Non-planar on-shell diagrams in $\mathcal{N} = 4$ Super Yang-Mills

- Amplitudes 2015 -

ETH Zürich

based on:

hep-th/1502.02034 - Franco, Galloni, BP, Wen

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09/07/15

Outline

1. Introduction

2. Grassmannian formulation

3. On-shell diagrams

Planar
vs
Non-planar

4. Conclusions

Introduction

Scattering amps in $\mathcal{N}=4$ SYM SU(N) $\Rightarrow$

- Maximally supersymmetric
- Conformal to all loops
- Integrable (N $\to \infty$)
- AdS/CFT

Planar limit: $N \to \infty$, with $\lambda = g_{\rm YM}^2 N$ fixed

$$\mathcal{A}_n = \sum_{\sigma \in S_n/\mathbb{Z}_n} \mathrm{Tr}(t^{a_{\sigma(1)}} t^{a_{\sigma(2)}} \dots t^{a_{\sigma(n)}}) \underbrace{A_n(\sigma(1), \sigma(2), \dots, \sigma(n))}_{\text{Partial amplitude (colour ordered)}}$$

(Finite N corrections $\propto$ multiple traces)

State of the art in the planar limit

Tree level:

- N^{k-2}MHV tree amplitudes with any k, n can be found recursively via the *BCFW recursion relation* [[Britto, Cachazo, Feng, Witten - 2005]]

- Tree-level amplitudes enjoy *Yangian* symmetry [[Drummond, Henn, Plefka - 2009]]

  [[Yangian = Superconformal + Dual Superconformal]]
  [[Drummond, Henn, Korchemsky, Sokatchev - 2008]]

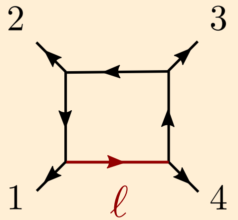
Loop level:

- Yangian symmetry broken due to IR divergences
- Loop integrand

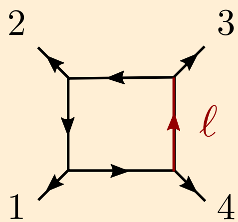
$$\int d^4\ell_1 \ldots d^4\ell_L \times \boxed{\text{Rational function of external and loop momenta}}$$

State of the art in the planar limit

Ambiguities:

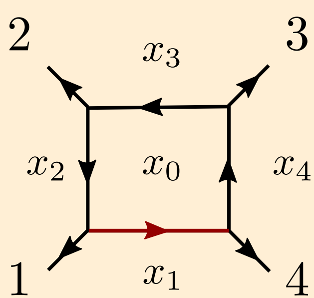


$$\leftrightarrow \quad \int d^4\ell \frac{1}{\ell^2(\ell+p_1)^2(\ell+p_1+p_2)^2(\ell-p_4)^2}$$



$$\leftrightarrow \quad \int d^4\ell \frac{1}{\ell^2(\ell+p_4)^2(\ell+p_1+p_4)^2(\ell-p_3)^2}$$

- Planar loop integrand well defined: dual variables x_i



$$= \int d^4x_0 \frac{1}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2}$$

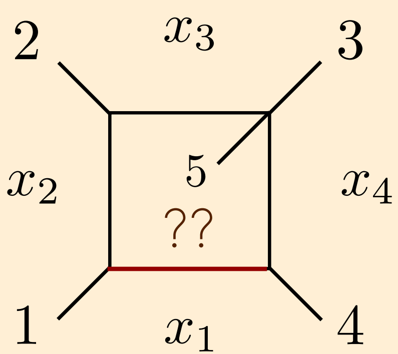
$$p_i = x_i - x_{i+1}$$
$$x_{ij} = x_i - x_j$$
$$\ell = x_{01}$$

State of the art in the planar limit

All-loop **integrand** determined by the all-loop recursion relation

[[Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka - 2010]]

- ~~Dual variables x_i allow different terms in recursion relation to be combined in a non-ambiguous way~~



Unavailable for non-planar integrands

↓

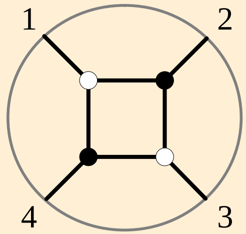
Non-planar integrand not well-defined

↓

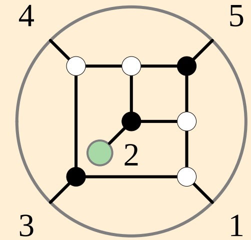
Consider instead **Leading Singularities**

[[Eden, Landshoff, Olive, Polkinghorne - 1966, Britto, Cachazo, Feng - 2004]]

Planar



Non-Planar



In the planar limit ∃ basis of
dual conformal integrands with
"unit leading singularity"
[[Arkani-Hamed, Bourjaily, Cachazo, Trnka - 2010]]

↓

LS are sufficient to determine
the all-loop integrand!

↓

All planar LS are residues of a
(positive) **Grassmannian** integral

↓

Positive Grassmannian
parametrised by
planar on-shell diagrams

Non-planar integrand
not well defined

↓

Consider non-planar LS

↓

Residues of a
Grassmannian integral

↓

Parametrised by
non-planar on-shell diagrams

OBS: Results for complete 4-pt integrands
up to 5-loops using max. cuts!
[[Bern, Carrasco, Johansson, Roiban - 2012,
+ Dixon - 2010]]

Grassmannian Formulation

[[Arkani-Hamed, Cachazo, Cheung, Kaplan - 2009]]
[[Mason, Skinner - 2009]]

Grassmannian formulation

[[Arkani-Hamed, Cachazo, Cheung, Kaplan - 2009]]

DEF: Grassmannian $Gr_{k,n}$ is the space of k-planes in $\mathbb{C}^n$

- Element of $Gr_{k,n}$: choose k n-vectors: $C_{\alpha a} = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{k1} & C_{k2} & \dots & C_{kn} \end{pmatrix}$
- $GL(k)$ gauge redundancy $\longrightarrow$ $\dim(Gr_{k,n}) = nk - k^2$
- Coordinates in $Gr_{k,n}$ $\longrightarrow$ Maximal minors (Plücker coords.)

$$\Delta_{i_1,i_2,\dots,i_k} = (i_1 i_2 \cdots i_k) = \det \begin{pmatrix} C_{1i_1} & C_{1i_2} & \cdots & C_{1i_k} \\ C_{2i_1} & C_{2i_2} & \cdots & C_{2i_k} \\ \vdots & & & \vdots \\ C_{ki_1} & C_{ki_2} & \cdots & C_{ki_k} \end{pmatrix}$$

