New techniques for computing ϵ-expansions for amplitudes

based on Stieberger, Puhlfürst, arXiv:1507.01582

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Outline:

1. Generalized hypergeometric functions and the Drinfeld associator
2. Generalized hypergeometric functions and recurrence relations

Generalized hypergeometric functions and the Drinfeld associator

- Knizhnik-Zamolodchikov (KZ) eq. with Lie algebra generators e_0, e_1 for the generating series $\Phi(x)$ of MPL:

$$\frac{d}{dx}\Phi(x) = \left(\frac{e_0}{x} + \frac{e_1}{1-x}\right)\Phi(x)$$

- There are unique solutions $\Phi_0 \overset{x\to 0}{\longrightarrow} x^{e_0}$ and $\Phi_1 \overset{x\to 1}{\longrightarrow} (1-x)^{-e_1}$
- Combination of unique solutions gives the Drinfeld associator $\Phi_{KZ}(e_0, e_1) = \Phi_1(x)^{-1}\Phi_0(x)$

with the expansion $\displaystyle\sum_{w\in\{e_0,e_1\}^\times} \zeta(w)w = 1 + \zeta_2[e_0, e_1] + \zeta_3([e_0,[e_0,e_1]] - [e_1,[e_0,e_1]]) + \dots$

- Definition of generalized hypergeometric functions:

$$y = {}_pF_{p-1}\left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_{p-1}\end{matrix}; x\right] = \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{p}(a_i)^m}{\prod_{j=1}^{p-1}(b_j)^m}\frac{x^m}{m!}, \quad (a)^n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad p \geq 1, \ a_s, b_r \in \mathbb{R},$$

- They satisfy Fuchsian equations with polynomials α_i, β_j symmetric in a_s, b_r :

$$-(1-x)\,\frac{d}{dx}(\theta^{n-1}y) + \sum_{i=1}^{n-1}\left(\alpha_i - \frac{\beta_i}{x}\right)\theta^i y + \alpha_0\, y = 0 \quad , \quad \theta = x\frac{d}{dx}$$

Generalized hypergeometric functions and the Drinfeld associator

... this is equivalent to a system of first order differential eqs. for $\vec{g}^T = (y, \theta y, \ldots, \theta^{(n-1)} y)$:

$$\frac{d\vec{g}}{dx} = \left(\frac{B_0}{x} + \frac{B_1}{1-x} \right) \vec{g}$$

- We construct two fundamental solutions $\vec{g}_0, \vec{g}_1$ and compare their asymptotic behaviour

$$\begin{array}{l} \vec{g}_0 \overset{x\to 0}{\longrightarrow} \gamma_0 \\ \vec{g}_1 \overset{x\to 1}{\longrightarrow} \gamma_1 \end{array} \quad \text{with} \quad \begin{array}{c} \Phi_0 \overset{x\to 0}{\longrightarrow} x^{B_0} \\ \Phi_1 \overset{x\to 1}{\longrightarrow} (1-x)^{-B_1} \end{array} \Rightarrow \begin{array}{l} \vec{g}_0 = \Phi_0 C_0 \\ \vec{g}_1 = \Phi_1 C_1 \end{array}, \text{ with connection matrices } C_0, C_1$$

- Insertion in the def. of the Drinfeld associator: $\Phi_{KZ}(B_0, B_1) = C_1 g_1^{-1} g_0 C_0^{-1}$

- $x \to 1$ gives: ${}_pF_{p-1}\left[{a_1,\ldots,a_p \atop 1+b_1,\ldots,1+b_{p-1}} ; 1 \right] = \Phi_{KZ}(B_0, B_1)|_{1,1}$

e.g.: $${}_2F_1\left[{a, b \atop 1+c} ; 1 \right] = \Phi_{KZ}\left[\begin{pmatrix} 0 & 1 \\ 0 & -c \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ ab & a+b-c \end{pmatrix} \right] \Bigg|_{1,1}$$

$${}_3F_2\left[{a_1, a_2, a_3 \atop 1+b_1, 1+b_2} ; 1 \right] = \Phi_{KZ}\left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -b_1 b_2 & -b_1 - b_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 a_2 a_3 & a_1 a_2 + a_1 a_3 + a_2 a_3 - b_1 b_2 & a_1 + a_2 + a_3 - b_1 - b_2 \end{pmatrix} \right]_{1,1}$$

