Single and double soft gluon and graviton theorems

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Single: with J. Broedel, M. de Leeuw and M. Rosso
`PRD90 (1406.6574) & PLB746 (1411.2230)`

Double: with T. Klose, T. McLoughlin, D. Nandan and G. Travaglini
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Overview

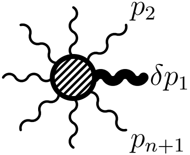
Renewed interest in universal properties of low energy gluon and graviton emissions. Novel factorization results down to the sub-(sub)-leading order in a soft momentum expansion.

Sparked by claimed connection to hidden infinite dimensional $\mathfrak{bms}_4$ symmetry of quantum gravity S-matrix [Cachazo, Strominger]

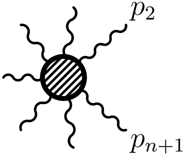
Plan

1. Novel subleading single soft theorems
2. Brief intro to extended $\mathfrak{bms}_4$ symmetry
3. Constraining soft theorems by symmetries and consistency
4. Double soft gluon and graviton theorems @ tree-level
5. Outlook

Single Soft Limits

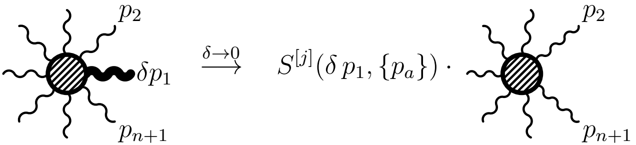


$$\xrightarrow{\delta\to 0} \quad S^{[j]}(\delta\, p_1, \{p_a\}) \cdot$$



Theorems of Low (1958) and Weinberg (1964)

Scattering amplitudes display universal factorization when a single photon (gluon) or graviton becomes soft: Parametrize soft momentum as $\delta\, q^\mu$ and take $\delta \to 0$



$$\boxed{\mathcal{A}_{n+1}(\delta\, q, p_1, \ldots, p_n) \underset{\delta\to 0}{=} S^{[0]}(\delta\, q, \{p_a\}) \cdot \mathcal{A}_n(p_1, \ldots, p_n) + \mathcal{O}(\delta^0)}$$

At tree-level with soft leg polarization $E_{\mu(\nu)}$:

$$S^{[0]}(\delta\, q, \{p_a\}) = \begin{cases} \displaystyle\sum_{a=1}^{n} \frac{1}{\delta} \frac{E_\mu\, p_a^\mu}{p_a \cdot q} & : \text{ photon} \quad \to \text{gluon (color ordered)} \\ \displaystyle\sum_{a=1}^{n} \frac{1}{\delta} \frac{E_{\mu\nu}\, p_a^\mu\, p_a^\nu}{p_a \cdot q} & : \text{ graviton} \end{cases}$$

Proof is elementary. Tree-level exact for gravity. IR divergent loop corrections in YM.

Subleading soft theorems

Universality & factorization extends to subleading order [Cachazo, Strominger][Low,Burnett,Kroll;Casali]

$$\boxed{\mathcal{A}_{n+1}(\delta\, q, p_1, \ldots, p_n) \underset{\delta\to 0}{=} S^{[j]}(\delta\, q, \{p_a\}) \cdot \mathcal{A}_n(p_1, \ldots, p_n) + \mathcal{O}(\delta^j)}$$

with soft operator

$$S^{[j]}(\delta\, q, \{p_a\}) = \begin{cases} \frac{1}{\delta} S^{(0)}_{\text{YM}} + S^{(1)}_{\text{YM}} & : \text{ Yang-Mills } (j=1) \\ \\ \frac{1}{\delta} S^{(0)}_{\text{G}} + S^{(1)}_{\text{G}} + \delta\, S^{(2)}_{\text{G}} & : \text{ Gravity } (j=2) \end{cases}$$

Explicit constructions (using BCFW, CHY) @ tree-level yield

$$S^{(1)\text{tree}}_{\text{YM}} = \frac{E_\mu\, q_\nu\, J_1^{\mu\nu}}{p_1 \cdot q} - \frac{E_\mu\, q_\nu\, J_n^{\mu\nu}}{p_n \cdot q} \qquad J_a^{\mu\nu} := p_a^\mu \partial_{p_a^\nu} + E_a^\mu \partial_{E_a^\nu} - \mu \leftrightarrow \nu$$

$$S^{(1)\text{tree}}_{\text{G}} = \sum_{a=1}^{n} \frac{(E \cdot p_a)\, E_\mu\, q_\nu\, J_a^{\mu\nu}}{p_a \cdot q} \qquad \text{writing} \quad E_{\mu\nu} = E_\mu E_\nu$$

$$S^{(2)\text{tree}}_{\text{G}} = \sum_{a=1}^{n} \frac{(E_\mu\, q_\nu\, J_a^{\mu\nu})^2}{p_a \cdot q}$$

arise from hidden symmetry?

Bondi-van der Burg-Metzner-Sachs (BMS) symmetry (1962)

- Study of classical gravitational waves: Expected Poincaré symmetry enlarged by BMS_4 group
- Acts at null infinity ($\mathcal{I}^{\pm}$) for asympt. flat space-times
- Coordinates: u (retarded time), r (radius), $x^A = \{\Theta, \phi\} \in S^2$ at $\mathcal{I}^{\pm}$

$$\boxed{ds^2 = e^{2\beta}\,\frac{V}{r}\,du^2 - 2e^{2\beta}\,du\,dr + g_{AB}(dx^A + U^A du)(dx^B + U^B du)}$$

Metric functions β, V, U^A, g_{AB} have fall-off conditions in r:

$$g_{AB} = r^2(d\Theta^2 + \sin^2\Theta\,d\phi^2) + \mathcal{O}(r), \quad \beta = \mathcal{O}(r^{-2}), \quad \frac{V}{r} = \mathcal{O}(r), \quad U^A = \mathcal{O}(r^{-2})$$

- BMS_4 group: Maps asymptotically flat space-times onto themselves

$$\boxed{\Theta' = \Theta'(\Theta, \phi) \qquad \phi' = \phi'(\Theta, \phi) \qquad u' = K(\Theta, \phi)\,(u - \alpha(\Theta, \phi))}$$

Where $(\Theta, \phi) \to (\Theta', \phi')$ is conformal transformation on S^2:

$$d\Theta'^2 + \sin^2\Theta' d\phi'^2 = K(\Theta, \phi)^2(d\Theta^2 + \sin^2\Theta\,d\phi^2)$$

- For $\Theta' = \Theta$ & $\phi' = \phi$ one has "supertranslations": $u' = u - \alpha(\Theta, \phi)$ with a general function $\alpha(\Theta, \phi)$.

