Multiple polylogarithms and Feynman integrals

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Topics

1. hyperlogarithms & iterated integrals
2. multiple polylogarithms
3. functional equations, symbols
4. parameter integrals
5. other stuff
6. coaction (Brown, Goncharov)

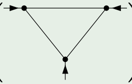
Some FI/amplitudes are expressible via multiple polylogarithms (MPL)

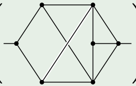
$$\mathrm{Li}_{n_1,\ldots,n_d}(z_1,\ldots,z_d) = \sum_{0<k_1<\cdots<k_d} \frac{z_1^{k_1}\cdots z_d^{k_d}}{k_1^{n_1}\cdots k_d^{n_d}}, \quad |z_i\cdots z_d| < 1$$

and their special values, e.g. multiple zeta values (MZV)

$$\zeta_{n_1,\ldots,n_d} = \mathrm{Li}_{n_1,\ldots,n_d}(1,\ldots,1).$$

Example ($p_1^2 = 1,\ p_2^2 = |z|^2,\ p_3^2 = |1-z|^2$)

$\Phi\Big($  $\Big) = \dfrac{4i\,\mathrm{Im}\left[\mathrm{Li}_2(z) + \log(1-z)\log|z|\right]}{z-\bar{z}}$

$\Phi\Big($  $\Big) = 252\zeta_3\zeta_5 + \frac{432}{5}\zeta_{3,5} - \frac{25056}{875}\zeta_2^4$

"expressible" means each term of the ε-expansion is a rational/algebraic linear combination of MPL whose arguments are rational/algebraic functions

iterated integrals (Chen 1973)

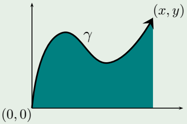
Take a manifold X and differential forms $\omega_1, \ldots, \omega_n \in \Omega^1(X)$. Integrating these along a path $\gamma \in C^1([0,1], X)$, we can construct functions (on γ):

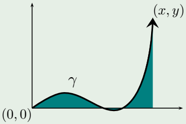
$$\int_\gamma \omega_n \cdots \omega_1 := \int_0^1 \gamma^*(\omega_n)(t_n) \int_0^{t_n} \cdots \int_0^{t_2} \gamma^*(\omega_1)(t_1)$$

1. If $\omega = \mathrm{d}f$ is exact, $\int_\gamma \omega = f(\gamma(1)) - f(\gamma(0))$ is boring.
2. Not all iterated integrals are homotopy invariant.

Example

Take $\omega = y\mathrm{d}x \in \Omega^1(\mathbb{R}^2)$, then $\int_\gamma \omega$ is the area between γ and the x-axis.





$\Rightarrow$ integrability condition (Chen), simplest case:

$$\int_\gamma \omega \quad \text{homotopy invariant} \Leftrightarrow \mathrm{d}\omega = 0$$

Hyperlogarithms (Poincaré 1884, Lappo-Danilevsky 1927)

Let $X = \mathbb{C} \setminus \Sigma$ for a finite set of points Σ. The regular, non-exact forms

$$\omega_\sigma = \mathrm{d}\log(z-\sigma) = \frac{\mathrm{d}z}{z-\sigma}$$

generate homotopy invariant iterated integrals, called hyperlogarithms.

Examples

$\int_\gamma \omega_0 = \log \frac{\gamma(1)}{\gamma(0)}$, $\int_0^z \omega_1 = -\operatorname{Li}_1(z)$, $\int_0^z \omega_0\omega_1 = -\operatorname{Li}_2(z)$

All MPL can be written this way:

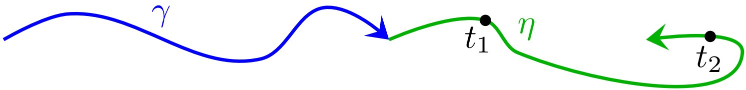
$$\int_0^z \omega_0^{n_d-1}\omega_{\sigma_d}\cdots\omega_0^{n_1-1}\omega_{\sigma_1} = (-1)^d \operatorname{Li}_{n_1,\ldots,n_d}\left(\frac{\sigma_2}{\sigma_1},\cdots,\frac{\sigma_d}{\sigma_{d-1}},\frac{z}{\sigma_d}\right)$$