In general:

$$B_0^p = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & -Q_{p-1}^p & -Q_{p-2}^p & \dots & -Q_2^p & -Q_1^p \end{pmatrix} , \; B_1^p = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ \Delta_p^p & \Delta_{p-1}^p & \dots & \Delta_2^p & \Delta_1^p \end{pmatrix}$$

with

$$\Delta_\alpha^p = P_\alpha^p - Q_\alpha^p \quad , \quad \alpha = 1, \dots, p \; ,$$

and the symmetric products of the parameters $a_1, \dots, a_p$ and $b_1, \dots, b_{p-1}$, respectively:

$$P_\alpha^p = \sum_{\substack{i_1, \dots, i_\alpha = 1 \\ i_1 < i_2 < \dots < i_\alpha}}^{p} a_{i_1} \cdot \dots \cdot a_{i_\alpha} \quad , \quad \alpha = 1, \dots, p \; ,$$

$$Q_\beta^p = \sum_{\substack{i_1, \dots, i_\beta = 1 \\ i_1 < i_2 < \dots < i_\beta}}^{p-1} b_{i_1} \cdot \dots \cdot b_{i_\beta} \quad , \quad \beta = 1, \dots, p-1 \quad , \quad Q_p^p = 0 \; .$$

Generalized hypergeometric functions and recurrence relations

Recurrence relations with non-commutative coefficients c_i, ($c_a c_b \neq c_b c_a$):

$$u_k = \sum_{\alpha=1}^{n} c_\alpha u_{k-\alpha} \quad \text{for} \quad k \geq n \quad \text{and with initial values} \quad u_k \quad \text{for} \quad 0 \leq k < n$$

$$\begin{aligned}
\text{e.g.:} \quad u_k &= c_1 u_{k-1} + c_2 u_{k-2}; \quad u_0 = 1; \quad u_1 = c_1 \\
u_2 &= c_1 u_1 + c_2 u_0 = c_1^2 + c_2 \\
u_3 &= c_1 u_2 + c_2 u_1 = c_1^3 + c_1 c_2 + c_2 c_1 \\
&\vdots \\
u_k &= \sum_{j_1 + 2j_2 = k} \{c_1^{j_1}, c_2^{j_2}\}
\end{aligned}$$

- we introduce - $\{c_1^{j_1}, c_2^{j_2}, \ldots, c_n^{j_n}\}$ - sum of distinct permutations, with j_i times the factor c_i

$$\begin{aligned}
\text{e.g.:} \quad \{c_1^2, c_2\} &= c_1^2 c_2 + c_1 c_2 c_1 + c_2 c_1^2 \\
\{c_1, c_2, c_3\} &= c_1 c_2 c_3 + c_1 c_3 c_2 + c_2 c_1 c_3 + c_2 c_3 c_1 + c_3 c_1 c_2 + c_3 c_2 c_1
\end{aligned}$$

- related to the shuffle product

$$\left\{c_1^{j_1}, c_2^{j_2}, \ldots, c_n^{j_n}\right\} = \underbrace{c_1 \ldots c_1}_{j_1 \text{ times}} ⧢ \underbrace{c_2 \ldots c_2}_{j_2 \text{ times}} ⧢ \ldots ⧢ \underbrace{c_n \ldots c_n}_{j_n \text{ times}}.$$

- We are interested in the ϵ-expansion: [Kalmykov et al., 0708.0803] [Boels, 1304.7918]

$$ {}_pF_{p-1}\left(\begin{matrix} \epsilon a_1, \ldots, \epsilon a_p \\ 1+\epsilon b_1, \ldots, 1+\epsilon b_{p-1} \end{matrix}; x \right) = \sum_{k=0}^{\infty} \epsilon^k u_k^p(x) $$

- It can be written in terms of GPL

$$ \mathcal{L}i_{\vec{n}}(x) = I(0)^{n_1-1} I(1) I(0)^{n_2-1} I(1) \ldots I(0)^{n_d-1} I(1) $$

$$ \text{with} \quad I(1)f(x) = \int_0^x \mathrm{d}t\, \frac{1}{1-t} f(t) \quad \text{and} \quad I(0)f(x) = \int_0^x \mathrm{d}t\, \frac{1}{t} f(t) $$