$\mathfrak{bms}_4$ algebra

- In standard complex coordinates $z = e^{i\phi}\cot(\Theta/2)$ conformal symmetry generated by Virasoro generators ("superrotations")

$$l_n = -z^{n+1}\,\partial_z \qquad \bar{l}_n = -\bar{z}^{n+1}\,\partial_{\bar{z}}$$

- Supertranslations generated by $\quad T_{m,n} = z^m\,\bar{z}^n\,\partial_u$
- Extended $\mathfrak{bms}_4$ algebra [Barnich, Troessart]

$$[l_n, l_m] = (m-n)\,l_{m+n} \qquad [\bar{l}_n, \bar{l}_m] = (m-n)\,\bar{l}_{m+n}$$

$$[l_l, T_{m,n}] = -m\,T_{m+l,n} \qquad [\bar{l}_l, T_{m,n}] = -n\,\bar{T}_{m,n+l}$$

- Poincaré subalgebra spanned by $\underbrace{l_{-1}, l_0, l_1; \bar{l}_{-1}, \bar{l}_0, \bar{l}_1}_{\text{Lorentz}} \quad \underbrace{T_{0,0}, T_{0,1}, T_{1,0}, T_{1,1}}_{\text{Translation}}$
- BMS$_4$ group maps gravitational wave solutions onto each other.
- Claim: | Supertranslations $\hat{=}\ S_{\mathsf{G}}^{(0)}$ | Superrotations $\hat{=}\ S_{\mathsf{G}}^{(1)}$ | [Cachazo, Strominger]

Status:

- Subleading soft theorems proven via
  - BCFW-recursion [Cachazo, Strominger; Casali]
  - CHY-formulae for tree amplitudes [Schwab, Volovich; Afkhani-Jeddi; Kalousios, Rojas; Zlotnikov]
  - Diagrammatics & Gauge invariance [Low, Burnett, Kroll; Bern, Davies, Di Vecchia, Nohle; White]
- Soft theorems hold at tree-level in all dimensions [Schwab, Volovich]
- Connection to BMS_4-algebra [Cachazo,Strominger] [He,Lysov,Kapec,Mitra,Pasterski,Pate,Strominger,Zhiboedov]
- Soft limits of string scattering amplitudes [Schwab; Bianchi, He, Huang, Wen; Di Vecchia,Marotta,Mojaza] [Bianchi,Guerrieri]
- Twistor string picture [Geyer, Lipstein, Mason; Adamo, Casali, Skinner; Lipstein]
- Subleading soft gluon emission from fermions [Luo, Mastrolia, Bobadilla]
- Double soft limits of gluons and scalars [Cachazo,He,Yuan; Volovich,Wen,Zlotnikov;Georgiou; Du,Luo]
- Double soft gluons and scalars from open strings [Di Vecchia,Marotta,Mojaza]
- Loop level structure: [Bern,Davies,Nohle,Di Vecchia; He,Huang,Wen]
  - Gravitons: No corrections at leading order, sub-leading and sub-subleading soft functions corrected at 1 respectively 2 loop order
  - Gluons: Already leading order soft function receives loop level corrections
- ...

Constraining soft theorems

$$\delta^{(D)}(\delta q + \sum_{i=1}^{n} p_i) \quad \text{vs.} \quad \delta^{(D)}(\sum_{i=1}^{n} p_i)$$

A subtle momentum conservation issue

- Write $\mathcal{A}_n(\{p_a\}) = \delta^{(D)}(\sum_{a=1}^{n} p_a)\, A_n(\{p_a\})$:

$$\boxed{\delta^{(D)}(\delta\, q + P)\, A_{n+1}(\delta\, q, \{p_a\}) \underset{\delta\to 0}{=} S^{[j]}(\delta\, q, \{p_a\})\, \delta^{(D)}(P)\, A_n(\{p_a\}) + \mathcal{O}(\delta^j)}$$

with $P = \sum_{a=1}^{n} p_a$ and $S^{[j]} = \frac{1}{\delta} S^{(0)} + S^{(1)} + \ldots$

- Variant A: State theorem on level of stripped amps, i.e.

$$A_{n+1}(\delta\, q, \{p_a\}) = S^{[j]}(\delta\, q, \{p_a\})\, A_n(\{p_a\})$$

& include prescription on how to secure momentum conservations, e.g. $p_a \to p_a + \delta\, \tilde{p}_a$ with $\sum_a p_a = 0 = \sum_a \tilde{p}_a$ (disfavored)

- Variant B: State theorem at the level of distributions! Is the natural path. Implies non-trivial commutator:

$$S^{[j]}(\delta\, q)\, \delta^{(D)}(P) = \delta^{(D)}(P + \delta\, q)\, \tilde{S}^{[j]}(\delta\, q)$$

In fact one finds $\tilde{S}^{[j]} = S^{[j]}$. (favored)

Consistency condition [Broedel, de Leeuw, JP, Rosso]

Relation at leading orders: $P = \sum_{a=1}^{n} p_a$

$$\left(\frac{1}{\delta}S^{(0)} + S^{(1)}\right)\delta^{(D)}(P) = \left(\delta^{(D)}(P) + \delta\, q\cdot\partial_P\delta^{(D)}(P)\right)\left(\frac{1}{\delta}\tilde{S}^{(0)} + \tilde{S}^{(1)}\right) + \mathcal{O}(\delta)$$

- No issue at leading order in δ:

$$S^{(0)} = \tilde{S}^{(0)} \qquad \& \qquad [S^{(0)}, \delta^{(D)}(P)] = 0$$

- Non-trivial commutator at NLO:

$$S^{(1)} = \tilde{S}^{(1)} + \chi \qquad \& \qquad [S^{(1)}, \delta^{(D)}(P)] = S^{(0)}\, q\cdot\partial_P\delta^{(D)}(P) + \delta^{(D)}(P)\,\chi$$

$\Rightarrow$ implies that $S^{(1)}(\delta\, q, \{p_a\})$ must contain differential operator ∂_{p_a}.