Notation and some special cases:

- $\int_0^z \omega_{\sigma_n}\cdots\omega_{\sigma_1} = I(0;\sigma_1,\ldots,\sigma_n;z) = G(\sigma_n,\ldots,\sigma_1;z)$ [Goncharov]
- $\Sigma = \{-1,0,1\}$ harmonic polylogarithms (HPL) [Remiddi & Vermaseren]
- $\Sigma = \{0,1,1-y,-y\}$ 2 dimensional HPL [Gehrmann & Remiddi]

Path concatenation

Let $\gamma \star \eta$ denote the concatenation of γ and η at $\gamma(1) = \eta(0) = (\gamma \star \eta)(\frac{1}{2})$:



To decompose

$$\int_{\gamma\star\eta} \omega_2\omega_1 = \int_\gamma \omega_2\omega_1 + \int_\eta \omega_2 \int_\gamma \omega_1 + \int_\eta \omega_2\omega_1,$$

split the interval

$$\underbrace{\{t_1 \le t_2\}}_{\int_{\gamma\star\eta}\omega_2\omega_1} = \underbrace{\{t_1 \le t_2 \le \tfrac{1}{2}\}}_{\int_\gamma \omega_2\omega_1} \cup \underbrace{\{t_1 \le \tfrac{1}{2} \le t_2\}}_{\int_\eta \omega_2 \int_\gamma \omega_1} \cup \underbrace{\{\tfrac{1}{2} \le t_1 \le t_2\}}_{\int_\eta \omega_2\omega_1}$$

More generally, the path concatenation formula reads

$$\int_{\gamma\star\eta} \omega_n \cdots \omega_1 = \sum_{k=0}^{n} \int_\eta \omega_n \cdots \omega_{k+1} \int_\gamma \omega_k \cdots \omega_1.$$

Path concatenation

$G(\vec{\sigma}; z)$ is analytic in $z = \gamma(1)$ on $\mathbb{C} \setminus \Sigma$, but multivalued.

Remember

Even though γ is suppressed in the notation $G(\vec{\sigma}; z)$ and $\int_0^z w$, these functions still depend on the homotopy class of γ.

If η is a closed loop with $\eta(0) = \eta(1) = 0$, analytic continuation $\mathcal{M}_\eta$ gives

$$\mathcal{M}_\eta \int_0^z \omega_n \cdots \omega_1 = \sum_{k=0}^{n} \int_0^z \omega_n \cdots \omega_{k+1} \int_\eta \omega_k \cdots \omega_1.$$

Note: This only adds lower weight functions (corresponding to prefixes).

Similar: change of basepoint

$$\int_b^z \omega_n \cdots \omega_1 = \sum_{k=0}^{n} \int_0^z \omega_n \cdots \omega_{k+1} \int_b^0 \omega_k \cdots \omega_1$$

Shuffle product

The shuffle product of two words

$$w_{n+m}\cdots w_{n+1} \sqcup\!\sqcup\, w_n \cdots w_1 = \sum_{\sigma} w_{\sigma(n+m)}\cdots w_{\sigma(1)}$$

is the sum of all their shuffles σ, i.e. permutations which preserve the relative order of letters in both factors:

$$\sigma^{-1}(1) < \cdots < \sigma^{-1}(n) \quad \text{and} \quad \sigma^{-1}(n+1) < \cdots < \sigma^{-1}(n+m).$$

For arbitrary words u and v, we find that ($\int_\gamma$ is linearly extended)

$$\left(\int_\gamma u\right)\cdot\left(\int_\gamma v\right) = \int_\gamma (u \sqcup\!\sqcup\, v).$$

Example

$$\int_\gamma \omega_3 \cdot \int_\gamma \omega_2\omega_1 = \int_\gamma (\omega_3\omega_2\omega_1 + \omega_2\omega_3\omega_1 + \omega_2\omega_1\omega_3)$$

$$\{t_3\}\times\{t_1 \le t_2\} = \{t_1 \le t_2 \le t_3\} \cup \{t_1 \le t_3 \le t_2\} \cup \{t_3 \le t_1 \le t_2\}$$

Singularities

Singularities when $\{\gamma(0), \gamma(1)\} \ni z \to \tau \in \{\infty\} \cup \Sigma$ are logarithmic: There exist (uniquely determined) functions $f_{k,w}(z)$, analytic at $z = \tau$, such that

$$\int_\gamma w = \sum_k \log^k(z-\tau) f_{k,w}(z).$$

Definition (logarithmic regularization)

The regularized limit is $\operatorname{Reg}_{z\to\tau} \int_\gamma w := f_{0,w}(\tau)$.