- The differential equation

$$ (1-x)\theta^{p-1} u_k^p(x) = \sum_{\alpha=1}^{p} \left[P_\alpha^p - \frac{1}{x} Q_\alpha^p \right] \theta^{p-\alpha} u_{k-\alpha}(x), \quad \theta = x\frac{d}{dx} $$

can be transformed into a recurrence relation: (take b.c. into account)

$$ u_k^p(x) = \sum_{\alpha=1}^{p} c_\alpha^p u_{k-\alpha}^p(x); \quad k \geq p $$

with the coefficients

$$ c_i^p = \Delta_i^p \; I(0)^{p-1} \; I(1) \; \theta^{p-i} - Q_i^p \; I(0)^i $$

Generalized hypergeometric functions and recurrence relations

Already known:

$$
{}_2F_1\left(\begin{matrix}-\epsilon a, \ \epsilon b \\ 1+\epsilon b\end{matrix}; x\right) = \sum_{k\geq 0} \epsilon^k v_k(x)
$$

$$
v_k(x) = \sum_{\alpha=1}^{k-1} (-1)^{k+1} a^{k-\alpha} b^{\alpha} \underbrace{\sum_{\beta} (-1)^{\beta} I(0)\{I(1)^{k-\alpha-1-\beta}, I(0)^{\alpha-1-\beta}, I(1,0)^{\beta}\} I(1)}_{= I(0)^{\alpha} I(1)^{k-\alpha} = \mathcal{L}i_{(\alpha+1, \{1\}^{k-\alpha-1})}(x)}
$$

with $I(a_1, a_2, \ldots, a_n) = I(a_1) I(a_2) \ldots I(a_n), \quad a_i \in \{0, 1\}$

New result:

$$
{}_3F_2\left(\begin{matrix}\epsilon a_1, \ \epsilon a_2, \ \epsilon a_3 \\ 1+\epsilon b_1, \ 1+\epsilon b_2\end{matrix}; x\right) = \sum_{k\geq 0} \epsilon^k w_k(x)
$$

$$
w_k(x) = \sum_{m_1+l_1+2(l_2+m_2)+3m_3=k-3} (-1)^{l_1+l_2} \Delta_1^{m_1} \Delta_2^{m_2} \Delta_3^{m_3+1} Q_1^{l_1} Q_2^{l_2}
$$

$$
\times \underbrace{I(0,0)\left\{I(0)^{l_1}, I(0,0)^{l_2}, I(1)^{m_1}, I(1,0)^{m_2}, I(1,0,0)^{m_3}\right\} I(1)}
$$

How to transform these generalized operator products to GPLs?

$$I(0)\left\{I(0)^{j_1}, I(0,0)^{j_2}, I(1)^{j_3}, I(1,0)^{j_4}, I(1,0,0)^{j_5}\right\} I(1) = \sum_{\substack{w=j_1+2j_2+j_3+2j_4+3j_5+2\\ d=j_3+j_4+j_5+1}} \mathcal{L}i_{\vec{n}}(x)\,\omega(\vec{n}; j_2, j_4, j_5)$$

with the weighting

$$\omega(\vec{n}; j_x, j_y, j_z) = \sum_{\substack{\vec{\beta}+\vec{\gamma}\leq 1\\ |\vec{\beta}|=j_y;\ |\vec{\gamma}|=j_z;\ \beta_1,\gamma_1=0}} \sum_{\substack{\vec{\alpha}\leq\lfloor\vec{n}-\vec{\beta}-2\vec{\gamma}/2\rfloor\\ |\vec{\alpha}|=j}} \binom{\vec{n}-\vec{\beta}-2\vec{\gamma}-\vec{\alpha}}{\vec{\alpha}}$$

We found the expansion in terms of GPLs for general p as well:

$${}_pF_{p-1}\left({\epsilon a_1,\ldots,\epsilon a_p \atop 1+\epsilon b_1,\ldots,1+\epsilon b_{p-1}}; x\right) = \sum_{k=0}^{\infty} \epsilon^k u_k^p(x)$$

$$u_k^p(x) = \sum_{\vec{l},\vec{m}} (-1)^{|\vec{l}|} (\Delta_1^p)^{m_1} (\Delta_2^p)^{m_2} \ldots (\Delta_{p-1}^p)^{m_{p-1}} (\Delta_p^p)^{m_p+1} (Q_1^p)^{l_1} (Q_2^p)^{l_2} \ldots (Q_{p-1}^p)^{l_{p-1}}$$

$$\times \sum_{\substack{w=k;\ n_1\geq p\\ d=m_1+m_2+\ldots+m_p+1}} \mathcal{L}i_{\vec{n}}(x)\,\omega_p(\vec{n}; \vec{l}; \vec{m})$$