- At NNLO (relevant for gravity): $S^{(2)} = \tilde{S}^{(2)} + \chi'$ &

$$[S^{(2)}, \delta^{(D)}(P)] = \frac{1}{2}\, S^{(0)}\,(q\cdot\partial_P)^2\delta^{(D)}(P) + q\cdot\partial_P\delta^{(D)}(P)\, S^{(1)} + \chi'\delta^{(D)}(P)$$

- $\Rightarrow$ ∂_{p_a} terms in $S^{(j)}$ are constrained by lower order $S^{(j'<j)}$ ops.
- Moreover, it turns out that $\chi = \chi' = 0$

Consistency condition [Broedel, de Leeuw, JP, Rosso]

Relation at leading orders: $P = \sum_{a=1}^{n} p_a$

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- $\Rightarrow$ ∂_{p_a} terms in $S^{(j)}$ are constrained by lower order $S^{(j'<j)}$ ops.
- Moreover, it turns out that $\chi = \chi' = 0$

Constraining subleading soft theorems I

Collect all known constraints on soft operators:

$$\boxed{\mathcal{A}_{n+1}(\delta\, q, E, \{E_a, p_a\}) \underset{\delta\to 0}{=} S^{[j]}(\delta\, q, E, \{E_a, p_a, \partial_{E_a}, \partial_{p_a}\}) \cdot \mathcal{A}_n(\{E_a, p_a\}) + \mathcal{O}(\delta^j)}$$

- Gauge invariances:

i) Soft leg: Invariance of $\mathcal{A}_{n+1}$ under shift $E_\mu \to E_\mu + q_\mu$: $\boxed{q \cdot \frac{\partial}{\partial E} S^{[j]} \sim 0}$

where $\sim$ indicates modulo Poincaré transformations

$$P^\mu := \sum_{a=1}^{n} p_a^\mu \qquad J^{\mu\nu} = \sum_{a=1}^{n} p_a^\mu \partial_{p_a^\nu} + E_a^\mu \partial_{E_a^\nu} - \mu \leftrightarrow \nu \qquad \text{as } (P^\mu, J^{\mu\nu})\mathcal{A}_n = 0$$

ii) Hard leg: As $p_a \cdot \frac{\partial}{\partial E} \mathcal{A}_n = 0$ we have $\boxed{p_a \cdot \frac{\partial}{\partial E_a} S^{[j]} \sim 0}$

Constraining subleading soft theorems II

$$\boxed{\mathcal{A}_{n+1}(\delta\, q, E, \{E_a, p_a\}) \underset{\delta\to 0}{=} S^{[j]}(\delta\, q, E, \{E_a, p_a, \partial_{E_a}, \partial_{p_a}\}) \cdot \mathcal{A}_n(\{E_a, p_a\}) + \mathcal{O}(\delta^j)}$$

- Distributional constraint: (as discussed)

$$S^{[j]}(\delta\, q)\, \delta^{(D)}(\sum_a p_a) = \delta^{(D)}(\delta\, q + \sum_a p_a)\, \tilde{S}^{[j]}(\delta\, q)$$

- Locality: $S^{(l)} = \sum_{a=1}^{n} S^{(l)}(q, E; E_a, p_a; \partial_{E_a}, \partial_{p_a})$

  "one leg at a time" as it would arise from a Ward identity. Is an assumption beyond tree-level

- Mass dimensions and loop counting:

$$D = 4: \qquad [g_{\mathsf{YM}}] = 0 \quad [\kappa] = -1 \quad [S^{[j]}_{\mathsf{YM}}] = -1 \quad [S^{[j]}_{\mathsf{G}}] = 0$$

Enforcing all constraints severely constrains the subleading soft functions!

Constraining subleading soft theorems II

$$\mathcal{A}_{n+1}(\delta\, q, E, \{E_a, p_a\}) \underset{\delta\to 0}{=} S^{[j]}(\delta\, q, E, \{E_a, p_a, \partial_{E_a}, \partial_{p_a}\}) \cdot \mathcal{A}_n(\{E_a, p_a\}) + \mathcal{O}(\delta^j)$$

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4D: Gauge theory

- Use spinor helicity: $q^\mu \to q^\alpha \tilde{q}^{\dot\alpha}$ & consider $(+)$ helicity soft gluon:
$E_\mu \to E^{(+)}_{\alpha\dot\alpha} = \dfrac{\mu_\alpha \, \tilde{q}_{\dot\alpha}}{\langle \mu q \rangle}$
- Ansatz: $S^{(1)}_{\mathsf{YM}} = \sum_{a=1}^{n} E^{(+)}_{\alpha\dot\alpha} \left[\Omega_a^{\alpha\dot\alpha\beta} \dfrac{\partial}{\partial \lambda_a^\beta} + \bar{\Omega}_a^{\alpha\dot\alpha\dot\beta} \dfrac{\partial}{\partial \tilde{\lambda}_a^{\dot\beta}} \right]$

$$\Omega_a^{\alpha\dot\alpha\beta} = \frac{c_1^{(a)}}{\langle a\, q\rangle [a\, q]} \lambda_a^\alpha \lambda_a^\beta \tilde{\lambda}_a^{\dot\alpha} \,,$$
$$\bar{\Omega}_a^{\alpha\dot\alpha\dot\beta} = \frac{\bar{c}_1^{(a)}}{\langle a\, q\rangle [a\, q]} \lambda_a^\alpha \tilde{\lambda}_a^{\dot\alpha} \tilde{\lambda}_a^{\dot\beta} + \frac{\bar{c}_2^{(a)}}{\langle a\, q\rangle [a\, q]} \lambda_q^\alpha \tilde{\lambda}_a^{\dot\alpha} \tilde{\lambda}_q^{\dot\beta} + \frac{\bar{c}_3^{(a)}}{\langle a\, q\rangle} \lambda_q^\alpha \delta^{\dot\alpha\dot\beta} \,.$$

(Locality, linear in $E^{(+)}$, first order in ∂_a and $\partial_{\dot\alpha}$, little-group scaling)

- Constraints: Gauge invariance $\mu_\alpha \to \mu_\alpha + \eta\, q_\alpha$

$$S^{(1)}_{\mathsf{YM}}[E_q \to q] = -\sum_{a=1}^{n} \left[c_1^{(a)} \, \lambda_a^\beta \frac{\partial}{\partial \lambda_a^\beta} + \bar{c}_1^{(a)} \, \tilde{\lambda}_a^{\dot\alpha} \frac{\partial}{\partial \tilde{\lambda}_a^{\dot\alpha}} \right] \qquad \Rightarrow c_1^{(a)} = \bar{c}_1^{(a)} = c$$