Example

$$\int_\gamma \omega_0 = \log \frac{z}{\gamma(0)} \qquad G(0;z) = \int_0^z \omega_0 := \operatorname*{Reg}_{\gamma(0)\to 0} \int_\gamma \omega_0 = \log(z)$$

These expansions and limits can be computed algorithmically using shuffles and rescalings like $G(\lambda\vec{\sigma}, \lambda z) = G(\vec{\sigma}; z)$ $(\sigma_1 \neq 0)$.

Differentials

By definition, $\partial_z G(\sigma_n, \vec{\sigma}; z) = \frac{1}{z-\sigma_n} G(\vec{\sigma}, z)$. But now consider the σ as variables themselves. Differentiating under the integral sign in

$$G(\sigma_n, \ldots, \sigma_1; z) = \int_0^z \frac{\mathrm{d}\, t_n}{t_n - \sigma_n} \int_0^{t_n} \frac{\mathrm{d}\, t_{n-1}}{t_{n-1} - \sigma_{n-1}} \cdots \int_0^{t_2} \frac{\mathrm{d}\, t_1}{t_1 - \sigma_1}$$

one finds

$$\mathrm{d}\, G(\vec{\sigma}; z) = \sum_{i=1}^{n} G(\cdots, \not{\sigma}_i, \cdots; z)\, \mathrm{d} \log \frac{\sigma_i - \sigma_{i-1}}{\sigma_i - \sigma_{i+1}} \qquad \sigma_0 := z, \sigma_{n+1} := 0$$

Equivalently, in the sum representation of MPL we see

$$\mathrm{d}\, \mathrm{Li}_{\vec{n}}(\vec{z}) = \sum_k \mathrm{Li}_{\vec{n}-\vec{e}_i}(\vec{z}) \frac{\mathrm{d} z_k}{z_k}$$

and can replace any occurring zeros (resulting from $n_k = 1$ say) by

$$\mathrm{Li}_{\vec{n}'}(\ldots, z_k z_{k-1}, \ldots) \frac{\mathrm{d} z_k}{1 - z_k} - \mathrm{Li}_{\vec{n}'}(\ldots, z_k z_{k+1}, \ldots) \frac{\mathrm{d} z_k}{z_k(1 - z_k)}.$$

Functional equations

Remember

Proving functional relations between MPL is easy - just differentiate!

Question

Can we write $\text{Li}_2(1-x)$ as a hyperlogarithm with argument x?

1. Take a derivative:

$$\partial_x \text{Li}_2(1-x) = \frac{1}{x-1} \text{Li}_1(1-x)$$

2. Express all MPL in the integrand as hyperlogs (recursion):

$$\text{Li}_1(1-x) = -\log(x) = -\text{G}(0;x)$$

3. integrate back:

$$\text{Li}_2(1-x) = C - \text{G}(1,0;x) = C - \log(1-x)\log(x) - \text{Li}_2(x)$$

4. Fix constant $C = \frac{\pi^2}{6} = \zeta_2$ via some limit, for example $x \to 1$

Products of hyperlogarithms basis

The representation in terms of hyperlogarithms is free of relations:

Lemma

All hyperlogarithms $G(\vec{\sigma}; z)$ are linearly independent functions of z (even with algebraic coefficients).