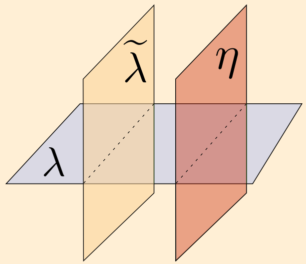
Plücker relations: **Ex:** $Gr_{2,4} \longrightarrow \Delta_{1,2}\Delta_{3,4} + \Delta_{1,3}\Delta_{4,2} + \Delta_{1,4}\Delta_{2,3} = 0$

Positive Grassmannian $Gr^+_{k,n} \longrightarrow \Delta_{i_1,i_2,\dots,i_k} > 0 \begin{cases} \forall\, C_{\alpha a} > 0 \\ i_1 < i_2 < \cdots < i_k \end{cases}$

Grassmannian formulation

[[Arkani-Hamed, Cachazo, Cheung, Kaplan - 2009]]

Planar LS are residues of the following integral over $Gr^{+}_{k,n}$



Ensure $\sum_i \lambda_i \widetilde{\lambda}_i = 0 \quad \sum_i \lambda_i \eta_i = 0$

$$\mathcal{L}_{n,k} = \frac{1}{\mathrm{Vol}(GL(k))} \int d^{k\times n} C_{\alpha a} \frac{\delta(C\cdot\widetilde{\lambda})\,\delta(C^{\perp}\cdot\lambda)\,\delta(C\cdot\eta)}{\underbrace{(1\ldots k)(2\ldots k+1)\ldots(n\ldots k-1)}}$$

Gauge fix k^2 entries of C

$k \times k$ consecutive minors of C

Ex: $(12\ldots k) = \det \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1k} \\ C_{21} & C_{22} & \cdots & C_{2k} \\ \vdots & & & \vdots \\ C_{k1} & C_{k2} & \cdots & C_{kk} \end{pmatrix}$

Grassmannian formulation

[[Arkani-Hamed, Cachazo, Cheung, Kaplan - 2009]]

$$\mathcal{L}_{n,k} = \frac{1}{\mathrm{Vol}(GL(k))} \int d^{k\times n} C_{\alpha a} \frac{\delta(C\cdot\widetilde{\lambda})\,\delta(C^{\perp}\cdot\lambda)\,\delta(C\cdot\eta)}{(1\ldots k)(2\ldots k+1)\ldots(n\ldots k-1)}$$

Poles when **consecutive** minors vanish

Non-planar

[[Galloni, Franco, BP, Wen - 2015]]

$$\mathcal{L}_{n,k} = \frac{1}{\mathrm{Vol}(GL(k))} \int d^{k\times n} C_{\alpha a} \frac{\delta(C\cdot\widetilde{\lambda})\,\delta(C^{\perp}\cdot\lambda)\,\delta(C\cdot\eta)}{(1\ldots k)(2\ldots k+1)\ldots(n\ldots k-1)} \times \mathcal{F}$$

To be discussed further soon!

$GL(k)$ invariance: cross ratio of minors

Ex: k=3 $\mathcal{F} = \dfrac{(123)(245)}{(124)(235)}$

No notion of ordering or positivity in non-planar case → $\cancel{Gr^{+}_{k,n}}$

On-shell diagrams

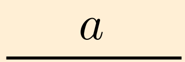
[[Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka - 2012]]

On-shell formulation

[[Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka - 2012]]

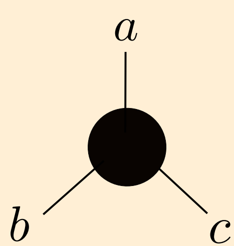
Trivalent bi-coloured graphs made of the building blocks:

Edges:



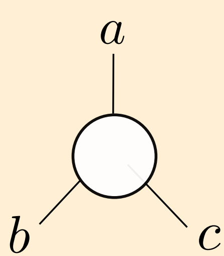
on-shell momentum $p_a = \lambda^a \widetilde{\lambda}^a$

Nodes



MHV amplitude

$\widetilde{\lambda}^a \propto \widetilde{\lambda}^b \propto \widetilde{\lambda}^c$ $\quad \leftrightarrow \quad Gr_{2,3}$

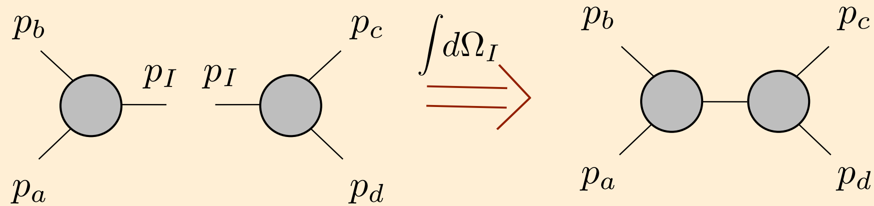


$\overline{\text{MHV}}$ amplitude

$\lambda^a \propto \lambda^b \propto \lambda^c$ $\quad \leftrightarrow \quad Gr_{1,3}$

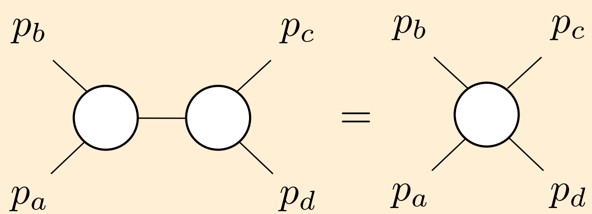
Constructing on-shell diagrams

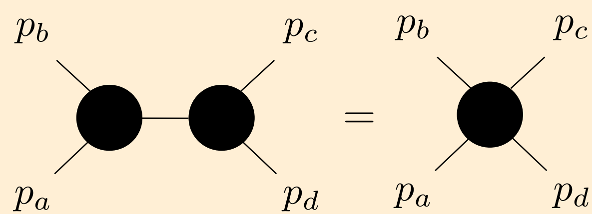
- To connect two nodes, integrate over on-shell phase space of edge in common:



$$d\Omega_I = \frac{d^2\lambda^I d^2\widetilde{\lambda}^I d^4\eta^I}{\underbrace{\mathrm{Vol}(GL(1))_I}_{\text{Little group}}}$$