4D: Gauge theory II

- Distributional constraint:

$$\sum_{a=1}^{n}\left[2c\frac{\langle\mu\, a\rangle}{\langle a\, q\rangle\langle\mu\, q\rangle}\lambda_a^{\alpha}\tilde{\lambda}_a^{\dot\alpha}+(\bar{c}_2^{(a)}+\bar{c}_3^{(a)})\frac{1}{\langle a\, q\rangle}\lambda_a^{\alpha}\tilde{q}^{\dot\alpha}\right]\frac{\partial}{\partial P^{\alpha\dot\alpha}}\delta^4(P)$$

$$\overset{!}{=}\underbrace{\frac{\langle n\, 1\rangle}{\langle n\, q\rangle\,\langle q\, 1\rangle}}_{S_{\mathsf{YM}}^{(0)}}\left(q^{\alpha}\tilde{q}^{\dot\alpha}\frac{\partial}{\partial P^{\alpha\dot\alpha}}\delta^4(P)\right)+\chi\,\delta^4(P)$$

Hence $c=\chi=0$ and $\bar{c}_2^{(a)}+\bar{c}_3^{(a)}=\begin{cases}1 \text{ for } a=1,n\\ 0 \text{ otherwise}\end{cases}$ using Schouten identity

- The unique result for subleading soft operator is

$$\boxed{S_{\mathsf{YM}}^{(0)}=\frac{\langle n\, 1\rangle}{\langle n\, q\rangle\,\langle q\, 1\rangle}}\quad\overset{\text{locality \& consistency}}{\Rightarrow}\quad\boxed{S_{\mathsf{YM}}^{(1)}=\frac{[\tilde{q}\tilde{\partial}_1]}{\langle q1\rangle}-\frac{[\tilde{q}\tilde{\partial}_n]}{\langle qn\rangle}}$$

- N.B: Does not prove the existence of subleading soft thm, but says that if a sub-leading universal soft factorization holds, it must be of this form.

4D: Gravity

Plus helicity soft graviton: $$\boxed{S_{\mathsf{G}}^{(0)} = \sum_{a=1}^{n} \frac{\langle xa\rangle \, \langle ya\rangle \, [qa]}{\langle xq\rangle \, \langle yq\rangle \, \langle aq\rangle}}$$ x & y reference spinors

- Analogous arguments: Local, first order ansatz

$$S_{\mathsf{G}}^{(1)} = \sum_{a=1}^{n} E^{(+)}_{\alpha\dot{\alpha}\beta\dot{\beta}} \left[\Omega_a^{\alpha\dot{\alpha}\beta\dot{\beta}\gamma} \frac{\partial}{\partial \lambda_a^{\gamma}} + \bar{\Omega}_a^{\alpha\dot{\alpha}\beta\dot{\beta}\dot{\gamma}} \frac{\partial}{\partial \tilde{\lambda}_a^{\dot{\gamma}}} \right]$$

  Ω_a & $\bar{\Omega}_a$ contain 4 local constants

- Again constraints (gauge invariance & distributional constraint) nail down subleading operator completely:

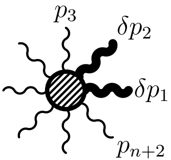
$$\Rightarrow \quad \boxed{S_{\mathsf{G}}^{(1)} = \frac{1}{2} \sum_{a=1}^{n} \frac{[a\, q]}{\langle a\, q\rangle} \left(\frac{\langle a\, x\rangle}{\langle q\, x\rangle} + \frac{\langle a\, y\rangle}{\langle q\, y\rangle} \right) [\tilde{q}\, \tilde{\partial}_a]}$$

- Same reasoning also fixes sub-subleading soft operator $S_{\mathsf{G}}^{(2)}$ in 4d.

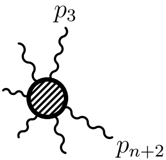
Summary

- Soft gluon & graviton emission displays universal factorization also at subleading order
- Claimed connection of leading and subleading soft gravtion theorems to extended BMS symmetry
- Rather elementary considerations strongly constrain subleading soft theorems:
  - YM: 1 free constant at subleading level
  - GR (tree): 2 free constants at subleading level, 3 at sub-subleading
- Constraining soft theorems @ loop-level: [Broedel,de Leeuw,JP,Rosso]
  - IR-divergent contributions: $S^{(0)}_{\mathsf{YM}}$, $S^{(1)}_{\mathsf{G}}$, $S^{(2)}_{\mathsf{G}}$ corrected
  - IR-finite factorized contributions: $S^{(1)}_{\mathsf{YM}}$ and $S^{(2)}_{\mathsf{G}}$ corrected (one-loop exact), but strongly constrained by our methods
  - IR-finite non-universal contributions: Open.

Double Soft Limits



$$\xrightarrow{\delta\to 0} \left\{ \begin{array}{c} \mathtt{CSL}(1,2,\{\mathtt{p_a}\}) \\ \mathtt{DSL}(1,2,\{\mathtt{p_a}\}) \end{array} \right\} \cdot$$



Motivation

- Soft behavior of S-matrix connected to symmetries $\Rightarrow$ potential for discovery of hidden symmetries of quantum gravity or YM S-matrix
- Soft scalar limits for massless Goldstone bosons of spontaneously broken symmetry

$$\lim_{\delta\to 0} \mathcal{A}_{n+1}(\phi^i(\delta q_1), 2, \ldots n+1) = 0 \qquad \text{[Adler]}$$

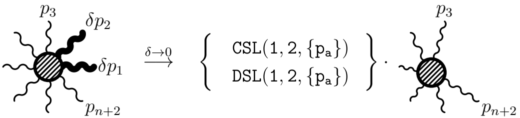
$$\lim_{\delta\to 0} \mathcal{A}_{n+2}(\phi^i(\delta q_1), \phi^j(\delta q_2), 3, \ldots n+2) = \sum_{a=3}^{n+2} \frac{p_a\cdot(q_1-q_2)}{p_a\cdot(q_1+q_2)} f^{ijk}\hat{T}_k \mathcal{A}_n(3, \ldots n+2)$$

  Symmetry algebra from double soft limit [Arkani-Hamed,Cachazo,Kaplan]
  Examples: Soft pions, Hidden $E_{7(7)}$ symmetry in $\mathcal{N}=8$ SUGRA
- Related works:
  - Scalars & fermions in $\mathcal{N}<8$ SUGRAs [Chu,Huang,Wen]
  - Scalars & photons in DBI, Galileon, Einstein-Maxwell-Scalar and NLSM [Cachazo,He,Yuan]
  - Double soft gluons [Volovich,Wen,Zlotnikov; Georgiou] from string theory [Di Vecchia,Marotta,Mojaza]

Ambiguities in taking a double soft limit

As single soft limit is non-vanishing for spin 1 & 2 double soft limit not unique.

