Amplitudes Conference

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Elliptic multiple zeta values and superstring one-loop amplitudes

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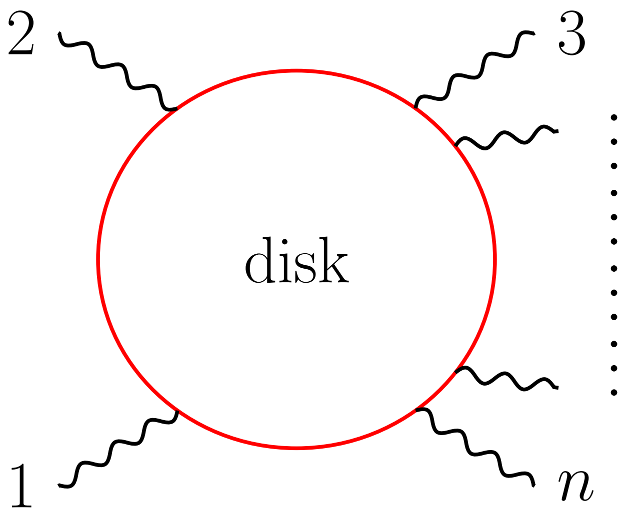
based on arXiv:1412.5535 & 1507.02254 with J. Brödel, C. Mafra, N. Matthes

09.07.2015

Motivation: Open string scattering amplitudes @ tree-level and one-loop

$\Longrightarrow$ iterated integrals over the worldsheet boundaries

genus zero

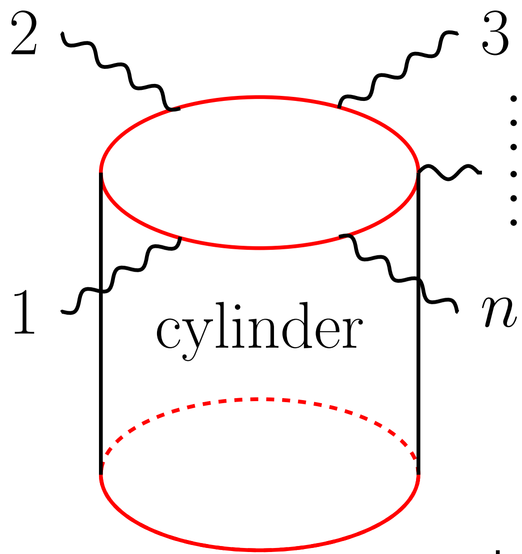


peel off YM trees $\downarrow$ expand in α'

$$\int\limits_{0\leq z_i\leq z_{i+1}\leq 1} \prod \frac{\mathrm{d}z_j}{z_j - n_j} = \text{MZVs}$$

[**earlier work**]

genus one



half a torus of modular parameter τ

$$\frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$$

$\downarrow$

$$\int\limits_{0\leq z_i\leq z_{i+1}\leq 1} \prod f^{(n_j)}(z_j)\mathrm{d}z_j = \text{eMZVs}$$

[**this talk**]

Outline

I. Definition of eMZVs and $f^{(n)}$

[Brown, Levin, Enriquez, . . .]

II. Superstring one-loop amplitudes

[Brödel, Mafra, Matthes, OS 1412.5535]

III. eMZV relations & indecomposables

[Brödel, Matthes, OS 1507.02254]

I. 1 Iterated integrals: genus-zero versus elliptic

Recall that multiple polylogarithms [Goncharov, see E. Panzer's talk]

$$G(a_1, a_2, \ldots, a_r; z) \equiv \int_0^z \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_r; t) \ , \qquad G(; z) \equiv 1$$

specialize to multiple zeta values (MZVs) upon $z \to 1$ and $a_j \to \{0, 1\}$:

$$\zeta(a_1, a_2, \ldots, a_r) \equiv (-1)^{\sum_{j=1}^r a_j} \int\limits_{0 \leq z_i \leq z_{i+1} \leq 1} \frac{\mathrm{d}z_1}{z_1 - a_1} \frac{\mathrm{d}z_2}{z_2 - a_2} \cdots \frac{\mathrm{d}z_r}{z_r - a_r}$$

Both are said to have weight $r \ \equiv$ number of integrations.