- Can construct more complicated diagrams
- Nodes of the same colour can be merged





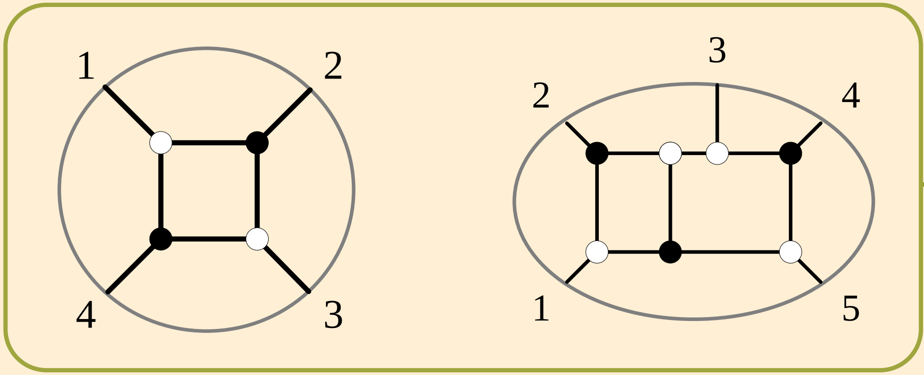
$\lambda^a \propto \lambda^b \propto \lambda^c \propto \lambda^d$

$\widetilde{\lambda}^a \propto \widetilde{\lambda}^b \propto \widetilde{\lambda}^c \propto \widetilde{\lambda}^d$

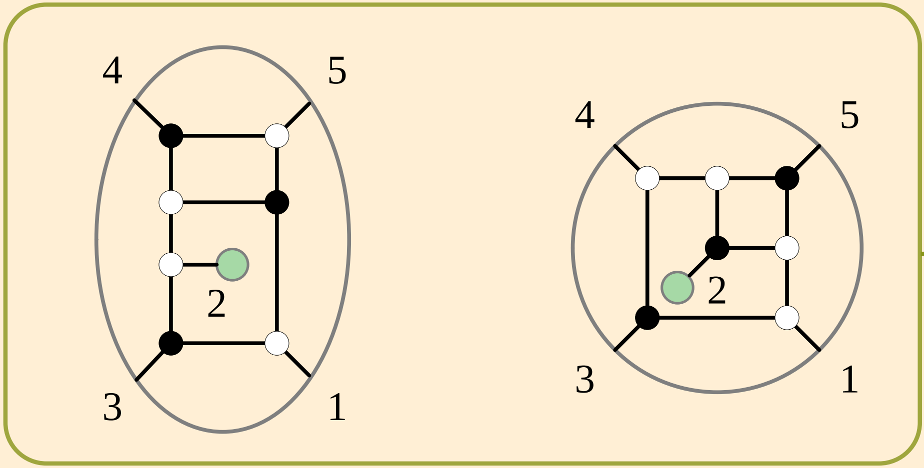
Every on-shell diagram can be made **bipartite**

Constructing on-shell diagrams

Examples:



Planar:
Can be embedded on a disk



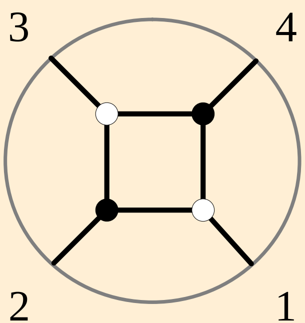
Non planar:
Can be embedded on a surface with multiple boundaries/ higher genus

Fusing Grassmannians

- An on-shell diagram with n_B black nodes, n_W white nodes and n_I internal edges is associated to $Gr_{k,n}$, where:

$$k = 2n_B + n_W - n_I$$

Ex:



$$n_B = 2 \qquad n_W = 2 \qquad n_I = 4$$

$$k = 2 \times 2 + 2 - 4 = 2$$

$$n = 4$$

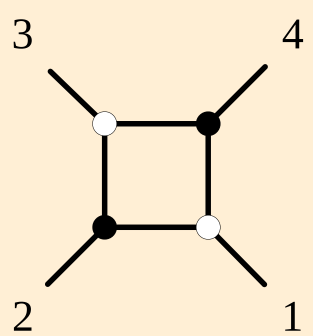
$$\boxed{Gr_{2,4}}$$

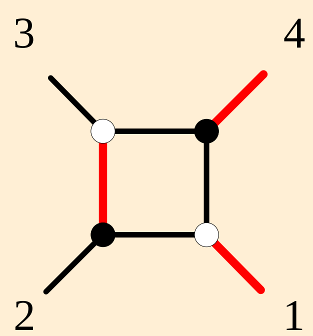
Boundary measurement

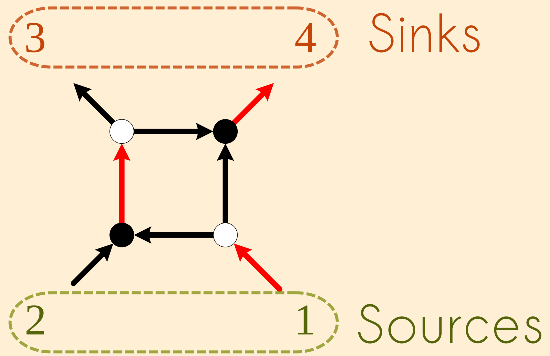
On-shell diagram

$C \in Gr_{k,n}$

Bipartite technology







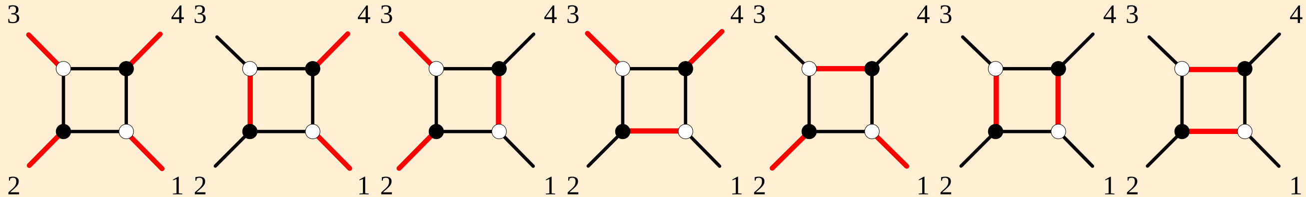
Perfect matching

Choice of edges such that every internal node is the endpoint of only one edge

Perfect orientation

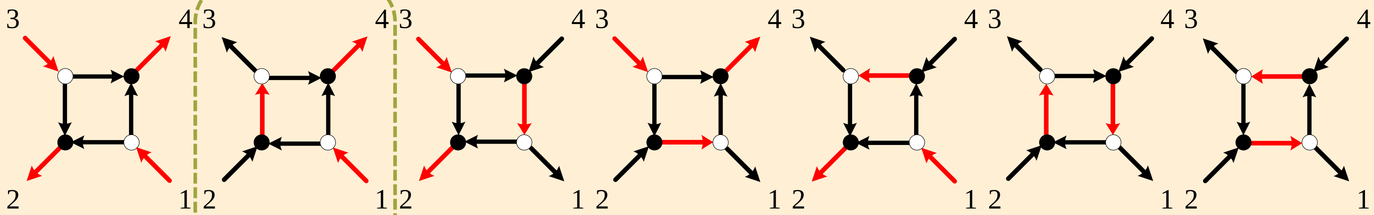
Orient edges in the perfect matching from Black to White. Black nodes have only one outgoing arrow, white nodes have only one incoming arrow

Perfect matchings:



p_1 p_2 p_3 p_4 p_5 p_6 p_7

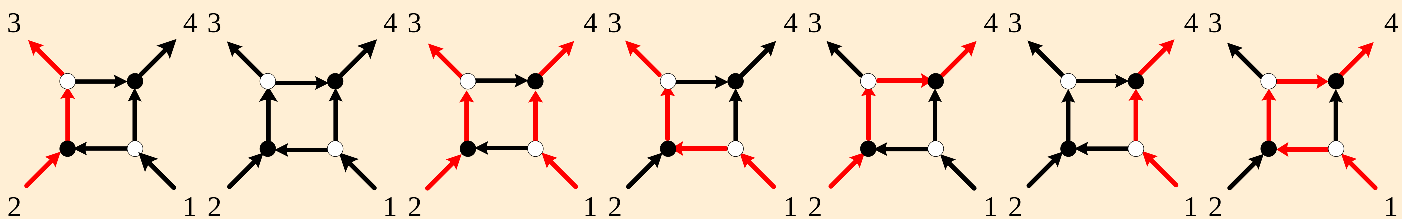
Oriented perfect matchings:



$\widetilde{p}_1$ $\widetilde{p}_2$ $\widetilde{p}_3$ $\widetilde{p}_4$ $\widetilde{p}_5$ $\widetilde{p}_6$ $\widetilde{p}_7$

Reference

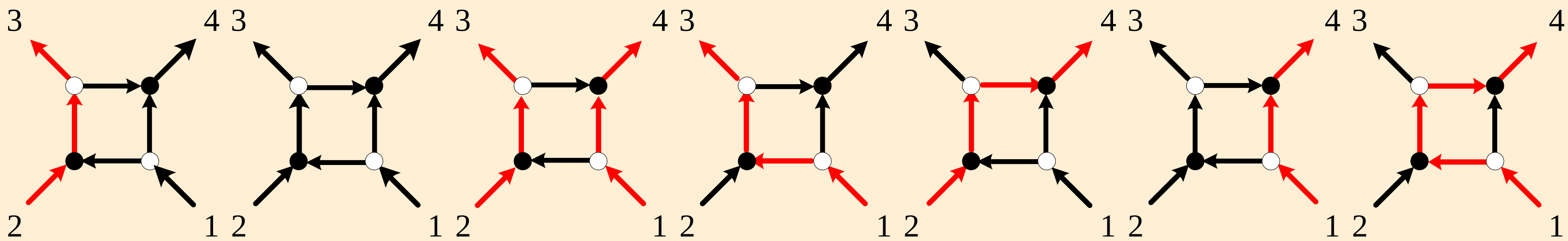
Flows:



$\mathfrak{p}_1$ $\mathfrak{p}_2$ $\mathfrak{p}_3$ $\mathfrak{p}_4$ $\mathfrak{p}_5$ $\mathfrak{p}_6$ $\mathfrak{p}_7$

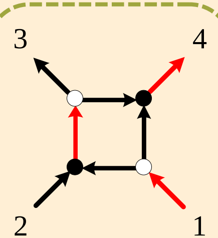
Boundary measurement

Flows:



$\mathfrak{p}_1$ $\mathfrak{p}_2$ $\mathfrak{p}_3$ $\mathfrak{p}_4$ $\mathfrak{p}_5$ $\mathfrak{p}_6$ $\mathfrak{p}_7$

Map between on-shell diagram and element of the Grassmannian



$\widetilde{p}_2$

Reference

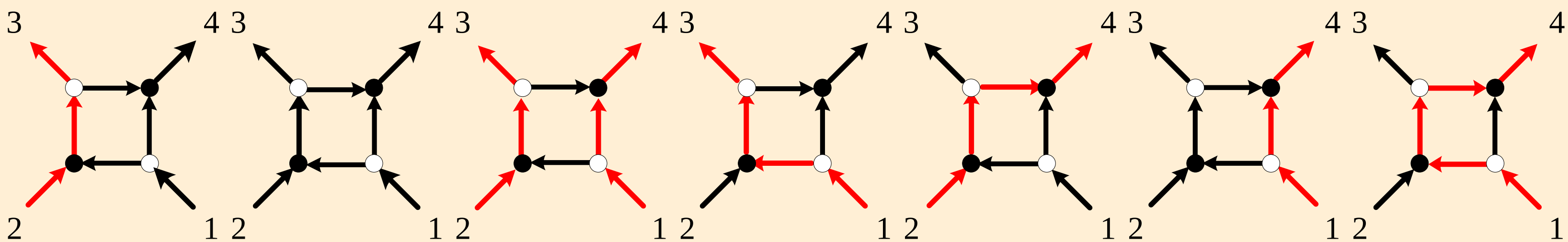
$$C_{ij} = \sum_{\Gamma\{i \rightsquigarrow j\}} (-1)^{s_\Gamma}\, \mathfrak{p}_{\{i \rightsquigarrow j\}}$$

Flows from i to j

Sign prescription

$$C = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 1 & 0 & \bullet & \bullet \\ 0 & 1 & \bullet & \bullet \end{pmatrix} \end{array}$$

Boundary measurement



$\mathfrak{p}_1$	$\mathfrak{p}_2$	$\mathfrak{p}_3$	$\mathfrak{p}_4$	$\mathfrak{p}_5$	$\mathfrak{p}_6$	$\mathfrak{p}_7$
$\{1,3\}$	$\{1,2\}$	$\{3,4\}$	$\{2,3\}$	$\{1,4\}$	$\{2,4\}$	$\{2,4\}$

Source set of perfect matching

$$C = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & -\mathfrak{p}_4 & -(\mathfrak{p}_6 + \mathfrak{p}_7) \\ 0 & 1 & \mathfrak{p}_1 & \mathfrak{p}_5 \end{pmatrix} \end{array}$$

$(-1)^{s_\Gamma}$ → Plücker coordinates are **positive** in **planar** case and are a **sum of flows with corresponding source set**.

