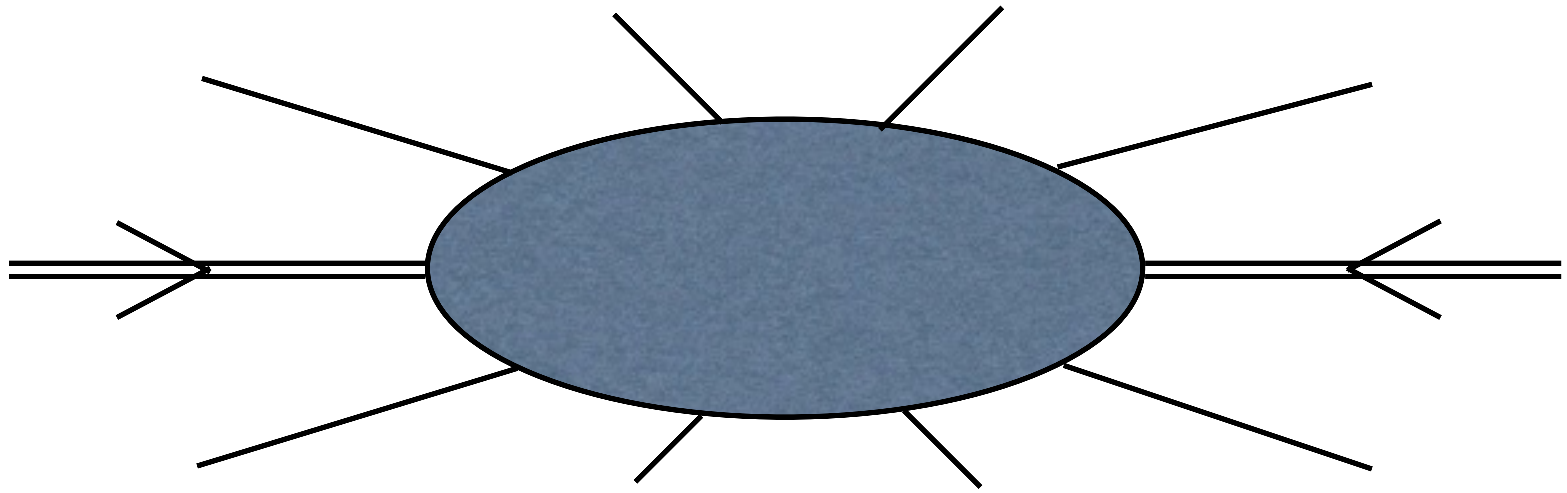
Amplitudes 2015

The Cluster Bootstrap for Amplitudes/Wilson loops

James Drummond

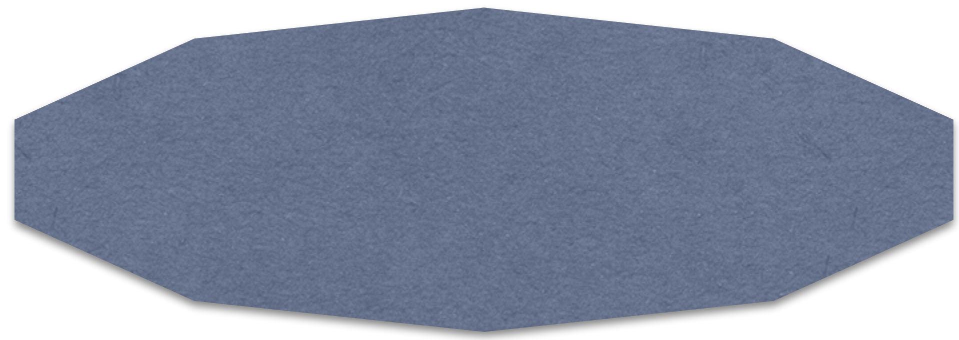
Based on 1412.3763 [JMD, Papathanasiou, Spradlin]

Motivation



- Analytic S-matrix programme
- Dream goal: define and calculate scattering amplitudes in terms of the analytic properties they obey.
- Singularities: poles and cuts correspond to physical processes.
- Properties like unitarity should heavily constrain the results.