There exist two natural ways:



- Consecutive soft limit:

$$\mathtt{CSL}(1,2)\mathcal{A}_n(3,\ldots,n+2) = \lim_{\delta_1\to 0}\lim_{\delta_2\to 0}\mathcal{A}_{n+2}(\delta_1\, q_1, \delta_2\, q_2, 3, \ldots, n+2)\Big|_{\delta_1=\delta_2=\delta}$$

ambiguity reflected in non-vanishing commutator:

$$\mathtt{aCSL}(1,2)\mathcal{A}_n(3,\ldots,n+2) = \tfrac{1}{2}\big[\lim_{\delta_1\to 0}, \lim_{\delta_2\to 0}\big]\,\mathcal{A}_{n+2}(\delta_1\, q_1, \delta_2\, q_2, 3, \ldots, n+2)\Big|_{\delta_1=\delta_2=\delta}$$

- Simultaneous soft limit: $\delta_1 = \delta_2 = \delta$

$$\mathtt{DSL}(1,2)\mathcal{A}_n(3,\ldots,n+2) = \lim_{\delta\to 0}\mathcal{A}_{n+2}(\delta\, q_1, \delta\, q_2, 3, \ldots, n+2)$$

This version used in scalar scenarios so far as there typically $\mathtt{CSL}(1,2) = 0$

Subleading double-soft functions: Results

Both double-soft functions diverge as $\frac{1}{\delta^2}$ at leading order

$$\mathtt{CSL}(1,2) = \sum_{i=0}^{I} \delta^{i-2}\,\mathtt{CSL}^{(i)}(1,2) \quad \text{and} \quad \mathtt{DSL}(1,2) = \sum_{i=0}^{I} \delta^{i-2}\,\mathtt{DSL}^{(i)}(1,2)$$

We have shown that universality extends at least to subleading order $I = 1$

Interesting to compare the two double soft limits:

- Same helicities of 1 & 2:

$$\mathtt{CSL}^{(0)}(1^h, 2^h) = \mathtt{DSL}^{(0)}(1^h, 2^h)$$

$$\mathtt{CSL}^{(1)}_{\mathsf{G}}(1^h, 2^h) = \mathtt{DSL}^{(1)}_{\mathsf{G}}(1^h, 2^h) \quad \text{but} \quad \mathtt{CSL}^{(1)}_{\mathsf{YM}}(1^h, 2^h) \neq \mathtt{DSL}^{(1)}_{\mathsf{YM}}(1^h, 2^h)$$

- Opposite helicities of 1 & 2:

$$\mathtt{CSL}^{(0)}_{\mathsf{G}}(1^h, 2^{\bar{h}}) = \mathtt{DSL}^{(0)}_{\mathsf{G}}(1^h, 2^{\bar{h}}) \quad \text{but} \quad \mathtt{CSL}^{(0)}_{\mathsf{YM}}(1^h, 2^{\bar{h}}) \neq \mathtt{DSL}^{(0)}_{\mathsf{YM}}(1^h, 2^{\bar{h}})$$

$$\mathtt{CSL}^{(1)}_{\mathsf{G}}(1^h, 2^{\bar{h}}) \neq \mathtt{DSL}^{(1)}_{\mathsf{G}}(1^h, 2^{\bar{h}}) \quad \text{and} \quad \mathtt{CSL}^{(1)}_{\mathsf{YM}}(1^h, 2^{\bar{h}}) \neq \mathtt{DSL}^{(1)}_{\mathsf{YM}}(1^h, 2^{\bar{h}})$$

Basis for (potential) extraction of $\mathfrak{bms}_4$ algebra.

Consecutive double soft limit: General structure

Consecutive double soft limit functions $\texttt{CSL}^{(i)}(1^{h_1},2^{h_2})$ follow from concatenation of single soft functions

$$\begin{aligned}\texttt{CSL}(1^{h_1},2^{h_2})A_{n-2}(3,\ldots,n) &:= \lim_{\delta_1\to 0}\lim_{\delta_2\to 0} A_n(\delta_1 q_1^{h_1},\delta_2 q_2^{h_2},3,\ldots,n)\\ &= S^{[1]}(\delta_2\, q_2^{h_2},\{1,3,\ldots,n\})\, S^{[1]}(\delta_1\, q_1^{h_1},\{3,\ldots,n\})\, A_{n-2}(3,\ldots,n)\end{aligned}$$

The first two orders:

$$\begin{aligned}\texttt{CSL}^{(0)}(1^{h_1},2^{h_2}) &= \frac{1}{\delta^2}\, S^{(0)}(q_2^{h_2},\{1,3,\ldots,n\})\, S^{(0)}(q_1^{h_1},\{3,\ldots,n\})\\ \texttt{CSL}^{(1)}(1^{h_1},2^{h_2}) &= \frac{1}{\delta}\Big(S^{(0)}(q_2^{h_2},\{1,3,\ldots,n\})\, S^{(1)}(q_1^{h_1},\{3,\ldots,n\})\\ &\quad + S^{(0)}(q_1^{h_1},\{3,\ldots,n\})S^{(1)}(q_2^{h_2},\{1,3,\ldots,n\})\\ &\quad + [\, S^{(1)}(q_2;\{1]\}), S^{(0)}(q_1)\,]\Big) \qquad \Leftarrow \text{contact term}\end{aligned}$$

Really nothing "new": Structure completely determined by single soft functions $S^{(j)}$.

Consecutive double soft limit: Color ordered gluons

- Leading order

$$\mathtt{CSL}^{(0)}(n,1^+,2^+,3)=\frac{\langle n3\rangle}{\langle n\,1\rangle\langle 12\rangle\langle 23\rangle}\qquad \mathtt{aCSL}^{(0)}(n,1^+,2^+,3)=0$$

$$\mathtt{CSL}^{(0)}(n,1^+,2^-,3)=\frac{\langle n\,3\rangle}{\langle n\,1\rangle[12][23]}\frac{[13]}{\langle 13\rangle}\qquad \mathtt{aCSL}^{(0)}(n,1^+,2^-,3)\neq 0$$

- Sub-leading order same helicity:

$$\mathtt{aCSL}^{(1)}(n,1^+,2^+,3)=\frac{1}{2\langle 12\rangle}\left[\left(\frac{\tilde\lambda_1^{\dot\alpha}}{\langle 23\rangle}-\frac{\tilde\lambda_2^{\dot\alpha}}{\langle 13\rangle}\right)\frac{\partial}{\partial\tilde\lambda_3^{\dot\alpha}}-\left(\frac{\tilde\lambda_1^{\dot\alpha}}{\langle 2\,n\rangle}-\frac{\tilde\lambda_2^{\dot\alpha}}{\langle 1\,n\rangle}\right)\frac{\partial}{\partial\tilde\lambda_n^{\dot\alpha}}\right]$$

