Hexagon Functions: Bootstrapping an NMHV Amplitude through Four Loops

arXiv:1308.2276, 1408.1505, plus work in progress

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Try this **ONE WEIRD TRICK**
to **CALCULATE FOUR-LOOP SCATTERING AMPLITUDES***

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*In planar N=4 super Yang-Mills, for six points

N=4 super Yang-Mills

- We are looking at $\mathcal{N} = 4$ super Yang-Mills in the planar limit, in $4 - 2\epsilon$ dimensions
- Maximally Helicity-Violating (MHV) component simplest $(- - + + ++)$, Next-to-MHV (NMHV) is what this talk will explore.
- Conformal symmetry enhanced by dual conformal symmetry: $\mathcal{N} = 4$ amplitudes can be interpreted as polygonal Wilson loops with corners defined in terms of the amplitude momenta, $k_i = x_i - x_{i+1}$.
- This led to understanding of IR divergences via BDS ansatz [Bern, Dixon, Smirnov '05].
- Dividing NMHV by MHV leads to IR-finite **Ratio Function**, with **transcendental weight two times the loop order**

Transcendental Functions

- Transcendental functions have fixed transcendental weight:
  $\pi^n, \quad \zeta_n, \quad \log^n z, \quad \mathrm{Li}_n(z)$, etc.
- Classical Polylogarithms:

$$\mathrm{Li}_n(z) = \int_0^z d\ln t_1 \int_0^{t_1} d\ln t_2 \ldots \int_0^{t_{n-1}} d\ln t_{n-1} \int_0^{t_n} d\ln(1-t_n)$$

- Transcendental functions fall into a more general class, with integrals over some set of rational functions:

$$\int_0^z d\ln\phi_1(t_1) \int_0^{t_1} d\ln\phi_2(t_2) \ldots \int_0^{t_n} d\ln\phi_n(t_n)$$

  Here n is the **transcendental weight**, while the ϕ_r are the **letters of the symbol**
- Final entry of the symbol corresponds to outermost integration ⇒ First derivative

Hexagon Functions

- Want to bootstrap things up through four loops, for six-particle amplitudes
- To do this, need functions germane to six-point dual conformally invariant processes: Hexagon Functions [Dixon, Drummond, MvH, Pennington 1308.2276]
- These functions depend on three dual conformally invariant cross ratios: u, v, w, or alternatively parity-odd variables y_u, y_v, y_w.

Construction

We construct functions with:

- Symbol entries from $\mathcal{S}_u = \{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$
- Physical branch cuts: first entry must be u, v, or w

From there, bootstrap!

- Derivatives of hexagon functions composed of hexagon functions of lower weight
- Fix transcendental constants with branch cuts

End up with basis of a few hundred irreducible functions.

Tree Level Ratio Function

- Out of momentum twistor four-brackets $\langle abcd \rangle = \epsilon_{RSTU} Z_a^R Z_b^S Z_c^T Z_d^U$, build the six superconformal R-invariants:

$$(f) = [abcde] = \frac{\delta^4(\chi_a \langle bcde \rangle + \text{cyclic})}{\langle abcd \rangle \langle bcde \rangle \langle cdea \rangle \langle deab \rangle \langle eabc \rangle}$$

- The tree-level ratio function then is:

$$\mathcal{P}^{(0)}_{\text{NMHV}} = (6) + (4) + (2) = (1) + (3) + (5)$$

Loop Level

- At loop level, R-invariants are dressed with permutations of two transcendental functions: an even parity function V, and and odd parity function $\tilde{V}$

$$\begin{aligned}\mathcal{P}_{\text{NMHV}} = \frac{1}{2}\Big[\quad & [(1)+(4)]V(u,v,w) + [(1)-(4)]\tilde{V}(u,v,w) \\ & + [(2)+(5)]V(v,w,u) - [(2)-(5)]\tilde{V}(v,w,u) \\ & + [(3)+(6)]V(w,u,v) + [(3)-(6)]\tilde{V}(w,u,v)\Big]\end{aligned}$$

Constraints

Lance Dixon and I tackled the ratio function at three loops [1408.1505], and with Andrew McLeod at four loops [to appear]. We began with a general ansatz of Hexagon Functions, then applied constraints:

- Symmetry:

$$V(w,v,u) = V(u,v,w) \quad \text{and} \quad \tilde{V}(w,v,u) = -\tilde{V}(u,v,w)$$

- "Gauge Freedom": Add a cyclically symmetric function to $\tilde{V}$

$$\frac{1}{2}\left[[(1)-(4)]\tilde{f}(u,v,w) - [(2)-(5)]\tilde{f}(u,v,w) + [(3)-(6)]\tilde{f}(u,v,w)\right]$$
$$= \frac{1}{2}\Big[[(1)+(3)+(5)] - [(2)+(4)+(6)]\Big]\tilde{f}(u,v,w) = 0$$

Constraints, Continued

- Final Entry Constraint: $\bar{Q}$ equation requires that R-invariant (1) can only multiply a function with final entries from

$$\left\{\frac{u}{1-u}, \frac{v}{1-v}, \frac{w}{1-w}, y_u, y_v, y_w, \frac{uw}{v}\right\},$$

while the other R-invariants multiply appropriate cyclic permutations [Caron-Huot, He '11].

