Feynman Integrals for QCD

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PRISMA

Amplitudes 2015
Zürich, 6. - 10. July 2015

Multi-loop Feynman integrals

$$I = \int \frac{\mathrm{d}^d k_1}{i\pi^{d/2}} \cdots \frac{\mathrm{d}^d k_L}{i\pi^{d/2}} \frac{1}{D_1^{a_1} \cdots D_N^{a_N}} \qquad a_i \in \mathbb{Z}, \quad D_1 = k_1^2 - m_1^2 \text{ etc.}$$

linear dependencies:

- integration-by-parts (IBP) identities [Tkachov, Chetyrkin '81]
- systematic reduction to master integrals [Laporta '00]
- think of it as linear vector space with some arbitrary basis

this talk: **two solving methods**

1. differential equations in kinematic invariants
2. direct integration of Feynman (Schwinger) parameters

Choice of basis

1. choose master integrals suitable for respective integration method
2. optimize functional basis for solution

Part I: Two-loop amplitudes for diboson production

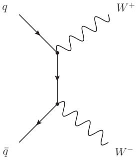
[Gehrmann, AvM, Tancredi, Weihs]

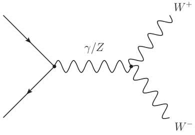
Vector boson pair production at LHC

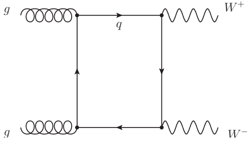
$pp \to VV' + X \to 4$ leptons $+ X$, where VV'=ZZ, W^+W^-, $\gamma^*\gamma^*$, $ZW^\pm$, $Z\gamma^*$, $W^\pm\gamma^*$

- sensitive to details of EWSB
- possible NP contributions at tree or loop level

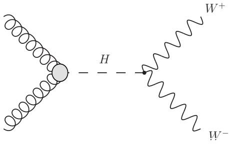
e.g. W^+W^- production:



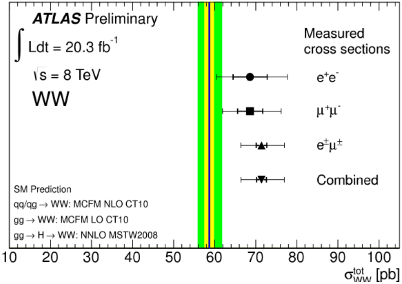


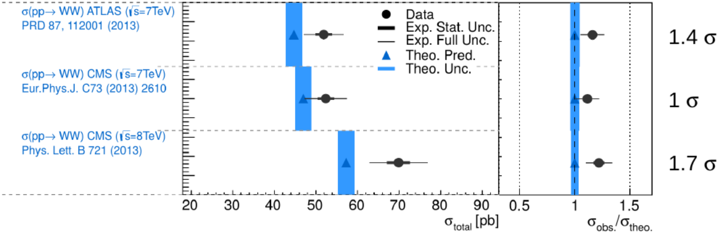


important background to Higgs signals:



2014 excess in *WW* production at LHC



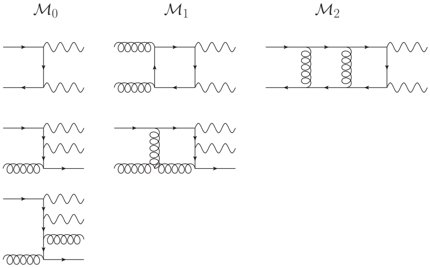


new physics ?

make sure to understand SM prediction !

ingredients for $VV' + X$ production at NNLO QCD:

	LO	NLO	NNLO
$2 \to 2$	$\mathcal{M}_0^* \mathcal{M}_0$ qq	$\mathcal{M}_0^* \mathcal{M}_1$ qq	$\mathcal{M}_0^* \mathcal{M}_2$, $\mathcal{M}_1^* \mathcal{M}_1$ qq, gg
$2 \to 3$	-	$\mathcal{M}_0^* \mathcal{M}_0$ qq, qg	$\mathcal{M}_0^* \mathcal{M}_1$ qq, qg
$2 \to 4$	-	-	$\mathcal{M}_0^* \mathcal{M}_0$ qq, qg, gg



note: some channels contribute only at higher orders:

- qg starting at NLO
- gg starting at NNLO $\to$ control error by computing N^{3}LO contributions from this channel

subtraction terms: up to 2 unresolved partons needed

- q_T **subtraction**: [Catani, Grazzini '07; Catani, Cieri, de Florian, Ferrera, Grazzini '13]
- N-jettiness subtraction: [Boughezal, Foecke, Liu, Petriello '15; Boughezal, Foecke, Giele, Liu, Petriello '15; Gaunt, Stahlhofen, Tackmann, Walsh '15]
- antenna subtraction: [Gehrmann-De Ridder, Gehrmann, Glover '05]
- sector-improved subtraction: [Czakon '10]

QCD approx. NNLO and electroweak NLO:

- *gg* initiated (one-loop only): [Binoth et al. ('05,'08); Duhrssen et al. ('05); Amettler et al. ('85); van der Bij, Glover ('88); Adamson, de Florian, Signer ('00)]
- high energy WW: [Chachamis, Czakon, Eiras ('08)]
- electroweak NLO: [Hollik, Meier (2004); Accomando, Denner, Meier ('05); Bierweiler, Kasprzik, Kühn, Uccirati ('12); Baglio, Ninh, Weber ('13); Billoni, Dittmaier, Jäger, Speckner ('13)]

QCD full NNLO (equal masses):

- master integrals: [Gehrmann, Tancredi, Weihs '13; Gehrmann, AvM, Tancredi, Weihs '14]
- amplitudes: [Gehrmann, AvM, Tancredi (unpublished)]
- ZZ@NNLO [Cascioli,Gehrmann,Grazzini,Kallweit,Maierhöfer,AvM,Pozzorini,Rathlev,Tancredi,Weihs'14]
- WW@NNLO [Gehrmann, Grazzini, Kallweit, Maierhöfer, AvM, Pozzorini, Rathlev, Tancredi '14]

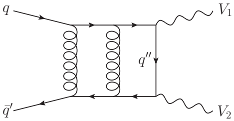
QCD full NNLO (different masses):