Can recursively bypass clashes $a_j = z$ via partial fraction:

$$G(a, 0, z; z) = G(0, 0, a; z) - G(0, a, a; z) - \zeta_2 G(a; z)$$

resting on: $$\frac{1}{(z-a)(z-b)} = \frac{1}{a-b} \left(\frac{1}{z-a} - \frac{1}{z-b} \right)$$

I. 1 Iterated integrals: genus-zero versus elliptic

Elliptic iterated integrals with suitable $f^{(n)}$ **[Brown, Levin]**

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_r \\ a_1 & a_2 & \dots & a_r \end{smallmatrix};z\right) \equiv \int_0^z \mathrm{d}t\, f^{(n_1)}(t-a_1)\, \Gamma\left(\begin{smallmatrix} n_2 & \dots & n_r \\ a_2 & \dots & a_r \end{smallmatrix};t\right) \ , \qquad \Gamma(;z) \equiv 1$$

specialize to elliptic multiple zeta values (eMZVs) upon $z \to 1$ and $a_j \to 0$:

$$\omega(n_1,n_2,\dots,n_r) \equiv \int\limits_{0\leq z_i \leq z_{i+1} \leq 1} f^{(n_1)}(z_1)\,\mathrm{d}z_1\, f^{(n_2)}(z_2)\,\mathrm{d}z_2 \ \cdots \ f^{(n_r)}(z_r)\,\mathrm{d}z_r \, .$$

Both are said to have length r and weight $\sum_{j=1}^r n_j$. **[Enriquez 1301.3042]**

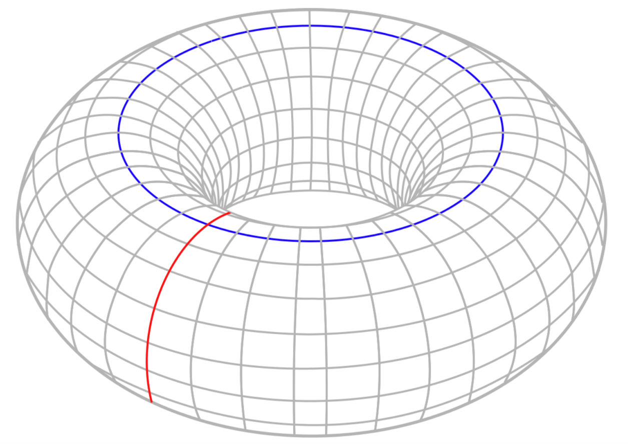
Can recursively bypass clashes $a_j = z$ via Fay relations ($C_{p,q} \in \mathbb{Z}$):

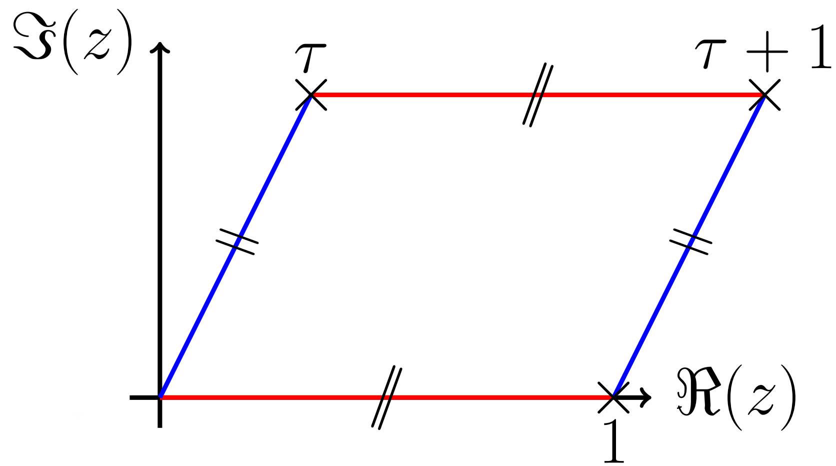
$$\Gamma\left(\begin{smallmatrix} 1 & 1 \\ z & 0 \end{smallmatrix};z\right) = 2\Gamma\left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix};z\right) + \Gamma\left(\begin{smallmatrix} 2 & 0 \\ 0 & 0 \end{smallmatrix};z\right) - 2\Gamma\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix};z\right) + \zeta_2$$

from $$f^{(m)}(z-a)f^{(n)}(z-b) = \sum_{p+q=m+n} C_{p,q} f^{(p)}(z-a)f^{(q)}(a-b) + \begin{smallmatrix} (m,a) \\ \updownarrow \\ (n,b) \end{smallmatrix}$$