Sign prescription

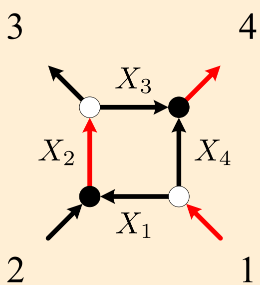
Ex: $\underbrace{\Delta_{1,2} = \mathfrak{p}_2}_{\{1,2\}}, \quad \Delta_{2,4} = \underbrace{\mathfrak{p}_6}_{\{2,4\}} + \underbrace{\mathfrak{p}_7}_{\{2,4\}}, \quad \Delta_{3,4} = \underbrace{\mathfrak{p}_3}_{\{3,4\}}$

Parametrising on-shell diagrams

Planar:

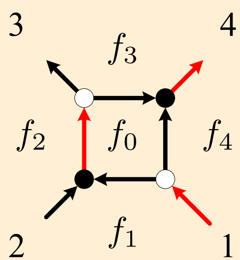
- On-shell dlog form: variables unfixed by delta-functions mapped to loop integration variables.
- # degrees of freedom of a planar on-shell diagram is $d = F - 1$

  # faces
- Bases for expressing flows: **Edges** and **Faces**



Edge variables:

$$\frac{dX_1}{X_1}\frac{dX_2}{X_2}\frac{dX_3}{X_3}\frac{dX_4}{X_4}\delta(C(X)\cdot\widetilde{\lambda})\,\delta(C(X)^{\perp}\cdot\lambda)\,\delta(C(X)\cdot\eta)$$



Face variables:

General for non-planar ?

$$\prod_{i=1}^{F} f_i = 1$$

$$\frac{df_1}{f_1}\frac{df_2}{f_2}\frac{df_3}{f_3}\frac{df_4}{f_4}\delta(C(f)\cdot\widetilde{\lambda})\,\delta(C(f)^{\perp}\cdot\lambda)\,\delta(C(f)\cdot\eta)$$

Generalised face variables

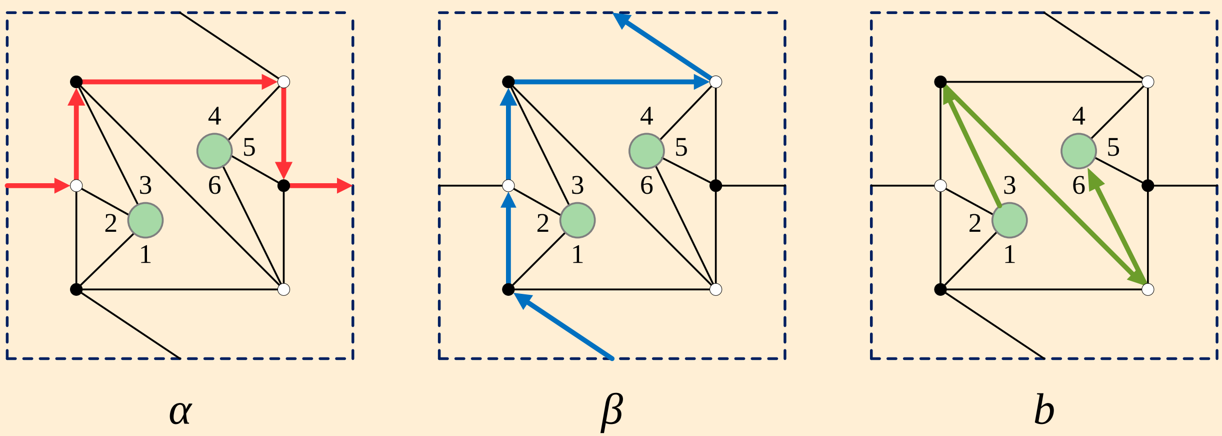
[[Galloni, Franco, BP, Wen - 2015]]

$$d = \underbrace{(F-1)}_{f_i} + \underbrace{(B-1)}_{b_a} + \underbrace{2g}_{\{\alpha_m, \beta_m\}} \;=\; F - \xi$$

F = # faces
B = # boundaries
g = genus

$f_i,\ i = 1, \ldots, F$	$\prod_{i=1}^{F} f_i = 1$	Faces
$b_a,\ a = 1, \ldots, B-1$		Paths connecting different boundaries
$\{\alpha_m, \beta_m\},\ m = 1, \ldots g$		Fundamental cycles

Ex: Genus 1



Generalised face variables

[[Galloni, Franco, BP, Wen - 2015]]

$$d = \underbrace{(F-1)}_{f_i} + \underbrace{(B-1)}_{b_a} + \underbrace{2g}_{\{\alpha_m, \beta_m\}} \quad = \; F - \xi$$