The recursive algorithm (differentiation & integration & limits) solves

Problem

Given some MPL $G(\vec{\sigma}(\vec{x}), z(\vec{x}))$ or $\mathrm{Li}_{\vec{n}}(\vec{z}(\vec{x}))$ whose arguments ($\vec{\sigma}, z$ or $\vec{z}$) are rational functions of variables $x_1, \ldots, x_n$, write it in the basis

$$\sum_{\vec{\sigma}_1,\ldots,\vec{\sigma}_n} G(\vec{\sigma}_1(x_2,\ldots,x_n); x_1) G(\vec{\sigma}_2(x_3,\ldots,x_n); x_2) \cdots G(\vec{\sigma}_n; x_n).$$

- completely algebraic (no numerics), programmed (`HyperInt`)
- in general not the shortest or "simplest" representation
- dependence on order of the variables $x_1, \ldots, x_n$

Higher dimensions: multiple polylogarithms

Idea: treat all variables equally; via higher dimensional iterated integrals

Example

$$\mathrm{d}\,\mathrm{Li}_{1,1}(z_1,z_2) = \frac{\mathrm{d}z_1}{1-z_1}\mathrm{Li}_1(z_2) - \frac{\mathrm{d}z_1}{z_1(1-z_1)}\mathrm{Li}_1(z_1z_2) + \frac{\mathrm{d}z_2}{1-z_2}\mathrm{Li}_1(z_1z_2)$$

$$= \frac{\mathrm{d}z_1}{1-z_1}\int_0^{z_2}\frac{\mathrm{d}z_2}{1-z_2} - \left(\frac{\mathrm{d}z_2}{1-z_2} - \frac{\mathrm{d}z_1}{z_1} - \frac{\mathrm{d}z_1}{1-z_1}\right)\int_0^{(z_1,z_2)}\mathrm{d}\log(1-z_1z_2)$$

$$= \int_{(0,0)}^{(z_1,z_2)}\left[\frac{\mathrm{d}z_1}{1-z_1}\frac{\mathrm{d}z_2}{1-z_2} + \left(\frac{\mathrm{d}z_1}{z_1} + \frac{\mathrm{d}z_1}{1-z_1} - \frac{\mathrm{d}z_2}{1-z_2}\right)\frac{z_1\mathrm{d}z_2 + z_2\mathrm{d}z_1}{1-z_1z_2}\right]$$

Corollary (MPL as iterated integrals)

Every $\mathrm{Li}_{\vec{n}}(\vec{z})$ of d arguments is an iterated integral on $\mathbb{C}^d \setminus \mathbf{V}\left(S_d^{MPL}\right)$ for the singularities (alphabet)

$$S_d^{MPL} = \{z_1,\ldots,z_d\} \cup \{1 - z_i \cdots z_j \colon\ 1 \leq i \leq j \leq d\}$$

[Brown & Bogner]

Higher dimensions & simplification

Every subset $S \subset \mathbb{Q}[z_1, \ldots, z_d]$ of polynomials determines a manifold

$$X = \mathbb{C}^d \setminus \mathbf{V}(S) = \mathbb{C}^d \setminus \bigcup_{f \in S} \{f = 0\}$$

which defines a space of iterated integrals. We consider homotopy invariant (Chen condition on w!) integrals $\int_\gamma w$ of forms $\mathrm{d}\log(f)$, $f \in S$.

Theorem (Kummer 1840)

Every iterated integral of at most three rational forms can be expressed as a sum of products of $\log, \mathrm{Li}_2$ *and* Li_3*.*

Conjecture (Duhr, Gangl & Rhodes)

To express every MPL up to weight 4*, it suffices to add the functions* $\mathrm{Li}_4, \mathrm{Li}_{2,2}$*. At weight* 5 *it suffices to add* Li_5 *and* $\mathrm{Li}_{2,3}$*.*

Method: basis Ansatz, fit coefficients via symbol (differentiation) or motivic coaction

Basepoints

Remember

The symbol w determines the function $\int_b^z w$ completely if one fixes a choice b for the basepoint.

- when b is forgotten, then the symbol w only captures the highest weight
- with b fixed, $w \mapsto \int_b^z w$ is injective
- the motivic coaction is not necessary
- different basepoints can make expressions simpler
- physics can sometimes hint a good basepoint

Programs

Symbol techniques are standard now, many people have programs which can do at least some of the work in an automatized way. Ask around; public: `HyperInt`, ...?... yours here!

Motivic coaction

Idea: abstract from the integral/function and define algebraic objects (motivic periods) which have more structure. The construction is complicated. [Brown,Dupont,Goncharov,...]