Motivation



- N=4 super Yang-Mills the simplest 4d gauge theory.
- Integrability in the planar limit gives even more structure.
- Duality between amplitudes and light-like Wilson loops.
- Analytic structure is more tractable.

Bootstrap programme

- Proceed experimentally:
- Observe that in perturbation theory amplitudes/Wilson loops are described by particular classes of functions.
- Make an ansatz in terms of these functions.
- Constrain ansatz with some physical input:
- Branch cuts (locality/unitarity), collinear limits, supersymmetry, OPE for Wilson loops, Regge limits for amplitudes…

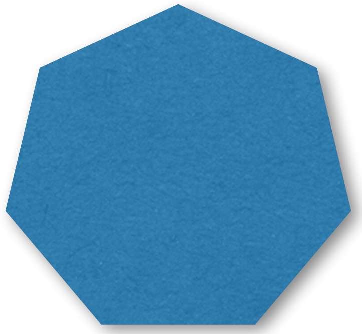
Hexagon amplitudes

[Dixon, JMD, Henn], [Dixon, JMD, von Hippel, Pennington],
[Dixon, JMD, Duhr, Pennington]

- Simplest amplitudes are the six-point ones.
- Functions appearing are polylogarithms on $\mathcal{M}_{0,6}$
- Impose physical branch cuts: Hexagon functions.
- Impose proper collinear behaviour.
- OPE (near collinear) limit/Regge limit data required to fix amplitude from three loops onwards.
- Fruitful interplay with integrable OPE approach [Sever's talk]

Cluster Algebras on G(4,n)

- Observation: singularities of two-loop results ([Caron-Huot]) coincide with A-coordinates of cluster algebras based on the Grassmannians G(4,n). [Golden, Goncharov, Spradlin, Vergu, Volovich]
- This allows us to expand the bootstrap programme to higher points.
- Today we analyse heptagon amplitudes (next simplest & finite set of A-coordinates).



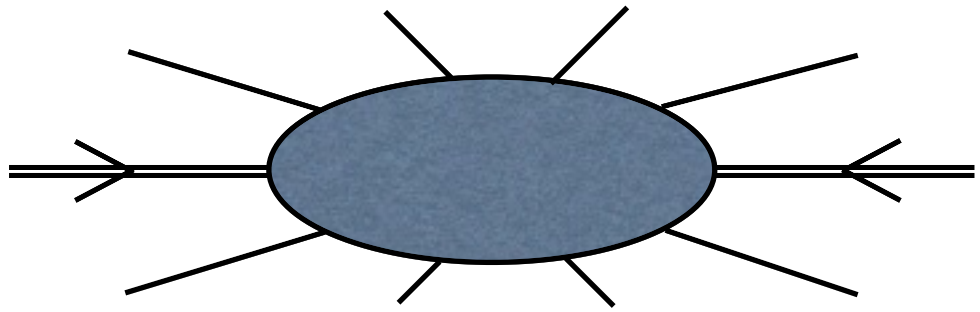
Scattering amplitudes

Amplitudes depend on:

On-shell (light-like) momenta $p_i^{\alpha\dot{\alpha}} = \lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}}$

Helicities $h_i \in \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$

Colour labels a_i



Planar limit: $$\mathcal{A}_{\text{full}} = \sum_{\sigma} \text{Tr}(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n)}}) \underbrace{\mathcal{A}(\sigma(1), \dots, \sigma(n))}_{\text{Colour-ordered partial amplitude}}$$

$\mathcal{A}(--++\dots+)$ (MHV)

$\mathcal{A}(---+\dots+)$ (NMHV)

Functions of momenta only

N=4 supersymmetry

On-shell supermultiplet

$$\Phi(\eta) = G_+ + \eta^A \Gamma_A + \tfrac{1}{2}\eta^A\eta^B S_{AB} + \tfrac{1}{3!}\eta^A\eta^B\eta^C \epsilon_{ABCD}\overline{\Gamma}^D + \tfrac{1}{4!}\eta^4 G_-$$