F = # faces
B = # boundaries
g = genus

$f_i,\; i = 1, \dots, F \qquad \prod_{i=1}^{F} f_i = 1$ Faces

$b_a,\; a = 1, \dots, B-1$ Paths connecting different boundaries

$\{\alpha_m, \beta_m\},\; m = 1, \dots g$ Fundamental cycles

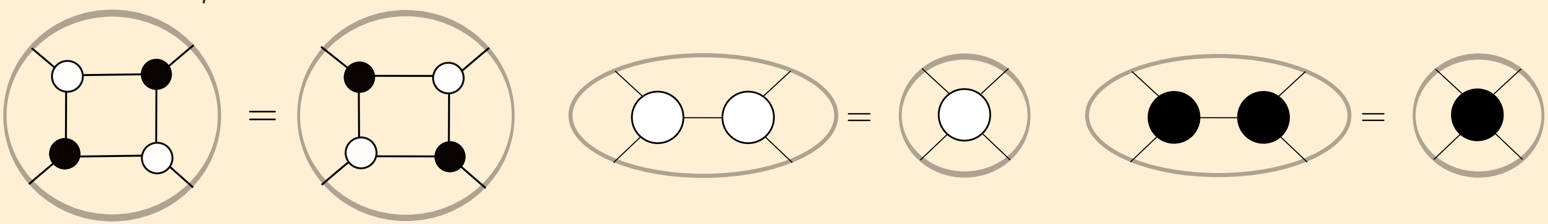
dlog on-shell form:

$$\frac{dX_1}{X_1}\frac{dX_2}{X_2}\cdots\frac{dX_d}{X_d} \quad \leftrightarrow \quad \prod_{i=1}^{F-1}\frac{df_i}{f_i}\prod_{a=1}^{B-1}\frac{db_a}{b_a}\prod_{m=1}^{g}\frac{d\alpha_m}{\alpha_m}\frac{d\beta_m}{\beta_m}$$

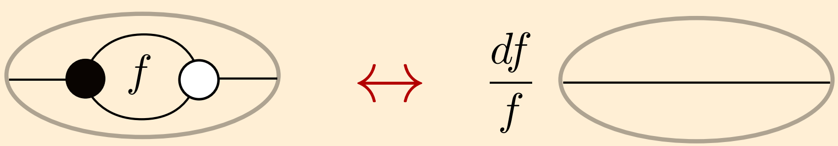
Reducibility & Equivalence: Planar

[[Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka - 2012]]

- Two on-shell diagrams that span the same region in the Grassmannian and have the same number of d.o.f are *equivalent.*



- If it is possible to remove an edge of a graph without sending any Plücker coord to zero, the graph is *reducible*.



Bubble deletion

- If it is impossible to remove an edge of a graph without sending some Plücker coord to zero, the graph is *reduced.*

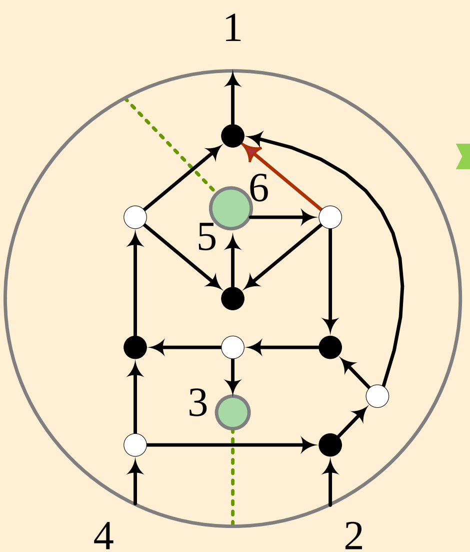
$\Longrightarrow$ (Positroid stratification of $Gr^{+}_{k,n}$)

Reducibility & Equivalence: Non-planar

[[Arkani-Hamed, Bourjaily, Cachazo, Postnikov, Trnka - 2014, Galloni, Franco, BP, Wen - 2015]]

A non-planar novelty:

- It is possible to remove an edge of a **reduced** graph without sending any Plücker coord to zero!



Recall: Deformation from planar Grassmannian integrand

$$\mathcal{F} = \frac{(346)^2(356)(123)(612)}{(136)(236)[(124)(346)(365) - (456)(234)(136)]}$$

Removal of an edge does not set any $\Delta_{i,j,k}$ to zero, but gives rise to the relation

$$\boxed{\Delta_{1,2,4}\Delta_{3,4,6}\Delta_{3,6,5} = \Delta_{4,5,6}\Delta_{2,3,4}\Delta_{1,3,6}}$$

Polytopes

[[Postnikov, Speyer, William - 2009, Franco, Galloni, Mariotti - 2013]]

Notions of equivalence/reduction can be rephrased in terms of polytopes:

Matching polytope:

Perfect matching $\leftrightarrow$ Flow $\leftrightarrow$ Point in **matching polytope**

Matroid polytope:

Perfect matchings with same source set $\leftrightarrow$ Point in **matroid polytope** $\leftrightarrow$ Plücker coord.

Flow: $$\mathfrak{p}_\mu = \prod_{i=1}^{F-1} f_i^{x_{i,\mu}} \prod_{j=1}^{B-1} b_j^{y_{j,\mu}} \prod_{m=1}^{g} \alpha_m^{z_{m,\mu}} \beta_m^{w_{m,\mu}}$$

Coord. in matching polytope:

$$(x_{1,\mu}, \ldots, x_{F-1,\mu}, y_{1,\mu}, \ldots, y_{B-1,\mu}, z_{1,\mu}, \ldots, z_{g,\mu}, w_{1,\mu}, \ldots, w_{g,\mu})$$

Coord. in matroid polytope:

$$(x_{1,\mu}, \ldots, x_{F_e,\mu})$$

(just external faces)

Polytopes

[[Postnikov, Speyer, William - 2009, Franco, Galloni, Mariotti - 2013]]

Matroid polytope:

Perfect matchings with same source set $\leftrightarrow$ Point in **matroid polytope** $\leftrightarrow$ Plücker coord.

Sign prescription in generalised boundary measurement must be consistent with

$\Delta_{i_1,i_2,\ldots,i_k} \leftrightarrow$ Sum of flows with source set $\{i_1, i_2, \ldots, i_k\}$ with coefficients ± 1

[[Gekhtman, Shapiro, Vainshtein - 2013]] $\longrightarrow$ **Annulus**

[[Franco, Galloni, Mariotti - 2013]] $\longrightarrow$ **Arbitrary *B*, genus zero**

[[Franco, Galloni, BP, Wen - 2015]] $\longrightarrow$ **Any graph**

Characterisation of on-shell diagrams

For **planar reduced** on-shell diagrams one can associate a permutation of external nodes that characterises equivalence classes.

↓ Non-planar diagrams **without extra constraints** on Plücker coordinates

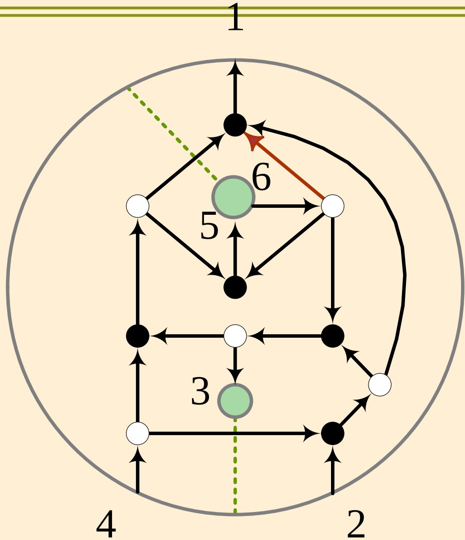
Two graphs are **equivalent** if they have the same matroid polytope and number of degrees of freedom.