Hint

In recent lectures at the IHES (available online), Brown constructed motivic Feynman periods in very general cases. Thus, a coaction is defined under very mild assumptions (in particular not restricted to polylogarithmic integrals).

Upshot: There exist algebraically/geometrically defined objects $I_G^{\mathfrak{m}}(\vec{s})$ for a wide class of Feynman graphs G and kinematics $\vec{s}$. These are elements of some algebra $\mathcal{H}^{\mathfrak{m}}$ with a period map

$$\text{per}\colon \mathcal{H}^{\mathfrak{m}} \longrightarrow \mathbb{C}$$

sending $I_G^{\mathfrak{m}}(\vec{s})$ to the actual Feynman integral (number) $I_G(\vec{s})$.

However, there is more structure: A coaction

$$\Delta\colon \mathcal{H}^{\mathfrak{m}} \longrightarrow \mathcal{H}^{\mathfrak{dr}} \otimes \mathcal{H}^{\mathfrak{m}}$$

which is very powerful to obtain relations, in particular for multiple polylogarithms (where differentiation does not buy us anything!).

Example

$$\Delta\zeta^{\mathfrak{m}}_{2,3} = 1 \otimes \zeta^{\mathfrak{m}}_{2,3} + \zeta^{\mathfrak{dr}}_{2,3} \otimes 1 + 2\zeta^{\mathfrak{dr}}_{3} \otimes \zeta^{\mathfrak{m}}_{2} + \zeta^{\mathfrak{dr}}_{2} \otimes \zeta^{\mathfrak{m}}_{3}$$

Note $\zeta^{\mathfrak{dr}}_2 = 0$ and that $\{\zeta^{\mathfrak{m}}_5, \zeta^{\mathfrak{m}}_2\zeta^{\mathfrak{m}}_3\}$ is a basis of motivic MZV in weight 5, therefore we conclude $\zeta^{\mathfrak{m}}_{2,3} = 2\zeta^{\mathfrak{m}}_2\zeta^{\mathfrak{m}}_3 + c\zeta^{\mathfrak{m}}_5$, with c undetermined.

An explicit formula for the coproduct/coaction for MPL is due to Goncharov/Brown.

Caution

If people speak about a coproduct $\Delta \longrightarrow \mathcal{H}^{\mathfrak{dr}} \otimes \mathcal{H}^{\mathfrak{dr}}$, they necessarily work with de Rham periods, which have no associated real period ($\zeta^{\mathfrak{dr}}_2 = 0$). For multiple polylogarithms, this means that multiples of $i\pi$ are dropped.

Motivic coaction

Caution

The similarity of both sides, $\mathcal{H}^{\mathfrak{dr}}$ and $\mathcal{H}^{\mathfrak{m}}$ in the case of MPL is rather misleading. In fact, by construction both sides are very different.

In the case of Feynman integrals, the $\mathfrak{m}$-side corresponds to Feynman integrals again, whereas the $\mathfrak{dr}$-side, at least empirically, relates to cuts of integrals, i.e. residues. [talk by Abreu]

Further points

- elliptic MZV and integrals [talk by Bogner,Schlotterer]
- numeric evaluation of MPL, e.g. Ginac, but also optimizations [Manteuffel, Tancredi]
- algebraic functions in the forms (root-valued letters)
- single-valued MPL
- cluster polylogarithms
- special values (number theory): relations between MZV, MPL at roots of unity,. . .
- . . .

Feynman integrals $\longrightarrow$ polylogarithms

1. differential equations in normal form [talk by Henn,von Manteuffel]

$$\mathrm{d}\vec{f}(\varepsilon,\vec{x}) = \varepsilon\left(\sum_{s\in S}\mathrm{d}\log(s)\mathbf{A}_s\right)\vec{f}(\varepsilon,\vec{x})$$

2. summation: extremely powerful computer algebra tools for sums [Ablinger, Raab, Schneider, Weinzierl. . .]
3. bootstrap, Ansätze, recursions [talk by Drummond, von Hippel]
4. integration of parameter integrals
5. . . .