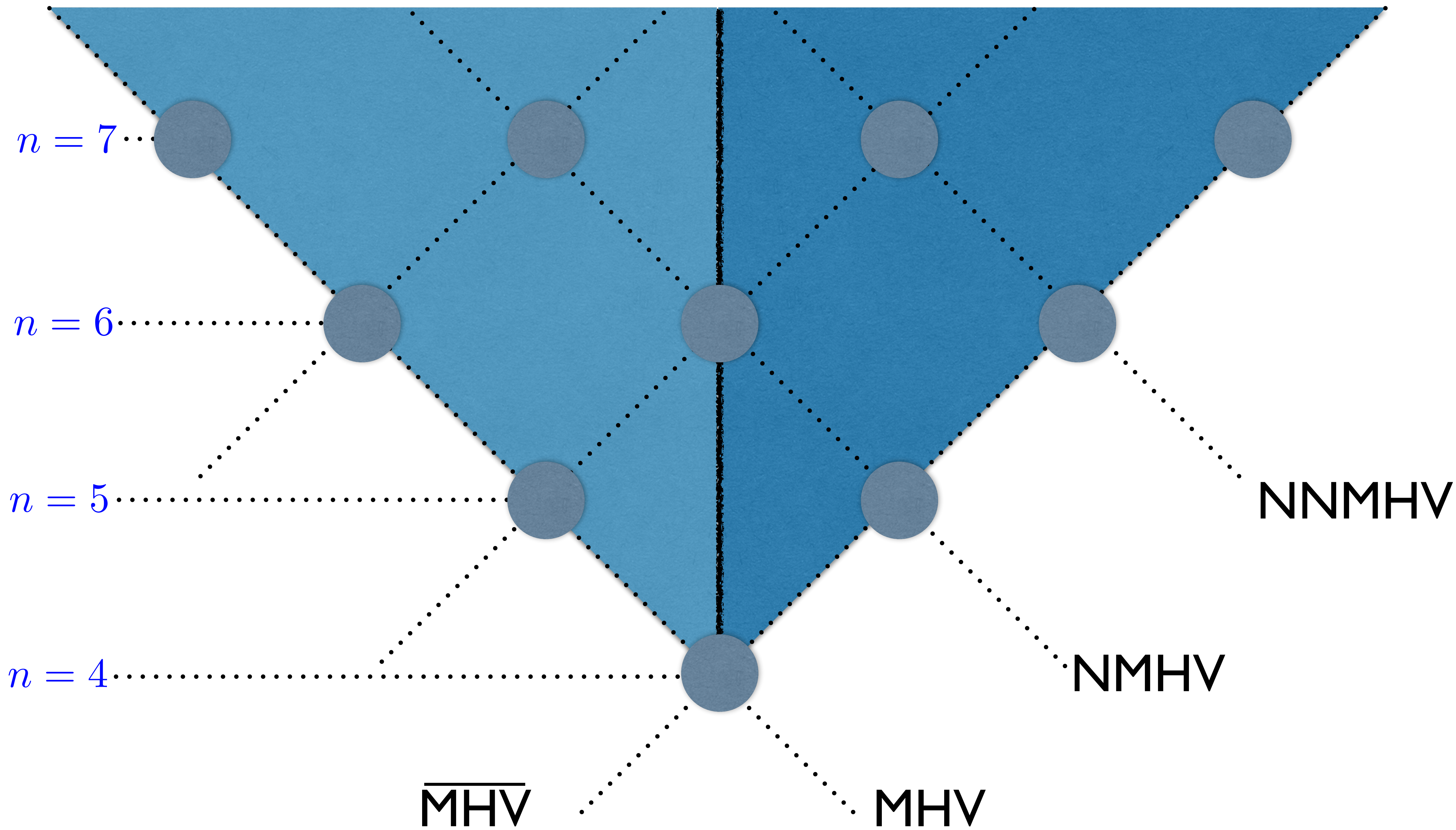
Supersymmetry generators

$$p^{\alpha\dot{\alpha}} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}, \qquad q^{\alpha A} = \lambda^\alpha \eta^A, \qquad \bar{q}^{\dot{\alpha}}_A = \tilde{\lambda}^{\dot{\alpha}} \frac{\partial}{\partial \eta^A}$$