An on-shell diagram "B" is a **reduction** of another diagram "A" if it is obtaind from "A" by deleting edges and it has the same matroid polytope.

A graph is **reduced** if it is impossible to remove edges while preserving the matroid polytope.

Constraints and polytopes

[[Galloni, Franco, BP, Wen - 2015]]



Before removal: 40 perfect matchings

$\Delta_{1,2,4}$	p_7, p_{35}	$\Delta_{4,5,6}$	p_{32}
$\Delta_{3,4,6}$	p_{23}	$\Delta_{2,3,4}$	p_{24}
$\Delta_{3,5,6}$	p_{37}	$\Delta_{1,3,6}$	p_{38}

After removal: 33 perfect matchings

Before and after removal:
$$p_{35}p_{23}p_{37} = p_{24}p_{32}p_{38}$$

After removal
p_7 disappears

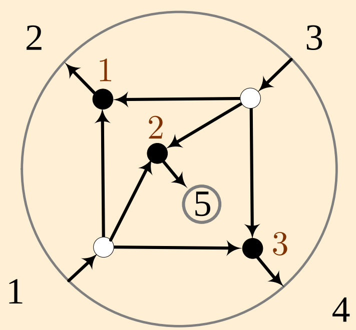
$$\Delta_{1,2,4}\Delta_{3,4,6}\Delta_{3,6,5} = \Delta_{4,5,6}\Delta_{2,3,4}\Delta_{1,3,6}$$

Finding $\mathcal{F}$ $\left(\mathcal{L}_{n,k} = \text{Planar} \times \mathcal{F}\right)$

MHV non-planar leading singularities:

[[Arkani-Hamed, Bourjaily, Cachazo, Postnikov, Trnka - 2014]]

- Every black node is connected to 3 external nodes either directly or via a white node
- $\exists \quad n-2$ black nodes



$$T = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 3 & 4 \end{pmatrix} \longrightarrow M = \begin{pmatrix} (23) & (31) & (12) & 0 & 0 \\ (35) & 0 & (51) & 0 & (13) \\ (34) & 0 & (41) & (13) & 0 \end{pmatrix}$$

$$\left(\Omega = \frac{d^{2\times n}C}{\text{Vol(GL(2))}} \left(\frac{\det(\widehat{M}_{i,j})}{(i\,j)} \right)^2 \frac{1}{\text{PT}^{(1)}\text{PT}^{(2)}\cdots\text{PT}^{(n_B)}} \right)$$

$$\boxed{\Omega = \frac{d^{2\times 5}C}{\text{Vol(GL(2))}} \frac{(13)^4}{(12)(23)(31)(13)(35)(51)(13)(34)(41)}}$$

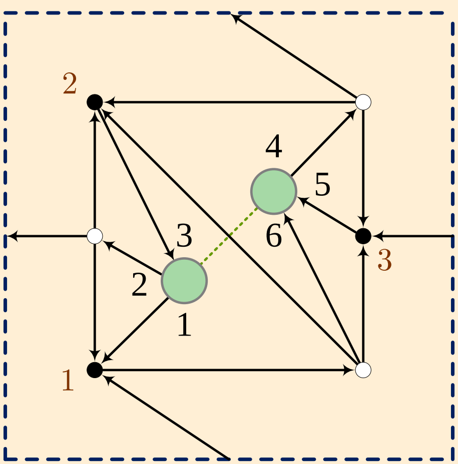
Finding $\mathcal{F}$

Strategy for higher MHV degree

[[Galloni, Franco, BP, Wen - 2015]]

Desired properties:

- Every black node is connected to $k+1$ external nodes either directly or via a white node
- $\exists\ \ n-k$ black nodes



$k = 3,\ d = 9$

$$T = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\begin{pmatrix} 1 & 6 & 4 & 2 \\ 3 & 2 & 4 & 6 \\ 5 & 4 & 2 & 6 \end{pmatrix} \quad M = \begin{pmatrix} (642) & (164) & 0 & (216) & 0 & (421) \\ 0 & (463) & (246) & (632) & 0 & (324) \\ 0 & (654) & 0 & (265) & (426) & (542) \end{pmatrix}$$

$$\Omega = \frac{d^{k\times n}C}{\mathrm{Vol}(\mathrm{GL}(k))}\left(\frac{\det(\widehat{M}_{a_1,\ldots,a_k})}{(a_1,\ldots,a_k)}\right)^k \frac{1}{\mathrm{PT}^{(1)}\mathrm{PT}^{(2)}\cdots\mathrm{PT}^{(n_B)}}$$

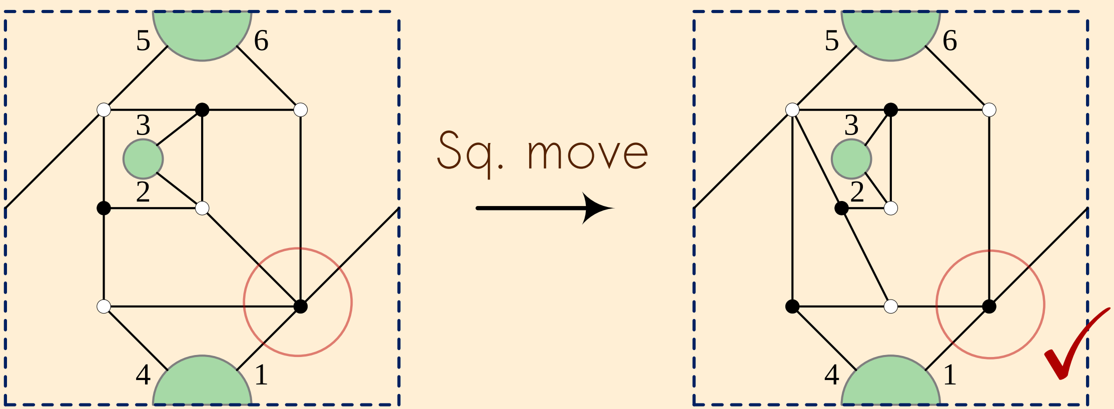
$$\boxed{\Omega = \frac{d^{3\times 6}C}{\mathrm{Vol}(\mathrm{GL}(3))}\frac{(246)^3}{(164)(421)(216)(324)(463)(632)(542)(265)(654)}}$$