Definite parameter integrals

They are everywhere [talk by Borowka]:

- hypergeometric functions ${}_pF_q(\cdots;z)$

$$ {}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\middle|z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\mathrm{d}t $$

- Appell's functions F_1, F_2, F_3, F_4

$$ F_3\left(\begin{matrix}\alpha,\alpha'\\\beta,\beta'\end{matrix}\middle|\gamma\middle|x\ y\right) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')} $$
$$ \times\int_0^1\int_0^{1-v} u^{\beta-1}v^{\beta'-1}(1-u-v)^{\gamma-\beta-\beta'-1}(1-ux)^{-\alpha}(1-vy)^{-\alpha'}\mathrm{d}u\mathrm{d}v $$

- Feynman integrals in Schwinger parameters
- Phase-space integrals [talk by Dulat]
- ...

Idea: If the representation is simple enough (linearly reducible), there is an order of integration for the parameters such that each intermediate partial integral is a MPL of rational arguments.

Schwinger parameters

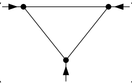
With the *superficial degree of divergence* $\mathrm{sdd} = |E(G)| - D/2 \cdot \mathrm{loops}(G)$,

$$\Phi(G) = \frac{\Gamma(\mathrm{sdd})}{\prod_e \Gamma(a_e)} \int_{(0,\infty)^E} \frac{1}{\psi^{D/2}} \left(\frac{\psi}{\varphi}\right)^{\mathrm{sdd}} \delta(1-\alpha_N) \prod_e \alpha_e^{a_e - 1} \mathrm{d}\alpha_e$$

Graph polynomials:

$$\mathcal{U} = \sum_T \prod_{e \notin T} \alpha_e \qquad \mathcal{F} = \sum_{F = T_1 \dot\cup T_2} q^2(T_1) \prod_{e \notin F} \alpha_e + \mathcal{U} \sum_e m_e^2 \alpha_e$$

Example: massless triangle

$\Phi\Big($  $\Big)$

$$= \int \frac{\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3}{(1+\alpha_2+\alpha_3)(\alpha_2\alpha_3 + z\bar{z}\alpha_3 + (1-z)(1-\bar{z})\alpha_2)}$$

$$= \frac{1}{z-\bar{z}} \int \left(\frac{\mathrm{d}\alpha_2}{\alpha_2+\bar{z}} - \frac{\mathrm{d}\alpha_2}{\alpha_2+z} \right) \log \frac{(\alpha_2+1)(\alpha_2+z\bar{z})}{(1-z)(1-\bar{z})\alpha_2}$$

Singularities of the original integrand: $S = \{\psi, \varphi\}$, i.e. at $\alpha_3 = \sigma_i$ for

$$\sigma_1 = -1-\alpha_2 \qquad \text{and} \qquad \sigma_2 = -\frac{\alpha_2(1-z)(1-\bar{z})}{\alpha_2+z\bar{z}}$$

After integrating α_1 from 0 to ∞, the integrand has singularities

$$S_3 = \{ \underbrace{1+\alpha_2}_{\sigma_1=0}, \underbrace{\alpha_2, 1-z, 1-\bar{z}}_{\sigma_2=0}, \underbrace{\alpha_2+z\bar{z}}_{\sigma_2=\infty}, \underbrace{z+\alpha_2, \bar{z}+\alpha_2}_{\sigma_1=\sigma_2 \ \text{(pinch)}} \}$$

With the same logic, predict the possible singularities after $\int_0^\infty \mathrm{d}\alpha_2$:

$$S_{3,2} = \{z, \bar{z}, 1-z, 1-\bar{z}, z-\bar{z}, z\bar{z}-1\}$$

Polynomial reduction [F. Brown]

Definition

Let S denote a set of polynomials, then S_e are the irreducible factors of

$$\left\{\text{lead}_e(f),\, f|_{\alpha_e=0},\, D_e(f)\colon\; f \in S\right\} \quad \text{and} \quad \{[f,g]_e\colon\; f,g \in S\}\,.$$

Lemma

If the singularities of F are cointained in S, then the singularities of $\int_0^\infty F \mathrm{d}\alpha_e$ are contained in S_e.