Superamplitude

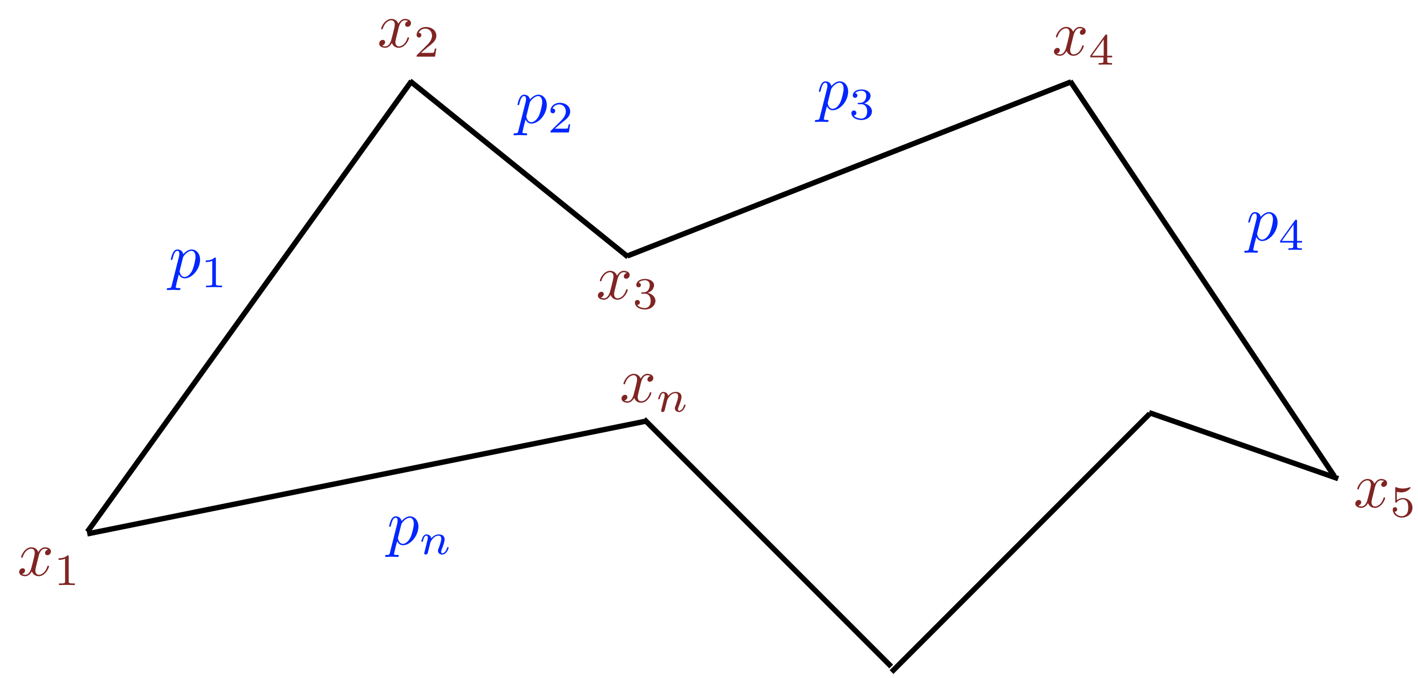
$$\mathcal{A}(\Phi_1, \ldots, \Phi_n) = \frac{\delta^4(p)\delta^8(q)}{\langle 12\rangle \ldots \langle n1\rangle}\left[\mathcal{P}^{(0)} + \mathcal{P}^{(4)} + \ldots \mathcal{P}^{(4n-16)}\right]$$

MHV expansion



Wilson loops

$$\left\langle \mathrm{tr}\, \mathcal{P} \exp \int_C A \right\rangle$$



- Naturally come with a dihedral symmetry.
- Colour-ordered MHV amplitudes and Wilson loops coincide.
  [Alday, Maldacena], [JMD, Korchemsky, Sokatchev], [Brandhuber, Heslop, Travaglini], [JMD,Henn,Korchemsky,Sokatchev],…
- Super Wilson loops for non-MHV amplitudes.
  [Mason, Skinner], [Caron-Huot]
- Conformal symmetry of Wilson loop is symmetry of amplitude.

Dual conformal symmetry

- Space of light-like polygons stable under conformal transformations.
- Conformal symmetry of Wilson loops broken by ultraviolet divergences.
- Divergences factorise and exponentiate.
- Interesting piece is the conformally invariant finite 'remainder'.