- master integrals: [Henn, Melnikov, Smirnov '14; Caola, Henn, Melnikov, Smirnov '14]; [Papadopoulos, Tammasini, Wever '14]; [Gehrmann, AvM, Tancredi '15]
- amplitudes $q\bar{q}' \to VV'$: [Caola, Henn, Melnikov, Smirnov '14]; [Gehrmann, AvM, Tancredi '15]
- amplitudes $gg \to VV'$: [Caola, Henn, Melnikov, Smirnov '15]; [AvM, Tancredi '15]
- $\gamma^*\gamma^*$@NNLO (partial): [Anastasiou, Cancino, Chavez, Duhr, Lazopoulos, Mistlberger, Müller '14]
- upcoming exact differential NNLO for various final states:
  see talks by [S. Kallweit, D. Rathlev, M. Wiesemann] at RadCor-LoopFest 2015

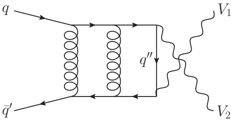
Feynman diagrams (generated with Qgraf [Nogueira])

***$q\bar{q}'$* channel** (just non-zero classes shown):

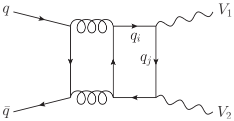
[A]



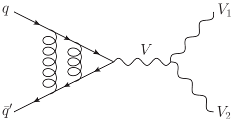
[B]



[C]

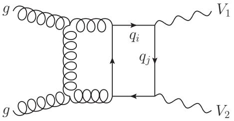


[F_V]

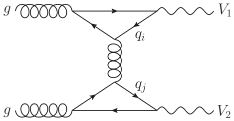


***gg* channel** ([B] and [F_V] do not contribute):

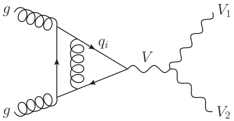
[A]



[B]



[F_V]



Lorentz structures for VV' amplitude

VV' amplitude:

$$S^{\mu\nu}(p_1, p_2, p_3) = \sum_j A_j(s, t, p_3^2, p_4^2)\, T_j^{\mu\nu}$$

$q\bar{q}'$ channel:

$$\begin{aligned}
T_1^{\mu\nu} &= \bar{u}(p_2)\, \not{p}_3 u(p_1)\, p_1^\mu p_1^\nu\,, & T_2^{\mu\nu} &= \bar{u}(p_2)\, \not{p}_3 u(p_1)\, p_1^\mu p_2^\nu\,,\\
T_3^{\mu\nu} &= \bar{u}(p_2)\, \not{p}_3 u(p_1)\, p_2^\mu p_1^\nu\,, & T_4^{\mu\nu} &= \bar{u}(p_2)\, \not{p}_3 u(p_1)\, p_2^\mu p_2^\nu\,,\\
T_5^{\mu\nu} &= \bar{u}(p_2)\, \gamma^\mu u(p_1)\, p_1^\nu\,, & T_6^{\mu\nu} &= \bar{u}(p_2)\, \gamma^\mu u(p_1)\, p_2^\nu\,,\\
T_7^{\mu\nu} &= \bar{u}(p_2)\, \gamma^\nu u(p_1)\, p_1^\mu\,, & T_8^{\mu\nu} &= \bar{u}(p_2)\, \gamma^\nu u(p_1)\, p_2^\mu\,,\\
T_9^{\mu\nu} &= \bar{u}(p_2)\, \gamma^\mu \not{p}_3 \gamma^\nu u(p_1)\,, & T_{10}^{\mu\nu} &= \bar{u}(p_2)\, \gamma^\nu \not{p}_3 \gamma^\mu u(p_1)\,.
\end{aligned}$$

gg channel:

$$\begin{aligned}
T_1^{\mu\nu} &= \epsilon_1 \cdot \epsilon_2\, g^{\mu\nu}\,, & T_2^{\mu\nu} &= \epsilon_1^\mu\, \epsilon_2^\nu\,, & T_3^{\mu\nu} &= \epsilon_1^\nu\, \epsilon_2^\mu\,, & T_4^{\mu\nu} &= \epsilon_1 \cdot \epsilon_2\, p_1^\mu\, p_1^\nu\,,\\
T_5^{\mu\nu} &= \epsilon_1 \cdot \epsilon_2\, p_1^\mu\, p_2^\nu\,, & T_6^{\mu\nu} &= \epsilon_1 \cdot \epsilon_2\, p_2^\mu\, p_1^\nu\,, & T_7^{\mu\nu} &= \epsilon_1 \cdot \epsilon_2\, p_2^\mu\, p_2^\nu\,, & T_8^{\mu\nu} &= \epsilon_2 \cdot p_3\, \epsilon_1^\mu\, p_1^\nu\,,\\
T_9^{\mu\nu} &= \epsilon_2 \cdot p_3\, \epsilon_1^\mu\, p_2^\nu\,, & T_{10}^{\mu\nu} &= \epsilon_2 \cdot p_3\, \epsilon_1^\nu\, p_1^\mu\,, & T_{11}^{\mu\nu} &= \epsilon_2 \cdot p_3\, \epsilon_1^\nu\, p_2^\mu\,, & T_{12}^{\mu\nu} &= \epsilon_1 \cdot p_3\, \epsilon_2^\mu\, p_1^\nu\,,\\
T_{13}^{\mu\nu} &= \epsilon_1 \cdot p_3\, \epsilon_2^\mu\, p_2^\nu\,, & T_{14}^{\mu\nu} &= \epsilon_1 \cdot p_3\, \epsilon_2^\nu\, p_1^\mu\,, & T_{15}^{\mu\nu} &= \epsilon_1 \cdot p_3\, \epsilon_2^\nu\, p_2^\mu\,, & T_{16}^{\mu\nu} &= \epsilon_1 \cdot p_3\, \epsilon_2 \cdot p_3\, g^{\mu\nu}
\end{aligned}$$

$$\begin{aligned}
T_{17}^{\mu\nu} &= \epsilon_1 \cdot p_3\, \epsilon_2 \cdot p_3\, p_1^\mu\, p_1^\nu\,, & T_{18}^{\mu\nu} &= \epsilon_1 \cdot p_3\, \epsilon_2 \cdot p_3\, p_1^\mu\, p_2^\nu\,,\\
T_{19}^{\mu\nu} &= \epsilon_1 \cdot p_3\, \epsilon_2 \cdot p_3\, p_2^\mu\, p_1^\nu\,, & T_{20}^{\mu\nu} &= \epsilon_1 \cdot p_3\, \epsilon_2 \cdot p_3\, p_2^\mu\, p_2^\nu\,.
\end{aligned}$$