  - We found for loops 1-3 this is even more constrained, used for new final entry condition

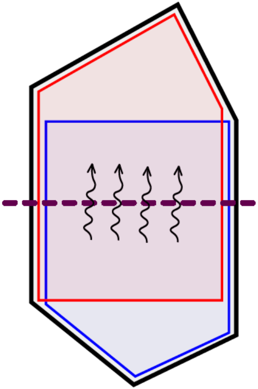
$$\left\{\frac{u}{1-u}, \frac{w}{1-w}, y_u y_w, y_v, \frac{uw}{v}\right\}.$$

- Spurious Pole Constraints: Unphysical poles should cancel. R-invariants (1) and (3) contain poles as $\langle 2456\rangle \to 0$, so we must have that

$$\left[V(u,v,w) - V(w,u,v) + \tilde{V}(y_u,y_v,y_w) - \tilde{V}(y_w,y_u,y_v)\right]_{\langle 2456\rangle=0} = 0$$

Near-Collinear Expansion

- The ratio function vanishes in the collinear limit.
- Basso, Sever, and Vieira calculate Wilson loops in $\mathcal{N} = 4$ sYM for finite coupling using integrability, expanding in GKP string states propagating across.
  - This corresponds to expansion around the collinear limit.
  - We use first-order data as constraints, second order as a check.



Constraints in Action

Constraint	$L = 1$	$L = 2$		$L = 3$		$L = 4$	
	even	even	odd	even	odd	even	odd
Integrable functions	10	82	6	639	122	5153	1763
(Anti)symmetry	7	50	2	363	39+10	2797	583+203
Final-entry conditions	3	14	1	78	$21 + 3$	487	$321 + 64$
Collinear vanishing	0	2	1	28	$21 + 3$	284	$321 + 64$
Spurious Pole	0	1		$4 + 3$		$180 + 64$	
Near-Collinear OPE	0	0		$0 + 3$		$0 + 64$	

The Function

- Can "show the function", but it's long, not illuminating, relies on defining lots of lower-weight functions.
- Better to look at plots.

Planes in v

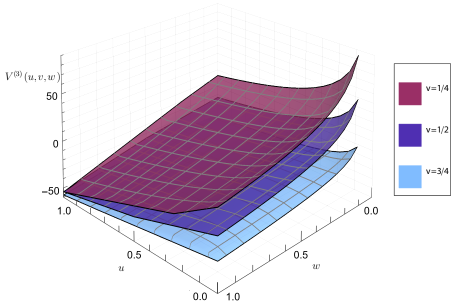


Figure: $V^{(3)}(u,v,w)$ evaluated on successive planes in v.

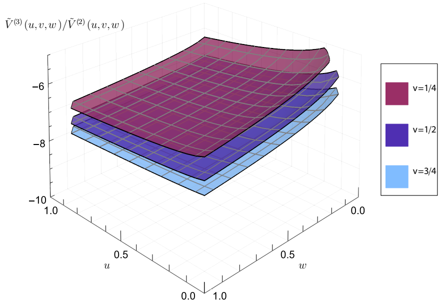


Figure: $\tilde{V}^{(3)}(u,v,w)/\tilde{V}^{(2)}(u,v,w)$ evaluated on successive planes in v.

Lines through the space

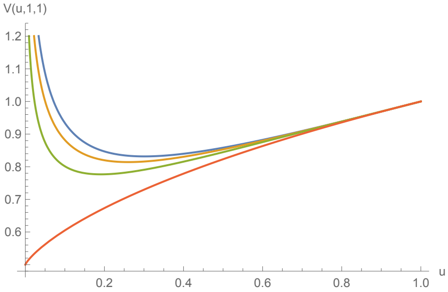


Figure: $V^{(4)}(u,1,1)$, $V^{(3)}(u,1,1)$, $V^{(2)}(u,1,1)$, and $V^{(1)}(u,1,1)$ normalized to one at $(1,1,1)$. One loop is in red, two loops is in green, three loops is in yellow, and four loops is in blue.

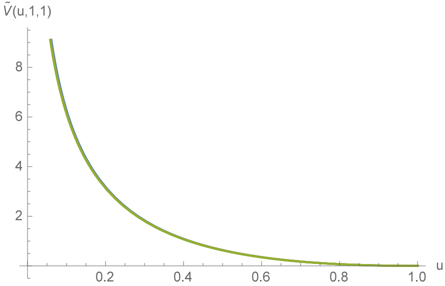


Figure: $\tilde{V}^{(4)}(u,1,1)$, $\tilde{V}^{(3)}(u,1,1)$ and $\tilde{V}^{(2)}(u,1,1)$ normalized so they have a $\ln^2 u$ term in the $u \to 0$ limit with coefficient one. Two loops is in green, three loops is in yellow, and four loops is in blue.

Conclusions and Open Questions

- We have bootstrapped up amplitudes at 6 points through 4 loops for both MHV and NMHV, with no need to know the integrands beforehand.
- Recently, Drummond, Papathanasiou, and Spradlin have found 3 loop 7 point MHV symbol in arXiv:1412.3763 [hep-th], more 7 point work ahead, potential to go beyond 7?
- BSV's calculation of the OPE provides an enormous amount of data. Even at first order, terms "want to be re-summed". Better understanding of this might lead to all-loop, all-kinematics picture.