Improvements

- Fubini: intersect over different orders
- Compatibility graphs

HyperInt: massive triangle

Graph polynomials:

```
> E:=[[2,3],[1,3],[1,2]]:
> M:=[[1,s1],[2,s2],[3,s3]],[m1^2,m2^2,m3^3]:
> psi:=graphPolynomial(E):
> phi:=secondPolynomial(E,M):
```

Polynomial reduction:

```
> L[{}]:=[{psi,phi}, {{psi,phi}}]:
> cgReduction(L, {s1,s2,s3,m1,m2,m3}, 2):
> L[{x[1],x[2]}][1];
```

$$\left\{ s_i + (m_j \pm m_k)^2, \sum_i s_i^2 - \sum_{i \neq j} s_i s_j \right\}$$

$$\cup \left\{ s_1 s_2 s_3 - \sum_i s_i^2 m_i^2 + \sum_i s_i (m_i^2 - m_j^2)(m_i^2 - m_k^2) + \sum_{i<j} s_i s_j (m_i^2 + m_j^2) \right\}$$

Linear reducibility

Definition

If for some order of variables (edges), all $S_{1,\dots,k}$ are linear in α_{k+1}, then S (the Feynman graph G with $S = \{\psi, \varphi\}$) is called linearly reducible.

Lemma

If S is linearly reducible, the integral $\prod_e \int_0^\infty \mathrm{d}\alpha_e \, f$ of any rational function f with singularities in S is a MPL with symbol letters in $S_{1,\dots,N}$.

Integration of linearly reducible integrands in terms of MPL can be automatized via hyperlogarithms.

HyperInt

- open source Maple program
- integration of hyperlogarithms
- transformations of MPL to $G(\cdots; z)$
  Note: No further simplification (e.g. rewrite as $\mathrm{Li}_{2,2}$) provided!
- polynomial reduction
- graph polynomials
- symbolic computation of constants (no numerics)

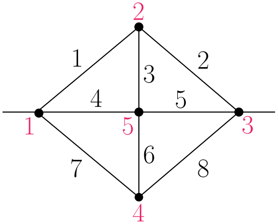
Example

```
> read "HyperInt.mpl":
> hyperInt(polylog(2,-x)*polylog(3,-1/x)/x,x=0..infinity):
> fibrationBasis(%);
```

$$\frac{8}{7}\zeta_2^3$$

computes $\int_0^\infty \mathrm{Li}_2(-x)\,\mathrm{Li}_3(-1/x)\mathrm{d}x = \frac{8}{7}\zeta_2^3$.

HyperInt: propagator




```
> E := [[2,1],[2,3],[2,5],[5,1],[5,3],[5,4],[4,1],[4,3]]:
> psi := graphPolynomial(E):
> phi := secondPolynomial(E, [[1,1], [3,1]]):
> add((epsilon*log(psi^5/phi^4))^n/n!,n=0..2)/psi^2:
> hyperInt(eval(%,x[8]=1), [seq(x[n],n=1..7)]):
> collect(fibrationBasis(%), epsilon);
```

$$\left(254\zeta_7 + 780\zeta_5 - 200\zeta_2\zeta_5 - 196\zeta_3^2 + 80\zeta_2^3 - \frac{168}{5}\zeta_2^2\zeta_3\right)\varepsilon^2 + \left(-28\zeta_3^2 + 140\zeta_5 + \frac{80}{7}\zeta_2^3\right)\varepsilon + 20\zeta_5$$

HyperInt: triangle

Graph polynomials:

```
> E:=[[1,2],[2,3],[3,1]]:
> M:=[[3,1],[1,z*zz],[2,(1-z)*(1-zz)]]:
> psi:=graphPolynomial(E):
> phi:=secondPolynomial(E,M):
```

Integration:

```
> hyperInt(eval(1/psi/phi,x[3]=1),[x[1],x[2]]):
> factor(fibrationBasis(%,[z,zz]));
```

$$(\mathrm{G}(z;1)\,\mathrm{G}(zz;0) - \mathrm{G}(z;0)\,\mathrm{G}(zz;1) + \mathrm{G}(zz;0,1) - \mathrm{G}(zz;1,0) + \mathrm{G}(z;1,0) - \mathrm{G}(z;0,1))/(z-zz)$$