I. 2 $f^{(n)}$ – doubly periodic extension of $\frac{\mathrm{d}z}{z}$ with Fay relation

Parametrization of elliptic curve $\equiv$ torus





Jacobi θ-function takes role of the identity map on the torus

$$\theta(z,\tau) \equiv \sin(\pi z) \prod_{n=1}^{\infty} (1-e^{2\pi i(n\tau+z)})(1-e^{2\pi i(n\tau-z)})$$

$$\text{uplift } \frac{1}{z} \to \partial_z \ln\theta(z,\tau) + \underbrace{2\pi i\,\frac{\Im(z)}{\Im(\tau)}}_{\text{sacrifice holomorphicity for } f^{(1)}(z,\tau)=f^{(1)}(z+\tau,\tau)} \equiv \overbrace{f^{(1)}(z,\tau) = f^{(1)}(z+1,\tau)}^{\text{poles like both } \frac{1}{z} \text{ and } \frac{1}{1-z}}$$

I. 2 $f^{(n)}$ – doubly periodic extension of $\frac{\mathrm{d}z}{z}$ with Fay relation

To go beyond

$$\text{uplift} \ \ \frac{1}{z} \ \rightarrow \ f^{(1)}(z,\tau) \equiv \partial_z \ln \theta(z,\tau) + 2\pi i \, \frac{\Im(z)}{\Im(\tau)} \ ,$$

check partial fraction $\Rightarrow$ new function $f^{(2)}$ without pole

$$\begin{array}{rcl} \dfrac{1}{(z-a)(z-b)} + \text{cyc}(z,a,b) & = & 0 \\ & \updownarrow & \\ f^{(1)}(z-a) f^{(1)}(z-b) + \text{cyc}(z,a,b) & = & f^{(2)}(z-a) + \text{cyc}(z,a,b) \end{array}$$

- similar identity for $f^{(1)}(z-a)f^{(2)}(z-b)$ yields new function $f^{(3)}$, etc.
- formally adjoin $f^{(0)} \equiv 1$
- drop the second argument τ of θ and $f^{(n)}$ here and henceforth

I. 2 $f^{(n)}$ – doubly periodic extension of $\frac{\mathrm{d}z}{z}$ with Fay relation

Generating fct.: non-holomorphic version of Kronecker-Eisenstein series

$$\Omega(z,\alpha) \equiv \exp\left(2\pi i \alpha \frac{\Im(z)}{\Im(\tau)}\right) \frac{\theta'(0)\theta(z+\alpha)}{\theta(z)\theta(\alpha)} = \sum_{n=0}^{\infty} \alpha^{n-1} f^{(n)}(z)$$

[Kronecker, Brown, Levin]

- double-periodic $\Omega(z,\alpha) = \Omega(z+\tau,\alpha) = \Omega(z+1,\alpha)$ thanks to $\frac{\Im(z)}{\Im(\tau)}$
- e.g. $f^{(2)}(z) \equiv \frac{1}{2}\left\{\left(\partial \ln\theta(z) + 2\pi i \frac{\Im(z)}{\Im(\tau)}\right)^2 + \partial^2 \ln\theta(z) - \frac{\theta'''(0)}{3\theta'(0)}\right\}$
- $\mathrm{Res}_{z=0}\Omega(z,\alpha) = 1 \;\Rightarrow$ no other poles than $f^{(1)}(z) \sim \frac{1}{z}$
- partial fraction generalizes to Fay relation:

$$\Omega(z_1,\alpha_1)\Omega(z_2,\alpha_2) = \Omega(z_1,\alpha_1+\alpha_2)\Omega(z_2-z_1,\alpha_2) + (1 \leftrightarrow 2)$$

II. One-loop superstring amplitude

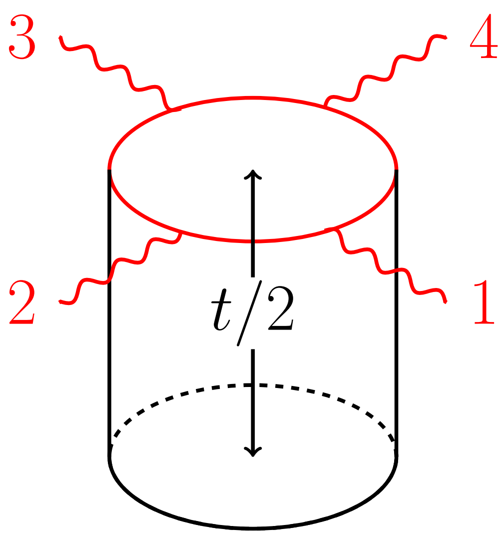
Focus on planar cylinder diagram at four-points:

$$A^{\text{1-loop}}_{\text{string}}(1,2,3,4) = s_{12}s_{23}A^{\text{tree}}_{\text{YM}}(1,2,3,4)\int_0^\infty \mathrm{d}t\; I_{1234}(s_{ij},\tau=it)$$