Helicity amplitudes for $q\bar{q}' \to V_1 V_2 \to l_5 \bar{l}_6 l_7 \bar{l}_8$

$$\mathcal{M}^{V_1 V_2}_{\lambda LL}(p_1, p_2; p_5, p_6, p_7, p_8) = i\,(4\pi\alpha)^2 \sum_j \frac{L^{V_1}_{l_5 l_6} L^{V_2}_{l_7 l_8} Q^{\lambda, V_1 V_2, [j]}_{q\,q'}}{D_{V_1}(p_3) D_{V_2}(p_4)} \; M^{[j]}_{\lambda LL}(p_1, p_2; p_5, p_6, p_7, p_8)$$

where M_{LLL} and M_{RLL} independent, others given by crossing relations. E.g.:

$$\begin{aligned}
M_{LLL}(p_1, p_2; p_5, p_6, p_7, p_8) = [1\,\not{p}_3\,2\rangle \Big\{ & E_1\,\langle 15\rangle\langle 17\rangle[16][18] \\
& + E_2\,\langle 15\rangle\langle 27\rangle[16][28] + E_3\,\langle 25\rangle\langle 17\rangle[26][18] \\
& + E_4\,\langle 25\rangle\langle 27\rangle[26][28] \; + E_5\langle 57\rangle[68] \Big\} \\
& + E_6\,\langle 15\rangle\langle 27\rangle[16][18] + E_7\,\langle 25\rangle\langle 27\rangle[26][18] \\
& + E_8\,\langle 25\rangle\langle 17\rangle[16][18] + E_9\,\langle 25\rangle\langle 27\rangle[16][28]\,,
\end{aligned}$$

Only 9 out of 10 independent form factors relevant for $d = 4$:

$$E_1 = A_1\,, \qquad E_6 = 2\,A_7 + \frac{2\,(u - p_3^2)}{s}\,(A_9 - A_{10})\,,$$

$$E_2 = A_2 + \frac{2}{s}\,(A_9 - A_{10})\,, \qquad E_7 = 2\,A_8 - \frac{2\,(t - p_3^2)}{s}\,(A_9 - A_{10})\,,$$

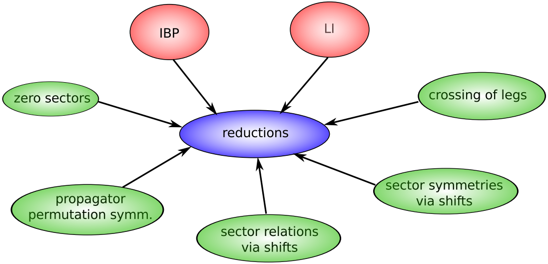
$$E_3 = A_3 - \frac{2}{s}\,(A_9 - A_{10})\,, \qquad E_8 = 2\,A_5 - \frac{2}{s}\left[(u - s - p_3^2)A_9 + (t - p_4^2)A_{10}\right]\,,$$

$$E_4 = A_4\,, \qquad E_9 = 2\,A_6 - \frac{2}{s}\left[(t - s - p_3^2)A_{10} + (u - p_4^2)A_9\right]\,.$$

$$E_5 = 2\,(A_9 + A_{10})\,,$$

Reduze 2 [AvM, C. Studerus]
arXiv:1201.4330, HepForge

based on: Reduze [Studerus '09],
GiNaC [Bauer, Frink, Kreckel '00],
Fermat [Lewis]

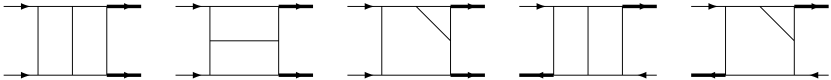


- distributed **Feynman integral reduction**
- advanced **shift finders**
- upcoming version features:
  - **bilinear propagators**
    (3-loop heavy flavour Wilson coefficients in DIS [Blümlein et al. '13-'14])
  - **phase space integrals**
    (soft-virtual N^3LO Higgs and DY [Li, AvM, Schabinger, Zhu '14])
  - **finite integral finder** + **dimension shifts**
    (dims & dots method [AvM, Panzer, Schabinger '14])
  - **family finder**, . . .

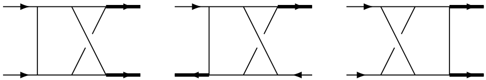
Master integrals for $q\bar{q}' \to VV'$ and $gg \to VV'$

84 master integrals (w/ products, w/o crossings)

planar two-loop master integrals



non-planar master integrals



An improved basis for differential equations

- method by [Kotikov '91]; [Gehrmann, Remiddi '99], relies on IBP reduction
- system of diff. eqns for basis integrals wrt external invariants ($\epsilon = (4-d)/2$):

$$\frac{\partial}{\partial s_i}\vec{M}(\epsilon, s) = \bar{\mathbf{A}}^{(s_i)}(\epsilon, s)\vec{M}(\epsilon, s)$$

- in certain cases proper choice of basis achieves [Kotikov '10]; [Henn '13]:

$$\bar{\mathbf{A}}^{(s_i)}(\epsilon, s) = \epsilon\,\mathbf{A}^{(s_i)}(s)$$

such that

$$\boxed{\mathrm{d}\vec{M}(\epsilon, s) = \epsilon \textstyle\sum_n \mathbf{A}^{(n)}\,\mathrm{d}\ln l_n(s)\ \vec{M}(\epsilon, s)}$$

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features:

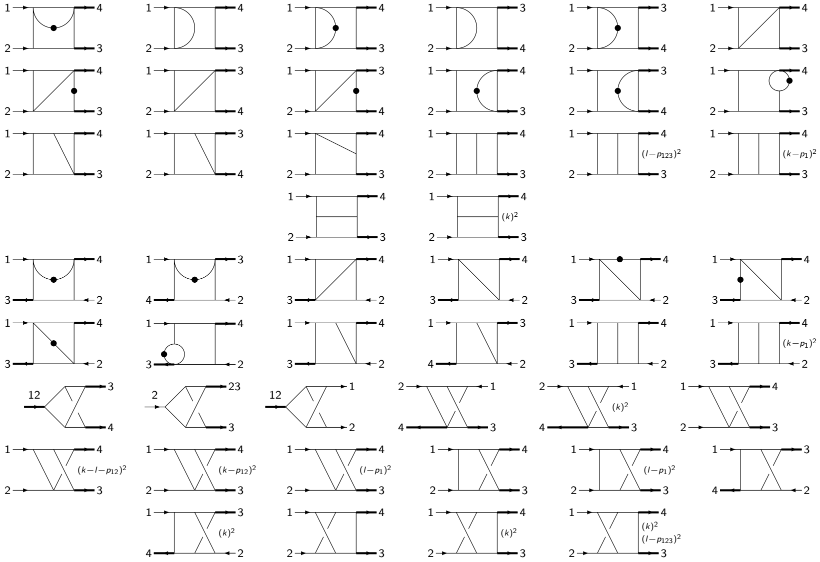
- full decoupling after expansion in ϵ:

$$\vec{M} = \vec{M}^{(0)} + \epsilon \vec{M}^{(1)} + \ldots$$

$$\mathrm{d}\vec{M}^{(k)}(s) = \sum_n \mathbf{A}^{(n)}\, \mathrm{d}\ln l_n(s)\ \vec{M}^{(k-1)}(s)$$