- Sub-leading order opposite helicity:

$$\begin{aligned}\mathtt{aCSL}^{(1)}(n,1^+,2^-,3)=&\frac{1}{2}\frac{1}{\langle 13\rangle^2}\frac{\langle 23\rangle}{[23]}-\frac{1}{2}\frac{1}{[n\,2]^2}\frac{[n\,1]}{\langle n\,1\rangle}\\&+\frac{1}{2}\frac{\tilde\lambda_1^{\dot\alpha}}{[12]}\left(\frac{1}{[n\,2]}\frac{[n\,1]}{\langle n\,1\rangle}\frac{\partial}{\partial\tilde\lambda_n^{\dot\alpha}}+\frac{1}{[23]}\frac{[13]}{\langle 13\rangle}\frac{\partial}{\partial\tilde\lambda_3^{\dot\alpha}}\right)\\&-\frac{1}{2}\frac{\lambda_2^{\alpha}}{\langle 12\rangle}\left(\frac{1}{\langle n\,1\rangle}\frac{\langle n\,2\rangle}{[n\,2]}\frac{\partial}{\partial\lambda_n^{\alpha}}+\frac{1}{\langle 13\rangle}\frac{\langle 23\rangle}{[23]}\frac{\partial}{\partial\lambda_3^{\alpha}}\right).\end{aligned}$$

Consecutive double soft limit: Gravitons

- Leading order

$$\texttt{CSL}^{(0)}(1^+,2^+) = \frac{1}{\langle 12\rangle^4}\sum_{a,b\neq 1,2}\frac{[2a][1b]}{\langle 2a\rangle\langle 1b\rangle}\langle 1a\rangle^2\langle 2b\rangle^2 \qquad \texttt{aCSL}^{(0)}(1^+,2^+) = 0$$

$$\texttt{CSL}^{(0)}(1^+,2^-) = \frac{1}{\langle 12\rangle^2[12]^2}\sum_{a,b\neq 1,2}\frac{\langle 2a\rangle[1b]}{[2a]\langle 1b\rangle}[1a]^2\langle 2b\rangle^2 \qquad \texttt{aCSL}^{(0)}(1^+,2^-) = 0$$

- Sub-leading order same helicity:

$$\texttt{CSL}^{(1)}(1^+,2^+) = \frac{1}{\langle 12\rangle^3}\sum_{a,b\neq 1,2}\frac{[2a][1b]}{\langle 2a\rangle\langle 1b\rangle}\langle 1a\rangle\langle 2b\rangle\left[\langle 2b\rangle\tilde\lambda_2^{\dot\alpha}\frac{\partial}{\partial\tilde\lambda_a^{\dot\alpha}} - \langle 1a\rangle\tilde\lambda_1^{\dot\alpha}\frac{\partial}{\partial\tilde\lambda_b^{\dot\alpha}}\right]$$

- Sub-leading order opposite helicity:

$$\texttt{aCSL}^{(1)}(1^+,2^-) = \frac{1}{2\langle 12\rangle[12]}\sum_{a\neq 1,2}\frac{[1a]^2\langle 2a\rangle^2}{\langle 1a\rangle^2[2a]^2}\langle a|q_{1\bar 2}|a] \qquad \text{(local!)}$$

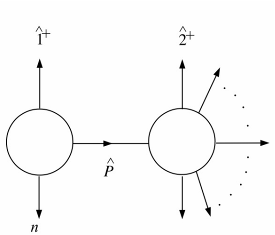
$$\texttt{sCSL}^{(1)}(1^+,2^-) = \frac{1}{2\langle 12\rangle[12]}\sum_{a\neq 1,2}\frac{[1a]^3\langle 2a\rangle^3}{\langle 1a\rangle[2a]}\left[\frac{1}{\langle a1\rangle[1a]}\left(1-\frac{\langle a2\rangle[2a]}{\langle a1\rangle[1a]}\right)+\frac{1}{\langle a2\rangle[2a]}\left(1-\frac{\langle a1\rangle[1a]}{\langle a2\rangle[2a]}\right)\right]$$

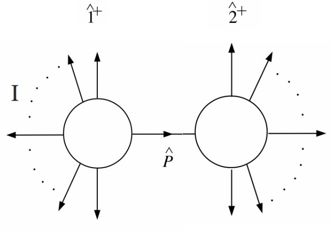
$$+\frac{1}{\langle 12\rangle^2[12]}\sum_{a,b\neq 1,2}\frac{\langle 2a\rangle[1b]}{[2a]\langle 1b\rangle}\left[\langle 2b\rangle^2[1a]\lambda_2^\alpha\frac{\partial}{\partial\lambda_a^\alpha} - \langle 1a\rangle^2[2b]\lambda_1^\alpha\frac{\partial}{\partial\lambda_b^\alpha}\right] \qquad \text{(sym. combination)}$$

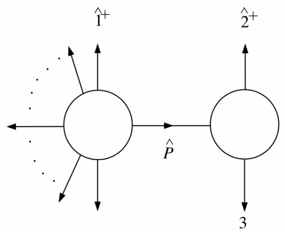
Simultaneous double soft limit from BCFW

Simultaneous double soft limit: $\boxed{\lambda_{1,2} \to \sqrt{\delta}\,\lambda_{1,2}\,,\, \tilde{\lambda}_{1,2} \to \sqrt{\delta}\,\tilde{\lambda}_{1,2}}$

Natural to consider a $\langle 12]$ shift: $\hat{\lambda}_1 = \lambda_1 + z\lambda_2 \qquad \hat{\tilde{\lambda}}_2 = \tilde{\lambda}_2 - z\tilde{\lambda}_1$







In generic (middle) situation the shift turns a soft leg into a hard leg as

$$z = -\frac{P_I^2 + \langle 1|P_I|1]\,\delta}{\delta\,\langle 2|P_I|1]} \sim \begin{cases} \frac{1}{\delta} & \text{for } P_I^2 \neq 0 \\ 1 & \text{for } P_I^2 = p_n^2 = p_3^2 = 0 \end{cases}$$

Simultaneous double soft limit from BCFW

Simultaneous double soft limit: $\lambda_{1,2} \to \sqrt{\delta}\,\lambda_{1,2}\,,\tilde{\lambda}_{1,2} \to \sqrt{\delta}\,\tilde{\lambda}_{1,2}$

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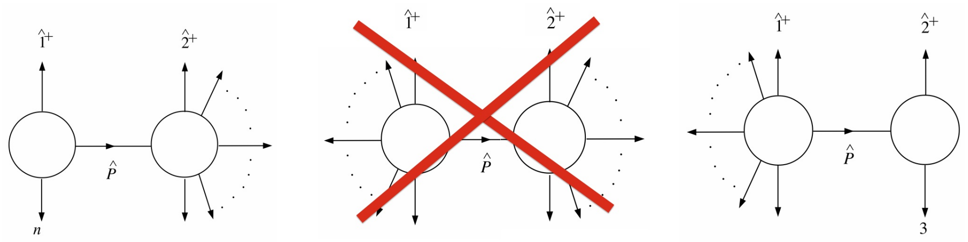

In generic (middle) situation the shift turns a soft leg into a hard leg as

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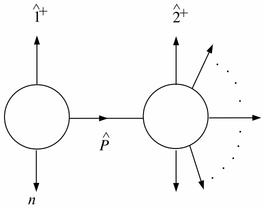
$\to$ At leading and sub-leading order only three-point factorized diagrams contribute! Origin of factorization and universality.