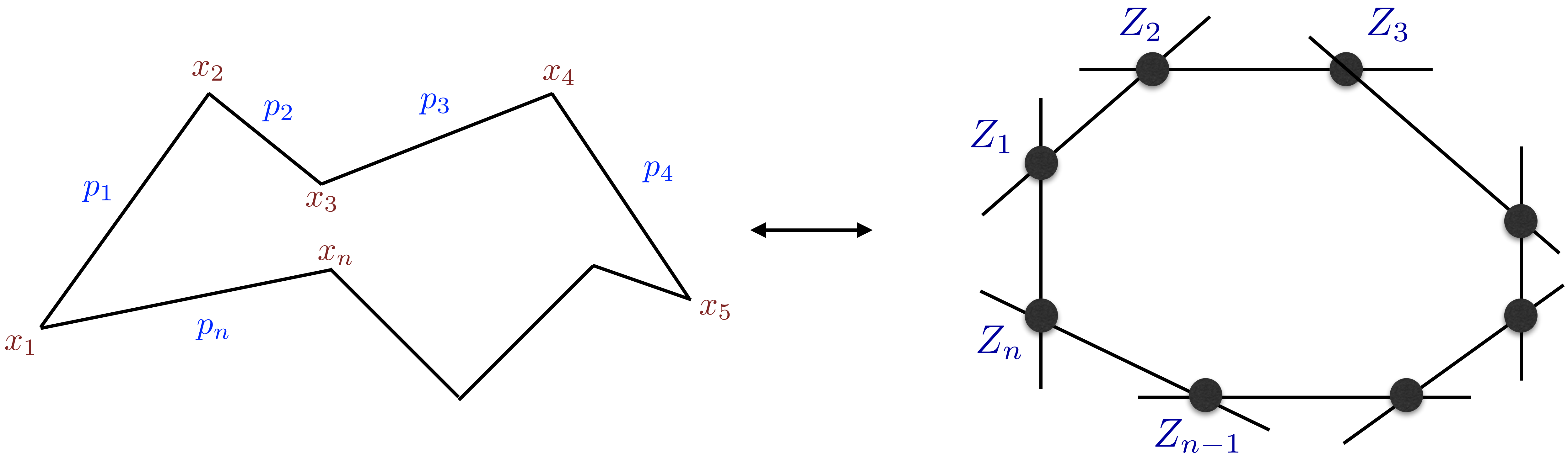
$$\left\langle \mathrm{tr}\, \mathcal{P} \exp \int_C A \right\rangle = \exp(\mathrm{UV\ div}) \exp R$$

- Divergences organised so that remainder begins at two loops in pert. theory.

- First conformal invariants at six points:
- Four and five points 'trivial'.

$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}\,, \quad v = \frac{x_{24}^2 x_{15}^2}{x_{14}^2 x_{25}^2}\,, \quad w = \frac{x_{35}^2 x_{26}^2}{x_{25}^2 x_{36}^2}\,.$$