- every term of ϵ expansion: multiple polylogs of uniform weight
- applies to phase space integrals [Höschele, Hoff, Ueda '14]; [AvM, Schabinger, Zhu '14]
- construction of canonical form: [Lee '14], see talk by [Tancredi]
- more applications: see talk by [Henn]

Master integrals for $q\bar{q}' \to VV'$ and $gg \to VV'$



Structure of result

vector of 111 master integrals in canonical basis with alphabet:

$$\{\bar{l}_1, \ldots, \bar{l}_{20}\} = \{2, \bar{x}, 1+\bar{x}, 1-\bar{y}, \bar{y}, 1+\bar{y}, 1-\bar{x}\bar{y}, 1+\bar{x}\bar{y}, 1-\bar{z}, \bar{z},$$
$$1+\bar{y}-2\bar{y}\bar{z}, 1-\bar{y}+2\bar{y}\bar{z}, 1+\bar{x}\bar{y}-2\bar{x}\bar{y}\bar{z}, 1-\bar{x}\bar{y}+2\bar{x}\bar{y}\bar{z},$$
$$1+\bar{y}+\bar{x}\bar{y}+\bar{x}\bar{y}^2-2\bar{y}\bar{z}-2\bar{x}\bar{y}\bar{z}, 1+\bar{y}-\bar{x}\bar{y}-\bar{x}\bar{y}^2-2\bar{y}\bar{z}+2\bar{x}\bar{y}\bar{z},$$
$$1-\bar{y}-\bar{x}\bar{y}+\bar{x}\bar{y}^2+2\bar{y}\bar{z}+2\bar{x}\bar{y}\bar{z}, 1-\bar{y}+\bar{x}\bar{y}-\bar{x}\bar{y}^2+2\bar{y}\bar{z}-2\bar{x}\bar{y}\bar{z},$$
$$1-2\bar{y}-\bar{x}\bar{y}+\bar{y}^2+2\bar{x}\bar{y}^2-\bar{x}\bar{y}^3+4\bar{y}\bar{z}+2\bar{x}\bar{y}\bar{z}+2\bar{x}\bar{y}^3\bar{z},$$
$$1-\bar{y}-2\bar{x}\bar{y}+2\bar{x}\bar{y}^2+\bar{x}^2\bar{y}^2-\bar{x}^2\bar{y}^3+2\bar{y}\bar{z}+4\bar{x}\bar{y}\bar{z}+2\bar{x}^2\bar{y}^3\bar{z}\}$$

in parametrisation which rationalizes root of Källén function $\sqrt{s^2+p_3^4+p_4^4-2(s\,p_3^2+p_3^2\,p_4^2+p_4^2\,s)}$:

$$s=\bar{m}^2(1+\bar{x})^2,\quad t=-\bar{m}^2\bar{x}((1+\bar{y})(1+\bar{x}\bar{y})-2\bar{z}\bar{y}(1+\bar{x})),\quad p_3^2=\bar{m}^2\bar{x}^2(1-\bar{y}^2),\quad p_4^2=\bar{m}^2(1-\bar{x}^2\bar{y}^2)$$

integrated in terms of:

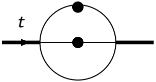
Multiple polylogarithms [Remiddi, Gehrmann]; [Goncharov]

$$G(a_1,a_2,\ldots,a_n;x)=\int_0^x dt\,\frac{dt}{t-a_1}G(a_2,\ldots,a_n;t),$$

- independent input for a couple of very simple bubbles and triangles
- remaining boundary functions fixed by regularity
- checked against `SecDec 2` [Borowka, Carter, Heinrich '12]
- symbol and more [Brown '11], [Duhr '12], [Duhr, Gangl, Rhodes '11], [Vollinga, Weinzierl '04]

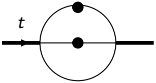
example result: (dots are squared propagators)

in **traditional functional basis**:

$-\epsilon^2 \bar{m}^{2\epsilon} t$ 

$$
\begin{aligned}
&= 1 \\
&\quad + \epsilon\big[-2G(-1,\bar{y}) - 2G(0,\bar{x}) - 2G(-1/\bar{y},\bar{x}) - 2G(((1+\bar{y})(1+\bar{x}\bar{y}))/(2(1+\bar{x})\bar{y}),\bar{z})\big] \\
&\quad + \epsilon^2\big[4G(0,\bar{x})G(((1+\bar{y})(1+\bar{x}\bar{y}))/(2(1+\bar{x})\bar{y}),\bar{z}) + 4G(-1/\bar{y},\bar{x})G(((1+\bar{y})(1+\bar{x}\bar{y}))/(2(1+\bar{x})\bar{y}),\bar{z}) \\
&\qquad + G(-1,\bar{y})(4G(0,\bar{x}) + 4G(-1/\bar{y},\bar{x}) + 4G(((1+\bar{y})(1+\bar{x}\bar{y}))/(2(1+\bar{x})\bar{y}),\bar{z})) + 4G(-1,-1,\bar{y}) \\
&\qquad + 4G(0,0,\bar{x}) + 4G(0,-1/\bar{y},\bar{x}) + 4G(-1/\bar{y},0,\bar{x}) + 4G(-1/\bar{y},-1/\bar{y},\bar{x}) \\
&\qquad + 4G(((1+\bar{y})(1+\bar{x}\bar{y}))/(2(1+\bar{x})\bar{y}),((1+\bar{y})(1+\bar{x}\bar{y}))/(2(1+\bar{x})\bar{y}),\bar{z})\big] \\
&\quad + O(\epsilon^3)
\end{aligned}
$$

in **optimized functional basis** for numerical evaluation:

$-\epsilon^2 m^{2\epsilon} t$ 

$$= 1 + \epsilon\big[-2\ln(l_1) - 2\ln(l_5)\big] + \epsilon^2\big[2\ln^2(l_1) + 4\ln(l_1)\ln(l_5) + 2\ln^2(l_5)\big] + O(\epsilon^3)$$