Simultaneous double soft limit: Gluons 1^+2^+

For same helicity gluons only one BCFW-diagram contributes:



- Leading order:

$$\mathtt{DSL}^{(0)}(n+2,1^+,2^+,3)=\frac{\langle n\,3\rangle}{\langle n1\rangle\langle 12\rangle\langle 23\rangle}=S^{(0)}(n,1^+,2)\;S^{(0)}(n,2^+,3)$$

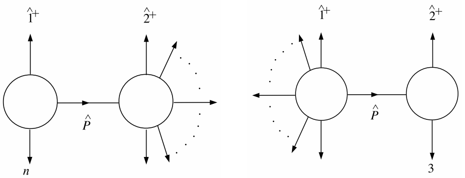
- Sub-leading order

$$\begin{aligned}\mathtt{DSL}^{(1)}(n,1^+,2^+,3)&=S^{(0)}(n,1^+,2)S^{(1)}(n,2^+,3)+S^{(0)}(1,2^+,3)S^{(1)}(n,1^+,3)\\&=-\frac{\langle n\,2\rangle}{\langle n\,1\rangle\langle 12\rangle}\left(\frac{1}{\langle 23\rangle}\tilde{\lambda}_2^{\dot\alpha}\frac{\partial}{\partial\tilde{\lambda}_3^{\dot\alpha}}+\frac{1}{\langle n\,2\rangle}\tilde{\lambda}_2^{\dot\alpha}\frac{\partial}{\partial\tilde{\lambda}_n^{\dot\alpha}}\right)\\&\quad-\frac{\langle 13\rangle}{\langle 12\rangle\langle 23\rangle}\left(\frac{1}{\langle 13\rangle}\tilde{\lambda}_1^{\dot\alpha}\frac{\partial}{\partial\tilde{\lambda}_3^{\dot\alpha}}+\frac{1}{\langle n\,1\rangle}\tilde{\lambda}_1^{\dot\alpha}\frac{\partial}{\partial\tilde{\lambda}_n^{\dot\alpha}}\right)\neq\mathtt{CSL}^{(1)}(+,+)\end{aligned}$$

Vanishing contact term

Simultaneous double soft limit: Gluons 1^+2^-

For mixed helicities now both BCFW-diagrams contribute:



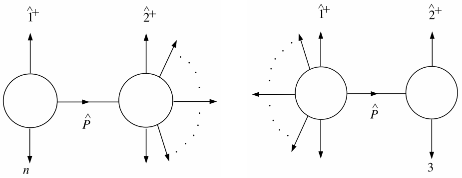
- Leading order:

$$\mathtt{DSL}^{(0)}(n,1^+,2^-,3) = S^{(0)}(1^+)\,S^{(0)}(\hat{2}^-) \;+\; S^{(0)}(2^-)\,S^{(0)}(\hat{1}^+)$$
$$= \frac{1}{\langle n|\,q_{12}\,|3]}\left[\frac{1}{2p_n\cdot\,q_{12}}\,\frac{[n\,3]\,\langle n\,2\rangle^3}{\langle 12\rangle\langle n\,1\rangle} \;-\; \frac{1}{2p_3\cdot q_{12}}\,\langle n\,3\rangle\,\frac{[31]^3}{[12][23]}\right]$$

"Non-local" structure: Hard particles are entangled.

Simultaneous double soft limit: Gluons 1^+2^-

For mixed helicities now both BCFW-diagrams contribute:



- Sub-leading order

$$
\begin{aligned}
\mathtt{DSL}^{(1)}(n,1^+,2^-,3) &= S^{(0)}(n,1^+,2)S^{(1)}(n,2^-,3) + S^{(0)}(3,2^-,1)S^{(1)}(n,1^+,3) \\
&+ \frac{\langle 23\rangle[13]}{[32]\langle 12\rangle}\frac{1}{(2p_3\cdot q_{12})}\lambda_2^{\alpha}\frac{\partial}{\partial\lambda_3^{\alpha}} + \frac{\langle n\,2\rangle[2\,n]}{[n\,1]\langle 12\rangle}\frac{1}{(2p_n\cdot q_{12})}\lambda_2^{\alpha}\frac{\partial}{\partial\lambda_n^{\alpha}} \\
&+ \frac{[n\,1]\langle 2n\rangle}{\langle 1\,n\rangle[21]}\frac{1}{(2p_n\cdot q_{12})}\tilde\lambda_1^{\dot\alpha}\frac{\partial}{\partial\tilde\lambda_n^{\dot\alpha}} + \frac{[31]\langle 32\rangle}{\langle 13\rangle[21]}\frac{1}{(2p_3\cdot q_{12})}\tilde\lambda_1^{\dot\alpha}\frac{\partial}{\partial\tilde\lambda_3^{\dot\alpha}} \\
&+ \mathtt{DSL}^{(1)}(n,1^+,2^-,3)|_c.
\end{aligned}
$$

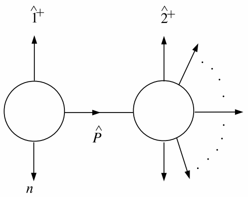
with contact term

$$
\mathtt{DSL}^{(1)}(n,1^+,2^-,3)|_c = \frac{\langle n\,2\rangle^2[1\,n]}{\langle n\,1\rangle}\frac{1}{(2p_n\cdot q_{12})^2} + \frac{[31]^2\langle 23\rangle}{[32]}\frac{1}{(2p_3\cdot q_{12})^2}
$$

Simultaneous double soft limit: Gravitons 1^+2^+

Moving to gravity:

Similar contributions as in gluonic case.
The other BCFW-diagrams vanish linearly in δ



