{76800}st+\frac{285839}{172800}t^2\right)+\frac{1916368326173}{71663616000}s^2+\frac{7258817218703}{71663616000}st+\frac{3175133834231}{35831808000}t^2\Big]$
31–60	$\frac{1}{\epsilon^4}\left[\frac{509381}{1990656}s^2+\frac{3991391}{1990656}st+\frac{242555}{331776}t^2\right]+\frac{1}{\epsilon^3}\left[\frac{50554927}{19906560}s^2+\frac{13023425}{1327104}st+\frac{8356667}{3317760}t^2\right]+\frac{1}{\epsilon^2}\Big[\zeta_2\left(\frac{509381}{995328}s^2+\frac{3991391}{995328}st+\frac{242555}{165888}t^2\right)+\zeta_3\left(\frac{990949}{57600}s^2+\frac{570691}{1382400}st-\frac{10906963}{691200}t^2\right)+\mathrm{S2}\left(\frac{1380997}{36864}s^2+\frac{9202651}{110592}st-\frac{821453}{27648}t^2\right)+\frac{65553264229}{3583180800}s^2+\frac{27992599379}{447897600}st+\frac{12366245939}{1194393600}t^2\Big]+\frac{1}{\epsilon}\Big[\zeta_5\left(\frac{10240481}{23040}s^2+\frac{96847583}{92160}st+\frac{3535453}{30720}t^2\right)+\zeta_4\left(\frac{816643}{76800}s^2-\frac{6008467}{307200}st-\frac{51227}{2048}t^2\right)-\zeta_3\left(\frac{8235182625383}{13436928000}s^2+\frac{25298224196579}{17915904000}st+\frac{11561841643253}{53747712000}t^2\right)+\zeta_2\left(\frac{174844657}{9953280}s^2+\frac{31428727}{663552}st-\frac{8072393}{1658880}t^2\right)+\mathrm{T1ep}\left(\frac{1380997}{497664}s^2+\frac{9202651}{1492992}st-\frac{821453}{373248}t^2\right)+\mathrm{S2}\left(\frac{2385329963}{16588800}s^2+\frac{1077896293}{3317760}st+\frac{1501624967}{2073600}t^2\right)+\mathrm{D6}\left(\frac{233051}{46080}s^2+\frac{4649023}{691200}st+\frac{77389}{172800}t^2\right)-\frac{273686499733}{2654208000}s^2-\frac{10212410685517}{35831808000}st-\frac{501121685203}{4777574400}t^2\Big]$
61–82	$-\frac{1}{\epsilon^4}\left[\frac{42499}{165888}s^2+\frac{148201}{110592}st+\frac{128515}{331776}t^2\right]+\frac{1}{\epsilon^3}\left[-\frac{27710299}{9953280}s^2-\frac{21805693}{2211840}st-\frac{29969953}{19906560}t^2\right]+\frac{1}{\epsilon^2}\Big[\zeta_3\left(-\frac{25627}{25600}s^2+\frac{1607353}{76800}st+\frac{3511933}{345600}t^2\right)+\zeta_2\left(-\frac{42499}{82944}s^2-\frac{148201}{55296}st-\frac{128515}{165888}t^2\right)+\mathrm{S2}\left(-\frac{10495}{512}s^2-\frac{673577}{18432}st+\frac{71441}{18432}t^2\right)-\frac{9650854177}{597196800}s^2-\frac{7458218987}{132710400}st-\frac{11252621119}{1194393600}t^2\Big]+\frac{1}{\epsilon}\Big[\zeta_5\left(-\frac{10327117}{46080}s^2-\frac{1788471}{5120}st+\frac{321979}{23040}t^2\right)+\zeta_4\left(-\frac{855529}{460800}s^2+\frac{10835777}{460800}st+\frac{892711}{76800}t^2\right)+\zeta_3\left(\frac{14908078591061}{53747712000}s^2+\frac{1396836736049}{2985984000}st-\frac{972377870569}{53747712000}t^2\right)+\zeta_2\left(-\frac{61714099}{4976640}s^2-\frac{35277233}{1105920}st-\frac{17110573}{9953280}t^2\right)+\mathrm{T1ep}\left(-\frac{10495}{6912}s^2-\frac{673577}{248832}st+\frac{71441}{248832}t^2\right)+\mathrm{S2}\left(\frac{113402161}{331776}s^2+\frac{1122715393}{1105920}st+\frac{119104427}{663552}t^2\right)+\mathrm{D6}\left(\frac{409}{3456}s^2+\frac{2269}{864}st+\frac{4169}{3456}t^2\right)+\frac{2736085919309}{35831808000}s^2+\frac{1462778758259}{7962624000}st+\frac{1166557609583}{71663616000}t^2\Big]$
sum	$-\frac{1}{\epsilon}su\frac{1}{72}\,(264\,\zeta_3-1)$