Twistors



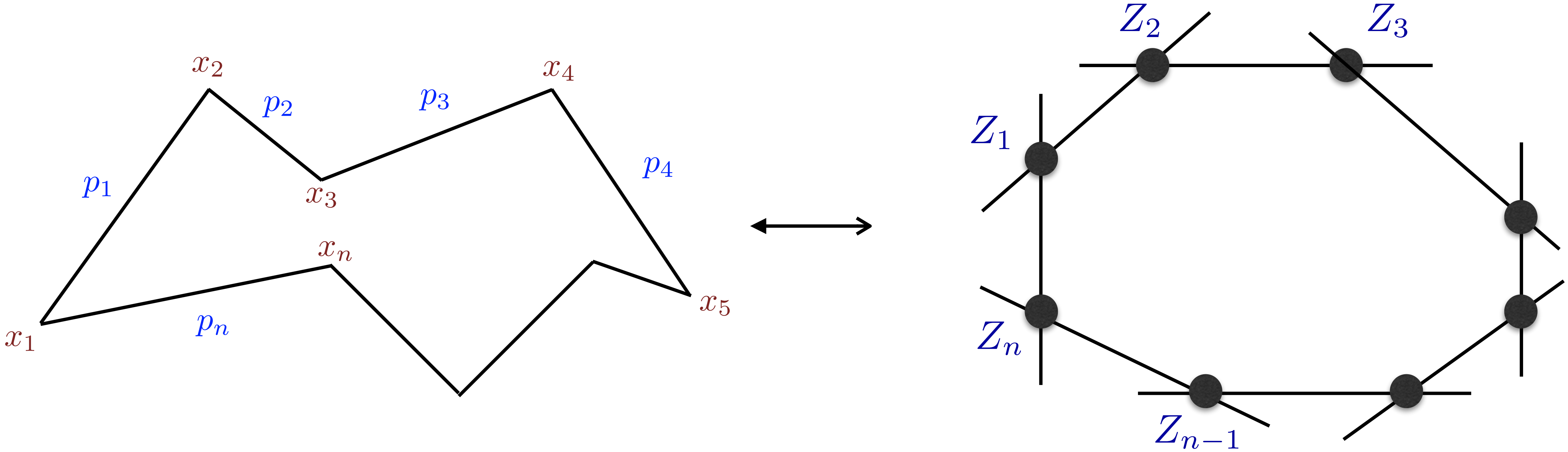
- Best to describe a sequence of intersecting null rays via twistors $Z_i \in \mathbb{CP}^3$

- Due to the relation to particle momenta, often called 'momentum twistors'. [Hodges]

- The corners of the loop map to lines in twistor space $x_i \sim \dfrac{Z_{i-1} \wedge Z_i}{\langle Z_{i-1} Z_i I \rangle}$

- Mandelstam variables $(p_i + p_{i+1} + \dots p_{j-1})^2 = (x_i - x_j)^2 = \dfrac{\langle Z_{i-1} Z_i Z_{j-1} Z_j \rangle}{\langle Z_{i-1} Z_i I \rangle \langle Z_{j-1} Z_j I \rangle}$

Twistors



Arrange the twistors into a (4 x n) matrix: (Z_i^A)

Gives a description of the Grassmannian $G(4, n)$

Kinematical space identified with: $\mathrm{Conf}_n(\mathbb{P}^3) = G(4, n)/(\mathbb{C}^*)^{n-1}$

Polylogarithms

[Chen], [Goncharov], [Brown], …

Classical polylogarithms: $\mathrm{Li}_n(x) = \int_0^x \frac{dt}{t}\mathrm{Li}_{n-1}(t)\,, \quad \mathrm{Li}_1(x) = -\log(1-x)$

More generally, polylogarithms in many variables:

$$df^{(k)} = \sum_\phi f_\phi^{(k-1)} d\log\phi\,, \quad f^{(1)} = \sum_\phi r_\phi \log\phi$$

The 'letters' ϕ run over a finite set of rational functions.

'Symbol' recursively defined $S(f^{(k)}) = \sum_\phi S(f_\phi^{(k-1)}) \otimes \phi\,, \quad S(\log\phi) = \phi$

Examples: $S(\mathrm{Li}_2(x)) = -[(1-x)\otimes x]\,, \quad S(\log^2 x) = 2[x\otimes x]$

Integrability: $d^2 f^{(k)} = 0 \quad \Longrightarrow \quad \sum_\phi df_\phi^{(k-1)} \wedge d\log\phi = 0$

Cluster algebras

The letters (singularities) will be dictated by cluster algebras associated to

$$\mathrm{Conf}_n(\mathbb{P}^3) = G(4,n)/(\mathbb{C}^*)^{n-1}$$

Cluster algebras: [Fomin, Zelevinsky]

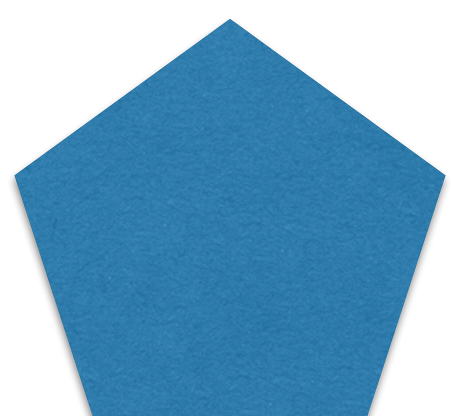
- Commutative algebras with distinguished set of generators (cluster variables).
- Variables grouped into overlapping sets (clusters).
- Clusters constructed from initial cluster via a process called ‘mutation’.