- Leading order:

$$\mathtt{DSL}^{(0)}(1^+,2^+) = S^{(0)}(1^+)\,S^{(0)}(2^+)$$

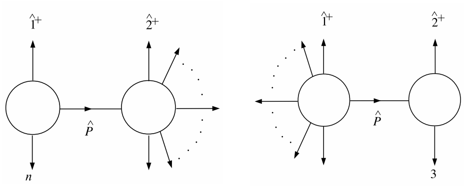
- Sub-leading order:

$$\begin{aligned}\mathtt{DSL}^{(1)}(1^+,2^+) = \frac{1}{\langle 12\rangle^3}\sum_{a,b\neq 1,2}\frac{[b1]\langle b2\rangle}{\langle 1b\rangle}\,\frac{\langle b|q_{12}|a]\,\langle 1a\rangle}{\langle 2a\rangle}\\ \left[\tilde{\lambda}_2^{\dot\alpha}\frac{\partial}{\partial\tilde{\lambda}_a^{\dot\alpha}} + \frac{\langle 1b\rangle}{\langle 2b\rangle}\tilde{\lambda}_1^{\dot\alpha}\frac{\partial}{\partial\tilde{\lambda}_a^{\dot\alpha}} - \frac{\langle 1a\rangle}{\langle 2b\rangle}\tilde{\lambda}_1^{\dot\alpha}\frac{\partial}{\partial\tilde{\lambda}_b^{\dot\alpha}}\right]\\ = S^{(0)}(1^+)\,S^{(1)}(2^+) + S^{(0)}(2^+)\,S^{(1)}(1^+)\end{aligned}$$

No contact term! Results identical to $\mathtt{CSL}(1^+,2^+)$.

Simultaneous double soft limit: Gravitons 1^+2^-

For mixed helicities again both BCFW-diagrams contribute:



- Leading order:

$$\mathtt{DSL}^{(0)}(1^+, 2^+) = S^{(0)}(1^+)\, S^{(0)}(2^-)$$

- Sub-leading order: (contact and non-contact terms)

$$\mathtt{DSL}^{(1)}(1^+, 2^-)|_{nc} = \frac{1}{q_{12}^4} \sum_{a,b\neq 1,2} \frac{[1a]^2\,[1b]\,\langle 2a\rangle\,\langle 2b\rangle^2}{\langle b1\rangle\,[2a]} \left(\frac{[12]}{[1a]} \lambda_2^\alpha \frac{\partial}{\partial\lambda_a^\alpha} - \frac{\langle 12\rangle}{\langle 2b\rangle} \tilde{\lambda}_1^{\dot\alpha} \frac{\partial}{\partial\tilde{\lambda}_b^{\dot\alpha}} \right)$$

$$= S^{(0)}(1^+)\, S^{(1)}(2^-) + S^{(0)}(2^-)\, S^{(1)}(1^+)\,.$$

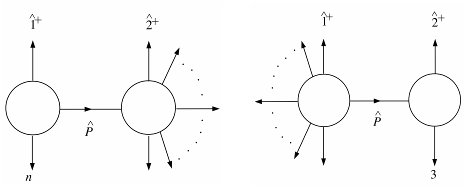
$$\mathtt{DSL}^{(1)}(1^+, 2^-)|_c = \frac{1}{q_{12}^2} \sum_{b\neq 1,2} \frac{[1b]^3\,\langle 2b\rangle^3}{[2b]\,\langle 1b\rangle} \frac{1}{2p_b\cdot q_{12}}$$

$\Leftarrow$ Difference to $\mathtt{CSL}(1^+, 2^+)$

Gravity looks simpler than gauge theory!

Simultaneous double soft limit: Gravitons 1^+2^-

For mixed helicities again both BCFW-diagrams contribute:



- Leading order:

$$\mathtt{DSL}^{(0)}(1^+,2^+) = S^{(0)}(1^+)\,S^{(0)}(2^-)$$

- Sub-leading order: (contact and non-contact terms)

$$\mathtt{DSL}^{(1)}(1^+,2^-)|_{nc} = \frac{1}{q_{12}^4}\sum_{a,b\neq 1,2}\frac{[1a]^2\,[1b]\,\langle 2a\rangle\,\langle 2b\rangle^2}{\langle b1\rangle\,[2a]}\left(\frac{[12]}{[1a]}\,\lambda_2^\alpha\,\frac{\partial}{\partial\lambda_a^\alpha} - \frac{\langle 12\rangle}{\langle 2b\rangle}\,\tilde{\lambda}_1^{\dot\alpha}\,\frac{\partial}{\partial\tilde{\lambda}_b^{\dot\alpha}}\right)$$

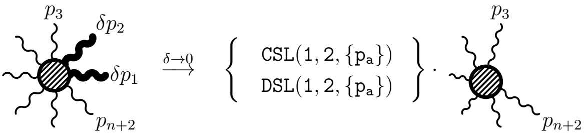
$$= S^{(0)}(1^+)\,S^{(1)}(2^-) + S^{(0)}(2^-)\,S^{(1)}(1^+)\,.$$

$$\mathtt{DSL}^{(1)}(1^+,2^-)|_{c} = \frac{1}{q_{12}^2}\sum_{b\neq 1,2}\frac{[1b]^3\,\langle 2b\rangle^3}{[2b]\,\langle 1b\rangle}\,\frac{1}{2p_b\cdot q_{12}}$$

$\Leftarrow$ Difference to $\mathtt{CSL}(1^+,2^+)$

Gravity looks simpler than gauge theory!

Summary: Double soft graviton and gluon limits



- Introduced two natural ways of taking double soft limit: Consecutive CSL and simultaneous DSL limits.
- Factorization & universality extends to the subleading order $\mathcal{O}(\frac{1}{\delta})$
- Depending on helicities of soft legs (same/different, gluons/gravitons) CSL and DSL agree or differ.
- Generically double soft gravity looks simpler than double soft gauge theory!

Outlook

- Multiple soft limits and the emergence of the $\mathfrak{bms}_4$ or Kac-Moody algebras from double soft amplitudes?
  Obstacle: Generic non-locality of $\mathtt{CSL}$ and $\mathtt{DSL}$.
- Are the $\mathtt{CSL}^{(1)}$ and $\mathtt{DSL}^{(1)}$ again determined by consistency from $\mathtt{CSL}^{(0)}$ and $\mathtt{DSL}^{(0)}$?
- Restate double soft gluons in non-color ordered form $\Rightarrow$ Nicer formulae
- Loop level structure?
- Multi soft limits?
- Possible application to speculative description of black hole formation as bound state of soft gravitons ("classicalization")? [Dvali,Gomez,Isermann,Lust,Stieberger]