- open superstring amplitudes at disk level: [Mafra, Schlotterer, Stieberger; 2011]

$$A^{(N)}_{\text{string}} = F^{(N)}(\alpha')A^{(N)}_{\text{YM}}$$

$A^{(N)}_{\text{string}}$ – N-point string amplitudes

$A^{(N)}_{\text{YM}}$ – corresponding YM amplitudes

$F^{(N)}(\alpha')$ – generalized Euler integrals

- There are 2 independent gen. Euler integrals at $N = 5$, e.g. ($s_{i,j} = \alpha'(p_i + p_j)^2$, $s_i = s_{i,i+1}$):

$$F_2^{(5)} = s_{13}s_{24}\frac{F^{(4)}(s_1,s_2)F^{(4)}(s_3,s_4)}{(1+s_1+s_2)(1+s_3+s_4)}\;{}_3F_2\left(\begin{matrix}1+s_1,\ 1+s_4,\ 1-s_{24}\\ 2+s_1+s_2,\ 2+s_3+s_4\end{matrix};1\right)$$

- α'-expansion in terms of MZVs $\zeta(\vec{n}) = \mathcal{L}i_{\vec{n}}(1)$.
- With kinematical part and the MZVs (contained in $f(j_1,\ldots,j_5)$) separated, the all-order expansion reads:

$$F_2^{(5)} = s_{13}s_{24}\underbrace{\sum_{j_1,\ldots,j_5\geq 0} s_1^{j_1}s_2^{j_2}s_3^{j_3}s_4^{j_4}s_5^{j_5}f(j_1,j_2,j_3,j_4,j_5)}_{\text{invariant w.r.t. cyclic permutation of } (s_1,\ldots,s_5)}$$

$\Rightarrow$ The products of MZVs in $f(j_1,\ldots,j_5)$ are invariant w.r.t. cycl. perm. of $(j_1,\ldots,j_5)$.

Identities for MZVs

$$F^{(4)}(s,u) = 1 - \sum_{k=2}^{\infty}(-1)^k \sum_{\alpha=1}^{k-1} s^{k-\alpha} u^{\alpha} \zeta(\alpha+1, \{1\}^{k-\alpha-1})$$

- $F^{(4)}(s,u)$ is invariant w.r.t. $s \leftrightarrow u$, which generates $\zeta(a+1, \{1\}^{b-1}) = \zeta(b+1, \{1\}^{a-1})$.

$$f(j_1, j_2, j_3, j_4, j_5) = \sum_{\vec{l}} \zeta'(l_1+1, \{1\}^{l_2-1}) \zeta'(l_3+1, \{1\}^{l_4-1}) \sum_{w=|\vec{j}|-|\vec{l}|+2} \zeta(\vec{n})\, \Omega(\vec{n}; \vec{j}-\vec{l})$$

with $\zeta'(a+1, \{1\}^{b-1}) = \zeta(a+1, \{1\}^{b-1})$ for $a, b \geq 1$; -1 for $a, b = 0$ and 0 else.

- $f(j_1, \ldots, j_5)$ is symmetric w.r.t cycl. permutations of $(j_1, \ldots, j_5)$, which generates e.g.:

a generalization of the sum theorem: $\displaystyle\sum_{\substack{w=c \\ d=a;\ n_1>b}} \zeta(\vec{n}) = \sum_{\substack{w=c \\ d=b;\ n_1>a}} \zeta(\vec{n})$

also: $\displaystyle\zeta(j_2+j_3+2, \{1\}^{j_1}) = -\sum_{l_1, l_2} \zeta'(l_1+1, \{1\}^{l_2}) \sum_{w=k-l_1-l_2} \zeta(\vec{n}) \omega(j_1-l_1, j_2-l_2, d, d-j_5-1)$,

with $$\omega(j_x, j_y, \delta_x, \delta_y) = \binom{\delta_x}{\delta_y} \binom{j_x+j_y-2\delta_y}{j_x-\delta_y} {}_2F_1\left[\begin{matrix}\delta_y - j_x,\ \delta_y - \delta_x \\ 2\delta_y - j_x - j_y\end{matrix}; 1\right]$$

Conclusion

Summary

- Two methods are provided to straightforwardly obtain compact and explicit expressions for ϵ-expansions of generalized hypergeometric functions.
- We established a connection between the Drinfeld associator and generalized hypergeometric functions, which reduces the computation of higher orders in the expansion of these functions to simple matrix multiplications.
- We found the general solution of linear recurrence relations with constant non-commutative coefficients.
- Using this general solution, we obtained a closed and compact expression for any order in the ϵ-expansion of generalized hypergeometric functions, which does not rely on its lower orders to be computed in advance.
- The all-order expansions are written explicitly in terms of GPLs.
- We obtained new representations for the $N = 5$ open string amplitude.
- New general identities for MZVs can be extracted from the all-order results.

Outlook

- Can we apply our methods to obtain expansion of other functions? (Feynman integrals, string amplitudes, ...)
- Are there different representations for the weightings in the sums of GPLs/MZVs? (Hypergeometric functions, Fibonacci numbers, ...)