A_2 example

- Cluster variables: $a_m \,, \quad m \in \mathbb{Z}$
- Initial cluster: $\{a_1, a_2\}$
- Clusters: $\{a_m, a_{m+1}\}$
- Mutation: $\{a_{m-1}, a_m\} \to \{a_m, a_{m+1}\}, \quad a_{m+1} = \dfrac{1+a_m}{a_{m-1}}$

Finite number of clusters:

$$a_3 = \frac{1+a_2}{a_1}\,, \quad a_4 = \frac{1+a_1+a_2}{a_1 a_2}\,, \quad a_5 = \frac{1+a_1}{a_2}\,, \quad a_6 = a_1\,, \quad a_7 = a_2$$



Topology of mutations is a pentagon.

Quivers

More generally, consider a quiver diagram, corresponding to a cluster.

Each cluster variable corresponds to node.

Mutation on node k yields a new quiver via the rules:

For each $i \to k \to j$

add new arrow $i \to j$

reverse all arrows to/from k

delete opposing pairs and returning arrows

$$a_k \to a'_k = \frac{1}{a_k}\left(\prod_{i\to k} a_i + \prod_{k\to j} a_j\right)$$

Sometimes finite, sometimes infinite.

A_2 Initial quiver $1 \longrightarrow 2$ becomes $1' \longleftarrow 2$

with $a_3 = a'_1 = \dfrac{1+a_2}{a_1}$

Grassmannian G(4,n)

Can associate a cluster algebra to the Grassmannian $G(4,n)$ [Scott]

Initial cluster given by specified set of 4-brackets $\langle ijkl \rangle$

Mutation generates homogeneous polynomials in 4-brackets

For $n = 6, 7$ algebras are finite (correspond to A_3 and E_6)

For $n \geq 8$ algebra is infinite.

Observation: [Golden, Goncharov, Spradlin, Vergu, Volovich]
known two-loop results show that letters are cluster A-coordinates.

Cluster bootstrap ansatz: letters are A-coordinates.

For hexagon: 9 A-coordinates,
For heptagon: 42 of them.

Hexagons

Mutations generate letters,

$$u_1 = \frac{\langle 1236\rangle\langle 3456\rangle}{\langle 1346\rangle\langle 2356\rangle}\,, \quad 1-u_1 = \frac{\langle 1356\rangle\langle 2346\rangle}{\langle 1346\rangle\langle 2356\rangle}\,, \quad y_1 = \frac{\langle 1345\rangle\langle 2456\rangle\langle 1236\rangle}{\langle 1235\rangle\langle 3456\rangle\langle 1246\rangle}$$

and those related by cyclic rotation of the labels.

Once obtained, any multiplicatively independent set of nine will do.

Topology of mutations is Stasheff polytope.

Can replace 4-brackets with 2-brackets: $\langle 1234\rangle \to \langle 56\rangle$

Space of functions identified with polylogarithms on $\mathcal{M}_{0,6}$

Heptagons

For heptagons we generate the following letters

$$a_{11} = \frac{\langle 1234\rangle\langle 1567\rangle\langle 2367\rangle}{\langle 1237\rangle\langle 1267\rangle\langle 3456\rangle}\,, \qquad a_{41} = \frac{\langle 2457\rangle\langle 3456\rangle}{\langle 2345\rangle\langle 4567\rangle}\,,$$

$$a_{21} = \frac{\langle 1234\rangle\langle 2567\rangle}{\langle 1267\rangle\langle 2345\rangle}\,, \qquad a_{51} = \frac{\langle 1(23)(45)(67)\rangle}{\langle 1234\rangle\langle 1567\rangle}\,,$$

$$a_{31} = \frac{\langle 1567\rangle\langle 2347\rangle}{\langle 1237\rangle\langle 4567\rangle}\,, \qquad a_{61} = \frac{\langle 1(34)(56)(72)\rangle}{\langle 1234\rangle\langle 1567\rangle}$$

and those obtained by cyclic rotation of the labels.

$$\langle a(bc)(de)(fg)\rangle \equiv \langle abde\rangle\langle acfg\rangle - \langle abfg\rangle\langle acde\rangle$$

Unlike in the hexagon case, the space of singularities depends on the choice of dihedral structure.