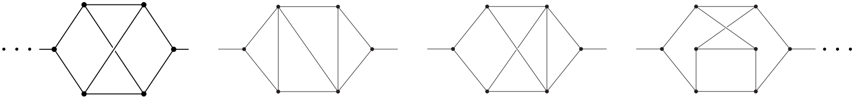
Polynomial reduction:

```
> L[{}]:=[{psi,phi}, {{psi,phi}}]:
> cgReduction(L):
> L[{x[1],x[2]}][1];
```

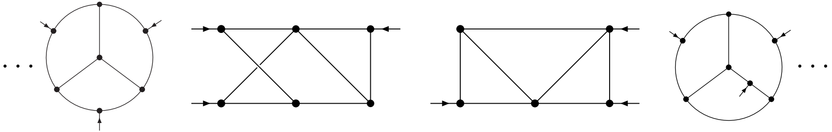
$$\{-1+z, -1+zz, -zz+z\}$$

Linearly reducible families (fixed loop order)

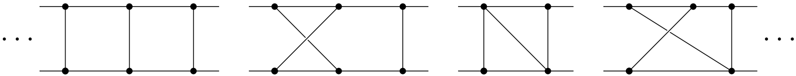
1. all ≤ 4 loop massless propagators (Panzer)



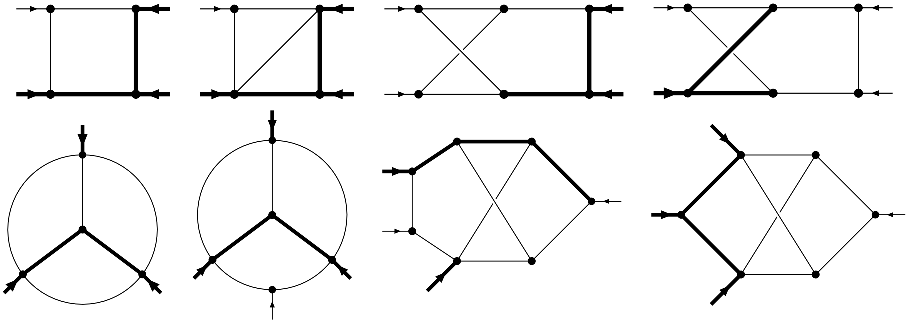
2. all ≤ 3 loop massless off-shell 3-point (Chavez & Duhr, Panzer)
   also in position space; give conformal 4-point integrals



3. all ≤ 2 loop massless on-shell 4-point (Lüders)

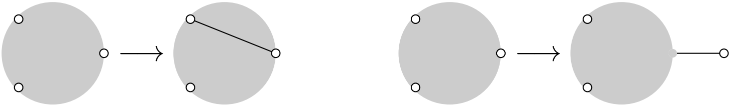


Linearly reducible massive graphs (some examples)



Linearly reducible families (infinite)

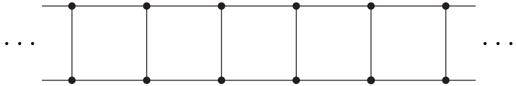
- 3-constructible graphs (3-point functions) [Brown, Schnetz, Panzer]



Theorem (Panzer)

All ε-coefficients of these graphs (off-shell) are MPL over the alphabet $\{z, \bar{z}, 1-z, 1-\bar{z}, z-\bar{z}, 1-z\bar{z}, 1-z-\bar{z}, z\bar{z}-z-\bar{z}\}$.

- minors of ladder-boxes (up to 2 legs off-shell)

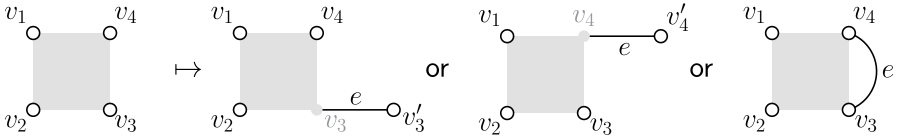


Theorem (Panzer)

All ε-coefficients of these graphs are MPL. For the massless case, the alphabet is just $\{x, 1+x\}$ *for* $x = s/t$.

4-point recursions

Start with the box or (double box) and repeat, in any order:



Example