OPTIMISED FUNCTIONAL BASIS

choose real valued $\ln l_i$, $\mathrm{Li}_n(R_1)$, $\mathrm{Li}_{2,2}(R_1, R_2)$ with

$$|R_1| < 1, \qquad |R_1 R_2| < 1$$

where R_i are power products of letters (e.g. $-l_1, l_3, -l_8/(l_1 l_3), \ldots$)

such that Li functions have convergent power series

$$\mathrm{Li}_n(R_1) = -\sum_{j_1=1}^{\infty} \frac{R_1^{j_1}}{j_1^n}, \qquad \mathrm{Li}_{2,2}(R_1, R_2) = \sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty} \frac{R_1^{j_1}}{(j_1+j_2)^2} \frac{(R_1 R_2)^{j_2}}{j_2^2}$$

features:

- symbol based rewriting [Goncharov, Spradlin, Vergu, Volovich '10]
- algorithmic argument construction [Duhr, Gangl, Rhodes '11]
- require absence of spurious letters
- **fast and stable** numerical evaluation:
  - **O(150ms) full off-shell helicity amplitudes**
  - O(35ms) equal mass interferences
  - orders of magnitude faster than traditional representation

Symbol based integration

- DGR basis: circumvents artifical linearisation of alphabet
- here: new parametrisation $s = m^2(1+x)(1+xy),\ t = -m^2 xz,\ p_3^2 = m^2,\ p_4^2 = m^2 x^2 y$:

$$\begin{aligned}\{l_1,\ldots,l_{17}\} = \{&x, 1+x, y, 1-y, z, 1-z, -y+z, 1+y-z, 1+xy, 1+xz, xy+z,\\ &1+y+xy-z, 1+x+xy-xz, 1+y+2xy-z+x^2yz,\\ &2xy+x^2y+x^2y^2+z-x^2yz, 1+x+y+xy+xy^2-z-xz-xyz,\\ &1+y+xy+y^2+xy^2-z-yz-xyz\}\end{aligned}$$

  very non-linear, but shorter than previous alphabet

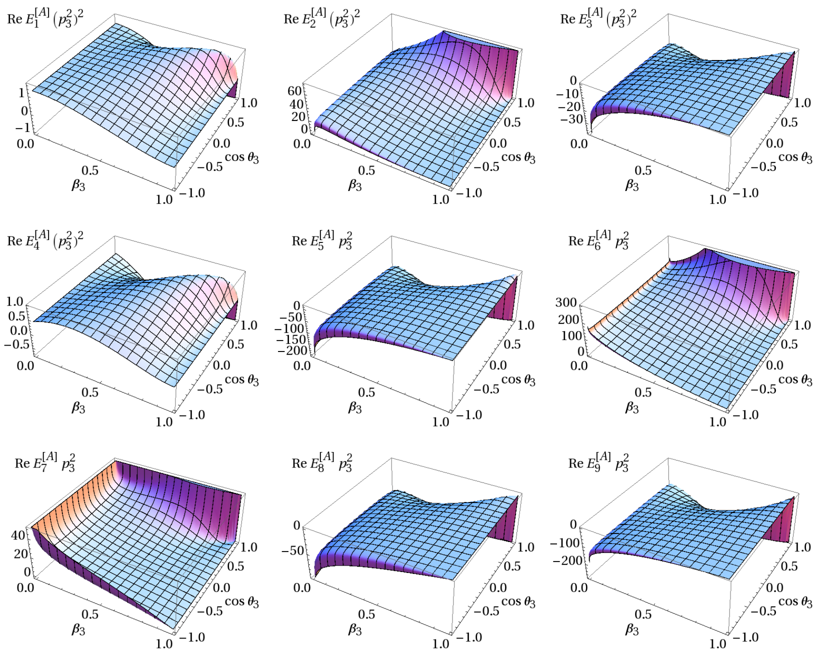
- skip "traditional integration" and "integrate symbol":

$$\mathrm{d}M_k = A^{(n)}\ M_{k-1}\,\mathrm{d}\ln l_n$$

$$\mathcal{S}(M_k) = A^{(n)}\ \mathcal{S}(M_{k-1}) \otimes l_n$$

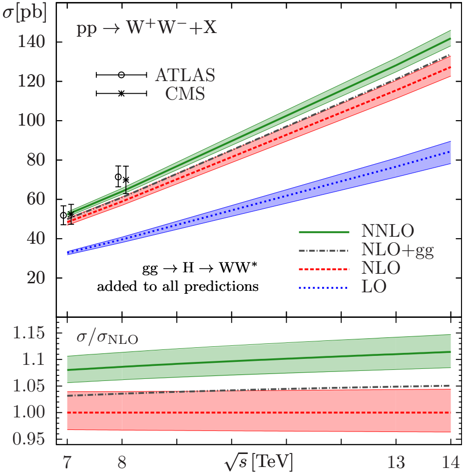
in $N = 4$: [Dixon, Drummond, Duhr, Pennington '14]
in SM Drell-Yan production: [AvM, Schabinger (to appear)]

helicity amplitudes for $q\bar{q}' \to VV'$ @ 2-loops [Gehrmann, AvM, Tancredi '15]



Result: W^+W^- production at NNLO

NNLO prediction significantly reduces tension with data:



[Gehrmann, Grazzini, Kallweit, Maierhöfer, AvM, Pozzorini, Rathlev, Tancredi '14]

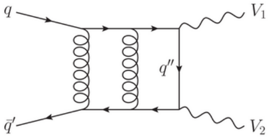
HTTPS://VVAMP.HEPFORGE.ORG

VVamp project

This is the web page of the VVamp project. We provide the two-loop helicity amplitudes for electroweak vector boson pair production and their decay into 4 leptons in quark-antiquark annihilation and in gluon-gluon fusion.

You can download our analytical results for the master integrals and the amplitudes. Moreover, we provide C++ implementations for the fast and reliable numerical evaluation of the amplitudes.