Naturally associated to the kinematic space of light-like Wilson loops.

Heptagon symbols

Heptagon symbols:

Now we want to build integrable words from the 42 heptagon letters

Locality: initial letters are a_{1i}

Symbol of heptagon Wilson loop remainder should be a heptagon symbol

Supersymmetry: final letters are $a_{2i}\,,\, a_{3i}$

Collinear limit: $R_n \to R_{n-1}$ $\qquad (i||i+1)$

Constructing symbols

Impose integrability by equating two decompositions of a word into integrable parts:

$$w^{(n)} = \underbrace{\sum c_{ij}^{(1)} w_i^{(k)} \otimes w_j^{(n-k)} = \sum c_{rs}^{(2)} w_r^{(k+1)} \otimes w_s^{(n-k-1)}}$$

These give homogeneous linear equations for $c_{ij}^{(1)}, c_{rs}^{(2)}$

$$A.C = 0$$

The calculation is just linear algebra but for rather large matrices.
Efficient algorithms for calculating the null spaces of integer matrices are very useful.

The calculation can be adapted easily for imposing conditions on initial and final entries simultaneously.

Results 1

Weight $k =$	1	2	3	4	5	6
Number of heptagon symbols	7	42	237	1288	6763	?
well-defined in the $7 \parallel 6$ limit	3	15	98	646	?	?
which vanish in the $7 \parallel 6$ limit	0	6	72	572	?	?
well-defined for all $i{+}1 \parallel i$	0	0	0	1	?	?
with MHV last entries	0	1	0	2	1	4
with both of the previous two	0	0	0	1	0	1

Table 1: Heptagon symbols and their properties.

The symbol of the two-loop remainder function is the only weight 4 heptagon symbol which is well-defined in all collinear limits.

There is a unique weight 6 heptagon symbol which obeys the final entry and is finite in all collinear limits.
We conclude this must be the symbol of the three-loop heptagon remainder.

Results 2

For comparison, hexagon symbols:

Weight $k =$	1	2	3	4	5	6
Number of hexagon symbols	3	9	26	75	218	643
well-defined (hence vanish) in the $6 \parallel 5$ limit	0	2	11	44	155	516
well-defined (hence vanish) for all $i{+}1 \parallel i$	0	0	2	12	68	307
with MHV last entries	0	3	7	21	62	188
with both of the previous two	0	0	1	4	14	59

Table 1: Hexagon symbols and their properties.

- In hexagon case must appeal to further input to fix the Wilson loop.
- OPE data or information from Regge limit required.
- Heptagon bootstrap more powerful than hexagon one!
- Hexagon can be recovered from heptagon by collinear limit.

Checks and Extensions

The Wilson loop admits an expansion around the collinear limit, similar to the operator product expansion for local operators in a CFT.

[Alday, Gaiotto, Maldacena, Sever, Vieira]

Further progress allows the prediction of the power suppressed terms in this limit.

[Basso, Sever, Vieira]

We find perfect agreement between this expansion and the series expansion of our symbol.

NMHV: We also find that the two-loop NMHV amplitude of Caron-Huot and He is the unique possible expression compatible with dihedral symmetry both before and after taking a collinear limit.

(up to adding the MHV expression multiplied by the tree amplitude).

Discussion and Outlook

Using the notion of cluster algebras we have a natural conjecture for the space of singularities for planar MHV amplitudes/Wilson loops to all orders.

We have tested this structure with a three-loop seven-point calculation.

Surprisingly, the bootstrap for the heptagon is actually more powerful than for the hexagon.

The heptagon calculate required no input from the Wilson loop OPE. This approach provides a test of OPE conjectures.
Perhaps an alternative formulation of integrability?

Intuitively it feels that the structure should be applicable more generally to light-like Wilson loops in any weakly coupled conformal gauge theory.

Requires further investigation…