Finding $\mathcal{F}$

Desired properties:

- Every black node is connected to $k+1$ external nodes either directly or via a white node ✗
- $\exists$ $n-k$ black nodes

a) Valency $v > k+1$



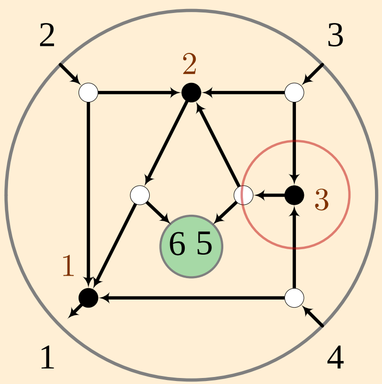
$k=3$

Finding $\mathcal{F}$

Desired properties:

- Every black node is connected to $k+1$ external nodes either directly or via a white node ✗
- $\exists \; n-k$ black nodes

b) Valency $v < k+1$



$$k = 3$$
$$d = 8$$

$$T = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} 1 & 2 & 6 & 4 \\ 2 & 3 & 5 & 6 \\ 5 & 3 & 4 & * \end{pmatrix}$$

Any row with $*$ implies linear dependence among remaining labels

$$(345)=0$$

Choose any other leg (e.g. 2)

$$\Omega = \frac{d^{3\times 6}C}{\mathrm{Vol}(\mathrm{GL}(3))} \frac{(264)^2(235)}{(126)(641)(412)(356)(562)(623)(342)(425)(345)}\Bigg|_{(345)=0}$$

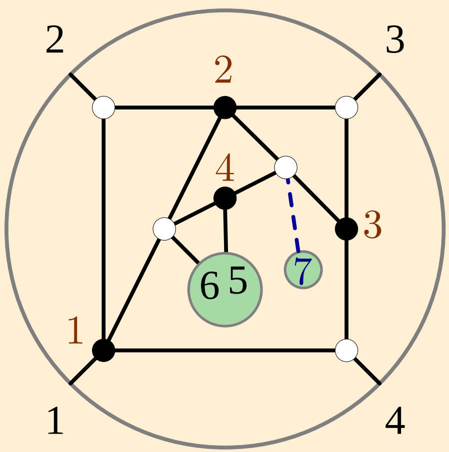
Finding $\mathcal{F}$

Desired properties:

- Every black node is connected to $k+1$ external nodes either directly or via a white node
- $\exists \; n-k$ black nodes ✗

\# white nodes surrounded by black nodes ($\alpha = 0, 1, \ldots$)

$$n_B = n - k + \alpha$$



$\alpha = 0$
$d = 10$

$$T = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{pmatrix} 1 & 2 & 6 & 4 \\ 2 & 3 & 7 & 6 \\ 7 & 3 & 4 & * \\ 5 & 6 & 7 & * \end{pmatrix} \Rightarrow (347) = (567) = 0$$

$$\Omega = \frac{d^{3\times 7}C}{\mathrm{Vol}(\mathrm{GL}(3))} \; I_{1,\ldots 6} \times \frac{1}{(347)(567)(725)}$$

$$\xrightarrow[c_{i7}=0]{\text{Res}} \boxed{\Omega = \frac{d^{3\times 6}C}{\mathrm{Vol}(\mathrm{GL}(3))} \frac{(246)^2}{(234)(345)(456)(612)(124)(146)(236)(256)}}$$

Summary

Non-planar on-shell diagrams

- Generalised face variables
- Boundary measurement for higher genus
- Equivalence and reductions in terms of polytopes
- Found diagrams that parametrise regions of the Grassmannian with extra constraints beyond Plücker relations

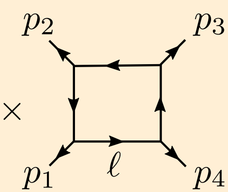
Concluding remarks & Outlook

1) Physical interpretation:

Planar: All tree level amplitudes and loop integrands via BCFW recursion relation. [[Britto, Cachazo, Feng, Witten - 2005]] [[Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka - 2010]]

dlog form of the loop integrand:

$$\mathcal{A}^{L=1} = \mathcal{A}^{L=0} \times$$



$$= \mathcal{A}^{L=0} \times \int d^4\ell \, \frac{(p_1+p_2)^2(p_1+p_3)^2}{\ell^2(\ell+p_1)^2(\ell+p_1+p_2)^2(\ell-p_4)^2}$$

$$d\log\left(\frac{\ell^2}{(\ell-\ell^*)^2}\right) d\log\left(\frac{(\ell+p_1)^2}{(\ell-\ell^*)^2}\right) d\log\left(\frac{(\ell+p_1+p_2)^2}{(\ell-\ell^*)^2}\right) d\log\left(\frac{(\ell-p_4)^2}{(\ell-\ell^*)^2}\right)$$

[[Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka - 2012]]

Non planar: Leading singularities of the loop integrand

? Non-planar loop integrand

? Non-planar Grassmannian formulation

[[Arkani-Hamed, Bourjaily, Cachazo, Trnka - 2014]]

[[Bern, Herrmannn, Litsey, Stankowicz, Trnka - 2014]]

Conjecture: Non-planar amps have only log singularities and no poles at infinity.

Concluding remarks & Outlook

2) Non-planar diagrams parametrise regions of $Gr_{k,n}$ with hidden relations between Plücker coordinates.
$\hookrightarrow$? Method for finding representative graph given a constraint

3) MHV non-planar leading singularities are sums of planar ones.
[[Arkani-Hamed, Bourjaily, Cachazo, Postnikov, Trnka - 2014]]

Same not true for non-MHV, however similar method can be used to find the deformation of the integrand $\mathcal{F}$.

4) Positive Grassmannian $Gr^{+}_{k,n} \rightarrow$ Amplituhedron
[[Arkani-Hamed, Trnka - 2013]]
? Non-planar generalisation

5) ? Possible application for form-factors on-shell diagrams
[[Frassek, Meidinger, Nandan, Wilhelm (2015) - see Matthias Wilhelm's poster]]