Quark channel



Reference

- Thomas Gehrmann, Andreas von Manteuffel, Lorenzo Tancredi: "*The two-loop helicity amplitudes for qq' -> V1V2 -> 4 leptons*", arXiv:1503.04812

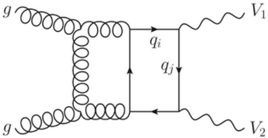
Downloads: amplitudes

- bare form factors exact in d: class A, class B, class C (Form format)
- finite form factors in qt-scheme: class A, class B, class C (Form format)
- relations for projectors: Aj of taui, taui of Aj (Form format)
- numerical implementation of form factors: qqvvamp package (C++, requires GiNaC)

Downloads: master integrals

- master integral definitions: Mathematica, Form format
- master integral traditional solutions: Mathematica, Form format
- master integral optimised solutions: Mathematica, Form format
- master integral crossing relations: Mathematica, Form format
- integral families, kinematics (in Reduze 2 format)

Gluon channel



Reference

- Andreas von Manteuffel, Lorenzo Tancredi: "*The two-loop helicity amplitudes for gg -> V1V2 -> 4 leptons*", arXiv:1503.08835

Downloads: amplitudes

- bare form factors exact in d: class A (Form format)
- finite form factors in qt-scheme: class A (Form format)
- relations for projectors: Pj of Tjdag (Form format)
- numerical implementation of form factors: ggvvamp package (C++, requires GiNaC)

Part II: A basis of finite Feynman integrals

[AvM, Panzer, Schabinger]

An improved basis for Feynman parameters

consider **Feynman parameter representation** of multi-loop integral

$$I = \frac{\Gamma\left(\nu - \frac{Ld}{2}\right)(-1)^{\nu}}{\prod_{i=1}^{N}\Gamma(\nu_i)}\left[\prod_{j=1}^{N}\int_0^{\infty}\mathrm{d}x_j\right]\delta(1-x_N)\,\mathcal{U}^{\nu-(L+1)d/2}\mathcal{F}^{-\nu+Ld/2}\prod_{k=1}^{N}x_k^{\nu_k-1}$$

where $\nu = \sum_i \nu_i$, ν_i denotes propagator multiplicity

presence of **subdivergencies** (= divergencies from Feynman parameter integrations) implies:

- can't directly expand in $\epsilon = (4-d)/2$
- no straight-forward analytical integration a la [Brown '08; Panzer '14]
- no straight-forward numerical integration

generic approaches to **singularity resolution**:

1. sector decomposition [Binoth, Heinrich '00], see talk by [Borowka]
2. polynomial exponent raising [Tkachov '96, Passarino '00]
3. regularising dimension shifts [Panzer '14]
4. New **basis of finite Feynman integrals** [AvM, Schabinger, Panzer '14]

Sector decomposition: shortcomings

calculate to $\mathcal{O}(\epsilon)$:

$$I(\epsilon) = \int_0^1 \mathrm{d}t \; t^{-1-\epsilon}(1-t)^{-1-2\epsilon}{}_2F_1(\epsilon, 1-\epsilon; -\epsilon; t)$$

decompose into sectors: split at (arbitrary) $t = 1/2$:

$$I_1(\epsilon) = \int_0^{1/2} \mathrm{d}t \; t^{-1-\epsilon}(1-t)^{-1-2\epsilon}{}_2F_1(\epsilon, 1-\epsilon; -\epsilon; t)$$

$$I_2(\epsilon) = \int_{1/2}^{1} \mathrm{d}t \; t^{-1-\epsilon}(1-t)^{-1-2\epsilon}{}_2F_1(\epsilon, 1-\epsilon; -\epsilon; t)\,.$$

rescale, expand in plus distributions, evaluate:

$$I_1(\epsilon) = -\frac{1}{\epsilon} - 1 + \left(3 + \frac{1}{3}\pi^2 - 8\ln(2)\right)\epsilon + \mathcal{O}\left(\epsilon^2\right)$$

$$I_2(\epsilon) = -\frac{1}{3\epsilon} + \frac{7}{3} + \left(-7 + \frac{1}{3}\pi^2 + 8\ln(2)\right)\epsilon + \mathcal{O}\left(\epsilon^2\right)\,.$$

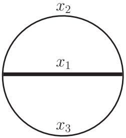
result:

$$I(\epsilon) = -\frac{4}{3\epsilon} + \frac{4}{3} + \left(-4 + \frac{2}{3}\pi^2\right)\epsilon + \mathcal{O}\left(\epsilon^2\right)\,.$$

note:

- split up of domain introduces **spurious terms** $\ln(2)$
- spurious order 5 polynomial denominators: [AvM, Schabinger, Zhu '13]
- destroys linear reducibility & prevents **analytical integration**

An example for subdivergencies



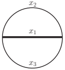
$$= \int \frac{\mathrm{d}^d k_1}{i\pi^{d/2}} \int \frac{\mathrm{d}^d k_2}{i\pi^{d/2}} \frac{1}{((k_1+k_2)^2 - m^2)\, k_1^2 k_2^2}$$

$$= -\Gamma(-1+2\epsilon) \int_0^\infty \mathrm{d}x_1 \delta(1-x_1) \int_0^\infty \mathrm{d}x_2 \int_0^\infty \mathrm{d}x_3\, \mathcal{U}^{-3+3\epsilon} \mathcal{F}^{1-2\epsilon}\,,$$

with Symanzik polynomials

$$\mathcal{U} = x_1x_2 + x_1x_3 + x_2x_3 \quad \text{and} \quad \mathcal{F} = m^2 x_1 \mathcal{U}\,.$$

- **can't expand integrand in ϵ:**



$$= -\left(m^2\right)^{1-2\epsilon} \frac{\Gamma(-1+2\epsilon)\Gamma(\epsilon)\Gamma(1-\epsilon)}{1-\epsilon}$$

$\Gamma(\epsilon)$ signals subdivergence

- **Euclidean** integrals: all divergencies from integration **boundaries**
- notation here: restrict to one or several parameters approaching **zero** (not infinity)

Systematic recognition of subdivergencies

- follow [Panzer '14]
- consider subsets

$$\{x_1, x_2\}, \quad \{x_1, x_3\}, \quad \{x_2, x_3\}, \quad \{x_1\}, \quad \{x_2\}, \quad \{x_3\}$$

- for each subset J consider **scaling with λ**:

$$J \to \lambda J$$

for integrand $P \equiv \mathcal{U}^{-3+3\epsilon}\mathcal{F}^{1-2\epsilon}$:

$$P \to P_{J_\lambda} = \lambda^{\deg_J(P)}\tilde{P} \qquad \text{where } \lim_{\lambda\to 0}\tilde{P} = \mathcal{O}(\lambda^0)$$

and the integral measure

$$\prod_{i=1}^{3} \mathrm{d}x_i \to \lambda^{|J|}\prod_{i=1}^{3}\mathrm{d}x_i$$

and read off:

convergence index

$$\omega_J(P) = |J| + \deg_J(P)\,,$$

$$\lim_{\epsilon\to 0}\omega_J(P) \leq 0 \quad \Leftrightarrow \quad \text{presence of non-integrable subdivergence}$$

Panzer's regularising shift

integrand can be regularized by dimension-shifts [Panzer '14]:

1. pick J for which $\lim_{\epsilon\to 0}\omega_J(P) \leq 0$
2. multiply by $1 = \int_0^\infty \mathrm{d}\lambda\ \delta(\lambda - x_J)$ with $x_J = \sum_{j\in J} x_j$
3. rescale $x_j \to \lambda x_j$ for all $j \in J$ and perform partial integration (surface term vanishes)
4. new integrand

$$P' = -\frac{1}{\omega_J(P)}\frac{\partial}{\partial\lambda}\tilde{P}\bigg|_{\lambda\to 1}.$$

   has improved convergence by design
5. iterate until no subdivergencies

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   $$P' = -\frac{1}{\omega_J(P)}\frac{\partial}{\partial\lambda}\tilde{P}\Big|_{\lambda\to 1}.$$

   has improved convergence by design
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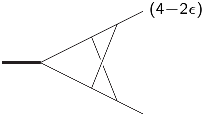
applicability in practice:

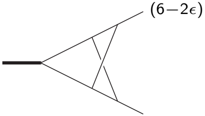
- problem: proliferation of terms
- solution: integration by parts (IBP) reductions

Our proposal: minimal dims & dots

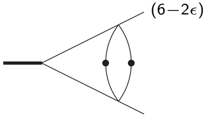
decompose wrt **quasi-finite basis**



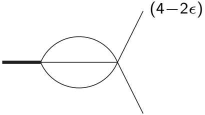
$$= \frac{4(1-\epsilon)(3-4\epsilon)(1-4\epsilon)}{\epsilon s^2}$$



$$- \frac{10 - 65\epsilon + 131\epsilon^2 - 74\epsilon^3}{\epsilon^3 s^2}$$



$$- \frac{14 - 119\epsilon + 355\epsilon^2 - 420\epsilon^3 + 172\epsilon^4}{(1-2\epsilon)\epsilon^3 s^3}$$



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basis consists of standard Feynman integrals, but

- in **shifted dimensions**
- with additional **dots** (propagators taken to higher powers)
- old reg. shifts generated $\mathcal{O}(10\mathrm{MB})$, here: 3 lines ! (more severe at higher loops)

Existence of quasi-finite basis

1. start with some basis B for topology and subtopologies
2. assume master b not quasi-finite and has integrand

$$P = \mathcal{U}^{\nu-(L+1)d/2}\mathcal{F}^{-\nu+Ld/2}\prod_{j=1}^{N} x_j^{\nu_j-1}, \qquad \text{where } \nu = \sum_{i=1}^{N}\nu_i$$

3. consider regularizating dimension shift:

$$P' = -\frac{1}{\omega_J(P)}\prod_{j=1}^{N} x_j^{\nu_j-1}\Bigg\{\Big(\nu-\tfrac{(L+1)d}{2}\Big)\mathcal{U}^{(\nu+L)-(L+1)(d+2)/2}\mathcal{F}^{-(\nu+L)+L(d+2)/2}\frac{\partial\widetilde{\mathcal{U}}}{\partial\lambda}\Big|_{\lambda\to 1}$$
$$+\,\mathcal{F}\text{ derivative term}\Bigg\},$$

   with $\mathcal{U}_{J_\lambda} = \lambda^{\deg_J(\mathcal{U})}\widetilde{\mathcal{U}}$

4. picking any monomial from $\frac{\partial\widetilde{\mathcal{U}}}{\partial\lambda}\big|_{\lambda\to1}$ or $\frac{\partial\widetilde{\mathcal{F}}}{\partial\lambda}\big|_{\lambda\to1}$ gives

   dimension-shifted and **dotted** integral with **improved convergence** !
5. choose one term such that new integral b' is independent of $B\setminus b$
6. replace $b\to b'$ and iterate until B free of subdivergences (quasi-finite)
7. optional: transition quasi-finite $\to$ finite integrals

Practical algorithm for basis construction

given the existence proof, forget about previous construction and just do:

Algorithm: construction of (quasi-)finite basis

- systematic scan for (quasi-)finite integrals with dim-shifts and dots
- IBP + dimensional recurrence for actual basis change

Practical algorithm for basis construction

given the existence proof, forget about previous construction and just do:

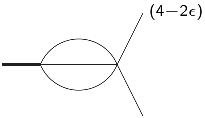
Algorithm: construction of (quasi-)finite basis

- systematic scan for (quasi-)finite integrals with dim-shifts and dots
- IBP + dimensional recurrence for actual basis change

remarks:

- computationally expensive part shifted to IBP solver (`Fire`, `Reduze`, `LiteRed`)
- efficient, easy to automate (implemented in dev. version of `Reduze` 2)
- any dim-shift good, e.g. shifts by [Tarasov '96], [Lee '10]
- see [Bern, Dixon, Kosower '93] for dim-shifted one-loop pentagon

Example 1a: non-planar two-loop vertex (quasi-finite)



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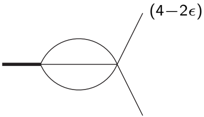

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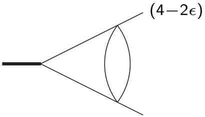
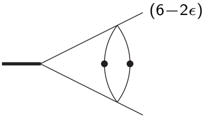


$= \frac{2-3\epsilon}{\epsilon}$



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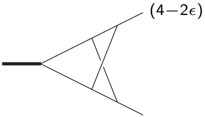
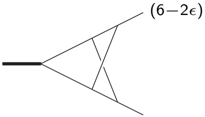
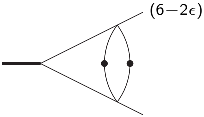


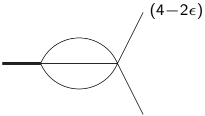
$= \frac{4(1-\epsilon)(3-4\epsilon)(1-4\epsilon)}{\epsilon s^2}$



$- \frac{10-65\epsilon+131\epsilon^2-74\epsilon^3}{\epsilon^3 s^2}$



$- \frac{14-119\epsilon+355\epsilon^2-420\epsilon^3+172\epsilon^4}{(1-2\epsilon)\epsilon^3 s^3}$



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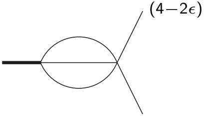
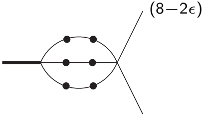
Example 1b: non-planar two-loop vertex (finite)



$$= -\frac{4s^2}{\epsilon(1-2\epsilon)}$$



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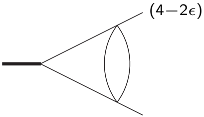
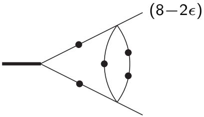


$$= \frac{2(2-3\epsilon)s}{\epsilon^2}$$



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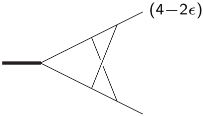
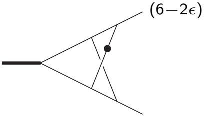
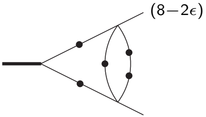