Bern, Davies, Dennen (2014)

N=5 Supergravity in D=4

- **Surprise**
  - **Second example of L=N-1 case UV finite, unexplained**
- **How much further can we go?**
  - **N=6 is out of reach (requires L=5 BCJ numerators)**
  - **We could look at L=4 N=4**
    - Above the critical L=N-1 line.
    - Multiple counterterms allowed by symmetry.
    - Solidly in the territory of amplitudes that *should* diverge

N=4 Supergravity in D=4

$$\mathcal{M}^{\text{4-loop}}_{\mathcal{N}=4}\Big|_{\text{div}} = \frac{1}{(4\pi)^8}\left(\frac{\kappa}{2}\right)^{10}\frac{1-264\zeta_3}{288\epsilon}stA^{\text{tree}}_{\mathcal{N}=4}\left(\mathcal{O}^{--++}+3\mathcal{O}^{-+++}+60\mathcal{O}^{++++}\right)$$

Bern, Davies, Dennen, Smirnov[2] (2013)

- Double copy makes SU(4) R-symmetry manifest
- Three distinct counterterms
- --++ is 4-graviton sector
- The latter two configurations would vanish if duality symmetry were not anomalous
  - E.g. in N>4 SG

- All three independent configurations have a similar divergence!
  - How much can we really read into this? There is very little information in the transcendental coefficient.

Matter Multiplets in the Loops

$$\mathcal{M}^{\text{4-loop}}_{\mathcal{N}=4}\Big|_{\text{div}} = \frac{1}{(4\pi)^8}\left(\frac{\kappa}{2}\right)^{10} \boxed{\frac{1-264\zeta_3}{288\epsilon}} st A^{\text{tree}}_{\mathcal{N}=4}\left(\mathcal{O}^{--++} + 3\mathcal{O}^{-+++} + 60\mathcal{O}^{++++}\right)$$

↓

$$\boxed{\frac{(n_V+2)}{4608}\left(\frac{6(n_V+2)n_V}{\epsilon^2} + \frac{(n_V+2)(3n_V+4) - 96(22-n_V)\zeta_3}{\epsilon}\right)}$$

- **Couple to matter multiplets to get more info**
- **Requires honest subtraction of subdivergences, since matter amplitudes diverge already at one loop**

Fischler (1979)

- **Kinematic factor is the same as pure SUGRA**
- **Transcendental constants factorize out**

The Structure of the Result

$$\mathcal{M}^{\text{4-loop}}_{\mathcal{N}=4}\Big|_{\text{div}} = \frac{1}{(4\pi)^8}\left(\frac{\kappa}{2}\right)^{10} \boxed{\frac{1-264\zeta_3}{288\epsilon}}\, st A^{\text{tree}}_{\mathcal{N}=4}\left(\mathcal{O}^{--++} + 3\mathcal{O}^{-+++} + 60\mathcal{O}^{++++}\right)$$

↓

$$\boxed{\frac{(n_V+2)}{4608}\left(\frac{6(n_V+2)n_V}{\epsilon^2} + \frac{(n_V+2)(3n_V+4) - 96(22-n_V)\zeta_3}{\epsilon}\right)}$$

- All three independent configurations still have a similar divergence
  - Peculiar because the nonanomalous sector should naively have a very different analytic structure. Not related by any supersymmetry Ward identities.
- Factorization of transcendental constants is (slightly) less trivial than it looks
  - ζ_4 and ζ_5 cancel away unexpectedly
- n_V dependence is apparently consistent with U(1) anomaly

U(1) Anomaly

Marcus (1985)

Carrasco, Kallosh, Tseytlin, Roiban (2013)

- **There is an anomaly in a U(1) subgroup of the SU(4) x SU(1,1) duality symmetry**
  - **Scalar degrees of freedom parameterize SU(1,1)/U(1)**
  - **Can gauge the U(1) to linearize the action of SU(1,1)**
    - scalars become complex doublet under global SU(1,1)
    - Pick up a phase under local U(1)

$$\Phi^\alpha \Phi_\alpha = 1 \qquad \Phi'_\alpha = e^{-i\gamma(x)} U_\alpha^\beta \Phi_\beta$$

- **Anomalous means different gauge choices for the U(1) give different theories at the quantum level**
  - **Theories differ by a local, finite term in the effective action**

Anomalies in unitarity cuts

- As pointed out by Carrasco, Kallosh, Tseytlin & Roiban, the anomalous sectors are poorly behaved and contribute to a four-loop UV divergence (unless somehow cancelled, as they are at three loops)
- Anomalous sector feeds poor UV behavior into MHV sector

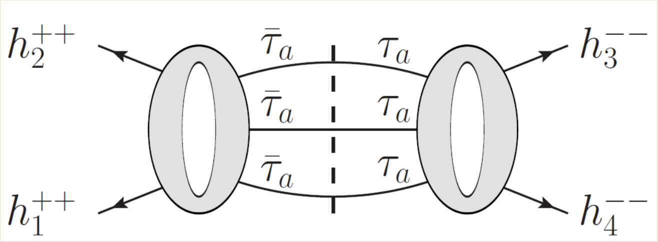


Figure from arXiv:1303.6219
Carrasco, Kallosh, Tseytlin, Roiban

- Key Feature: Each anomaly insertion gives a factor of (n_V+2)
  - This cut contributes $(n_V+2)^2$ times a two-loop integral
  - To get ζ_3 requires a three-loop integral, which leaves only enough room for one anomaly insertion.

Connection Between Sectors?

$$\mathcal{M}^{\text{4-loop}}_{\mathcal{N}=4}\Big|_{\text{div}} = \frac{1}{(4\pi)^8}\left(\frac{\kappa}{2}\right)^{10} \boxed{\frac{1-264\zeta_3}{288\epsilon}} st A^{\text{tree}}_{\mathcal{N}=4}\left(\mathcal{O}^{--++} + 3\mathcal{O}^{-+++} + 60\mathcal{O}^{++++}\right)$$

↓

$$\boxed{\frac{(n_V+2)}{4608}\left(\frac{6(n_V+2)n_V}{\epsilon^2} + \frac{(n_V+2)(3n_V+4) - 96(22-n_V)\zeta_3}{\epsilon}\right)}$$

- **Result looks consistent with being entirely due to the anomaly**
  - **$(n_V+2)^2$ for rational numbers**
  - **$(n_V+2)\,\zeta_3$ consistent with a single anomaly insertion**
- **Bottom line: This divergence looks specific to *N* = 4 SG, and likely due to the anomaly.**
  - **Though the high loop order prevents a detailed analysis**

Quantum (in)equivalence

- In light of a forthcoming paper, one might worry that we've done something bad in N=4:

Bern, Cheung, Chi, Davies, Dixon, Nohle (to appear)

- In the presence of anomalies, UV divergence and renormalization scale dependence can become unlinked
  - Then UV divergence isn't a good proxy for $\log(M^2)$ terms
  - Two-loop calculations in Einstein gravity show this explicitly for the conformal anomaly

* See also Zvi Bern's talk at Strings

- Not a problem in our case – verification of subdivergence cancellations
  - Although U(1) duality symmetry is anomalous in N=4, UV is still linked to $\log(M^2)$

Enhanced Cancellations

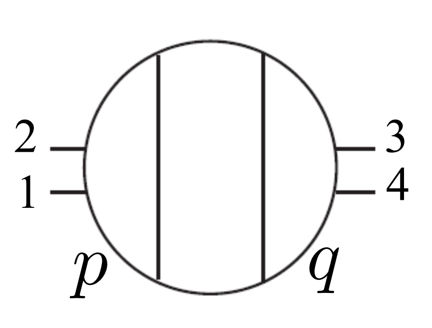
Enhanced Cancellations

- We've seen three examples of vanishing of expected UV divergences.
- How exotic are these cancellations?

- Among the symmetry considerations:
  - Björnsson & Green developed a first-quantized formulation of N=8 sugra based on Berkovits' pure spinors
  - BEFKMS Used duality symmetry to exhaustively rule out N=8 counterterms until 7 loops (L=N-1)
  - Bossard, Howe, Stelle, Vanhove extended the L=N-1 counterterm to N=4, 5 & 6

Enhanced Cancellations

- All paths lead to the same conclusion: L=N-1 has an allowed counterterm
- From a unitarity perspective, symmetry arguments essentially expose the power counting on maximal cuts



- N=4 sugra is (pure YM) x (N=4 SYM)
- Pure YM is already log divergent on the maximal cut of the ladder diagram
- No amount of standard symmetry transformations can push these log divergences into other diagrams

- Any cancellations beyond these, such as those *between* different diagrams on maximal cuts, we term "enhanced cancellations"
  - Enhanced cancellations are not present in non-abelian gauge theories

Summary

- Known explicit examples of enhanced cancellations
  - D=4, N=4 pure sugra is finite at 3 loops
  - D=4, N=5 pure sugra is finite at 4 loops
  - D=5 half-maximal sugra is finite at 2 loops
- No known standard symmetry explanation for any of these
- Enhanced cancellations appear to be a qualitatively new mechanism controlling the UV
  - Though still mysterious!
  - Awaits bold new ideas