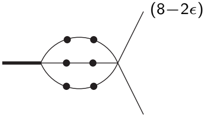
$$= -\frac{4(1-\epsilon)(1-4\epsilon)}{\epsilon s}$$



$$-\frac{2(2-3\epsilon)(5-21\epsilon+14\epsilon^2)}{\epsilon^4 s}$$



$$+\frac{4(2-3\epsilon)(7-31\epsilon+26\epsilon^2)}{\epsilon^4(1-2\epsilon)s}$$



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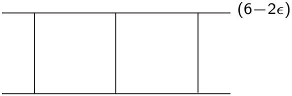
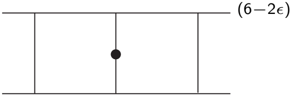
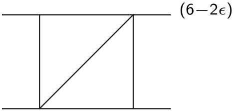
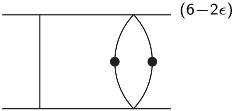
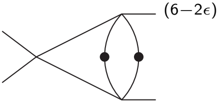
Example 2: massless planar double box family

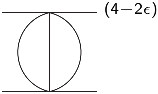
$b_1 =$ 

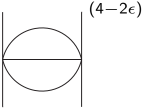
$b_2 =$ 

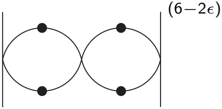
$b_3 =$ 

$b_4 =$ 

$b_5 =$ 

$b_6 =$ 

$b_7 =$ 

$b_8 =$ 

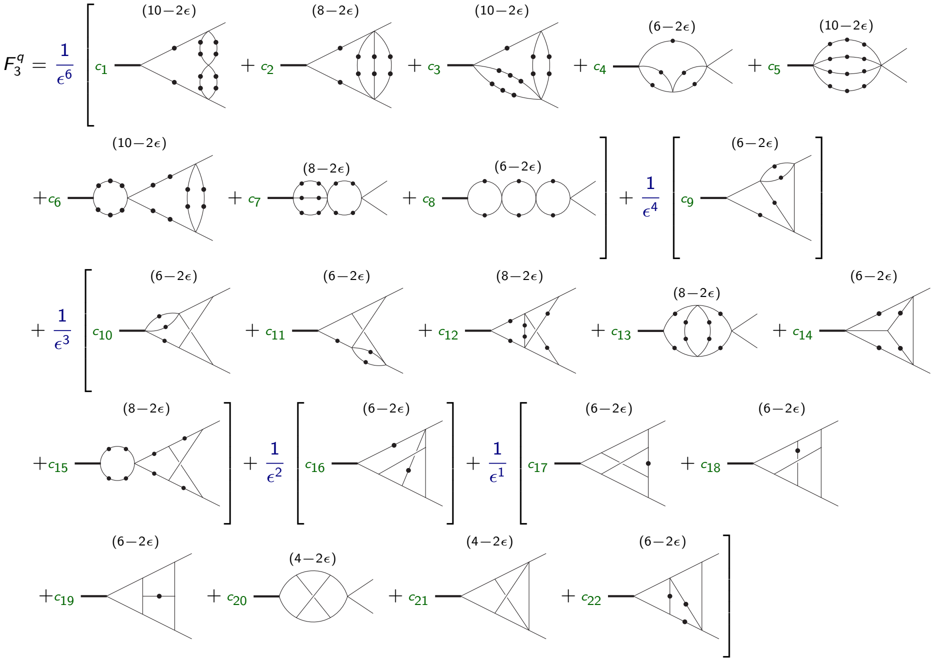
Example 3: Three-loop form factor

- massless quark and gluon form factors:
  - simplest objects to study IR properties of QCD

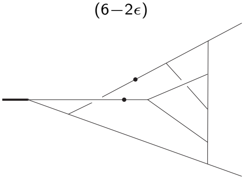
- master integrals:
  - [Gehrmann, Heinrich, Huber, Studerus '06]
  - [Heinrich, Huber, Maître '07]
  - [Heinrich, Huber, Kosower, V. Smirnov '09]
  - [Lee, A. Smirnov, V. Smirnov '10]
  - [Baikov, Chetyrkin, A. Smirnov, V. Smirnov, Steinhauser '09]
  - [Lee, V. Smirnov '10] $\Leftarrow$ the only complete weight 8
  - [Henn, A. Smirnov, V. Smirnov '14] (diff. eqns.)

- form factor @ 3-loops:
  - [Baikov, Chetyrkin, A. Smirnov, V. Smirnov, Steinhauser '09]
  - [Gehrmann, Glover, Huber, Ikizlerli, Studerus '10, '10]

Example 3: Three-loop form factor [AvM, Panzer, Schabinger; to appear]



- example: a non-planar 12-line top level topology @ 4-loops
- analytical result with HypInt [Panzer]:



$$= \frac{18}{5}\zeta_2^2\zeta_3 - 5\zeta_2\zeta_5 + \mathcal{O}(\epsilon) \qquad \approx 3.18074 + \mathcal{O}(\epsilon)$$

- numerical result with Fiesta [A. Smirnov]: $3.18082 + \epsilon\, 58.8288 + \mathcal{O}(\epsilon^2)$

Numerical evaluations

advantages of (quasi-)finite basis:

- straight-forward to integrate numerically (in principle)
- no blow up in number of numerical integrations (speed, stability)
- no cancellation of spurious structures (stability)

experiments with numerical evaluations:

- naive straight-forward implementation works already quite well
- convenient: employ existing sector decomposition programs `Fiesta`, `SecDec` and `sector_decomposition`
- (quasi-)finite integrals: faster & more reliable

Conclusions

differential equations:

- powerful analytical method for multiscale integrals in QCD
- refinement via normal form basis (if applicable)
- essential: systematic treatment of multipole polylogs (symbol etc)
- optimize functional basis for result
- NNLO prediction for diboson production at LHC

basis of finite integrals:

- simple and efficient method for singularity resolution in multi-loop integrals
- analytical integrations: quasi-finite integrals are Feynman integrals (dim-shifted, dotted)
- numerical integrations: faster and more stable evaluations
- application: massless QCD form factors