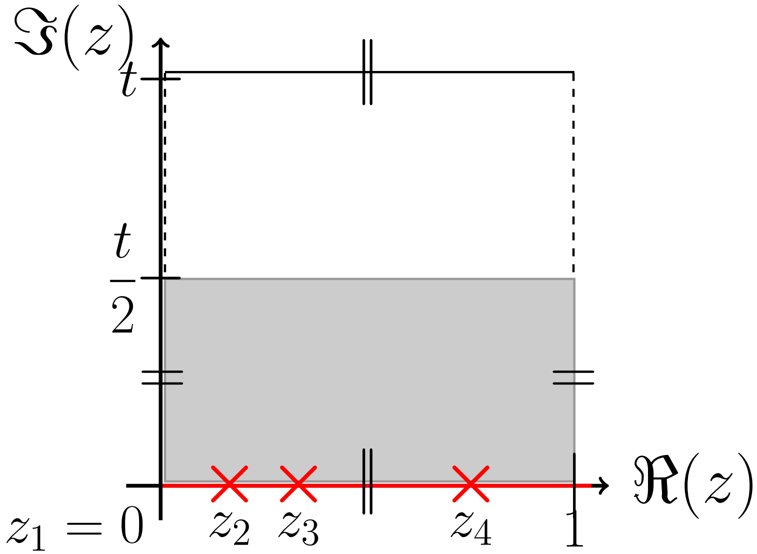
$$I_{1234}(s_{ij},\tau) = \int_0^1 \mathrm{d}z_4 \int_0^{z_4} \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2\; \exp\left(\sum_{i<j}^{4} s_{ij} P(z_i - z_j,\tau)\right)\bigg|_{z_1=0}$$

[Brink, Green, Schwarz 1982]

with dim'less $s_{ij} \equiv \alpha'(k_i+k_j)^2$ and worldsheet propagator $\partial P = f^{(1)}$.



parametrized as →



Analytic α'-dependence from expanding the exponentials

$$I_{1234}(s_{ij},\tau) = \int_0^1 \mathrm{d}z_4 \int_0^{z_4} \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2 \prod_{i<j}^{4} \sum_{n_{ij}=0}^{\infty} \frac{\big[s_{ij}P(z_i-z_j,\tau)\big]^{n_{ij}}}{n_{ij}!}\Big|_{z_1=0}$$

and integrating $P(z_i-z_j) = \int_{z_j}^{z_i} \mathrm{d}x \; f^{(1)}(x-z_j)$ order by order in α'.

Each monomial in s_{ij} is accompanied by eMZVs, e.g.

$$s_{ij}^0 \;\leftrightarrow\; \int_0^1 \mathrm{d}z_4 \; f^{(0)}(z_4) \int_0^{z_4} \mathrm{d}z_3 \; f^{(0)}(z_3) \int_0^{z_3} \mathrm{d}z_2 \; f^{(0)}(z_2) = \omega(0,0,0)$$

after formally inserting $f^{(0)} = 1$ as well as

$$\left.\begin{matrix} s_{12} \\ \\ s_{13} \end{matrix}\right\} \leftrightarrow \int_0^1 \mathrm{d}z_4 \; f^{(0)} \int_0^{z_4} \mathrm{d}z_3 \; f^{(0)} \int_0^{z_3} \mathrm{d}z_2 \; f^{(0)} \left\{\begin{matrix} \int_0^{z_2} \mathrm{d}x \; f^{(1)}(x) \\ \\ \int_0^{z_3} \mathrm{d}x \; f^{(1)}(x) \end{matrix}\right.$$

$$\Longrightarrow \quad s_{12} \;\leftrightarrow\; \omega(1,0,0,0) \; , \qquad s_{13} \;\leftrightarrow\; \underbrace{\omega(1,0,0,0)}_{\text{from } 0\leq x\leq z_2} + \underbrace{\omega(0,1,0,0)}_{\text{from } z_2\leq x\leq z_3}$$

At higher order ...

$$s_{12}s_{23} \leftrightarrow \int_0^1 \mathrm{d}z_4 \int_0^{z_4} \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2 \left(\int_{z_3}^{z_2} \mathrm{d}x \, f^{(1)}(x - z_3) \right) \left(\int_0^{z_2} \mathrm{d}y \, f^{(1)}(y) \right)$$

$$= - \int_0^1 \mathrm{d}z_4 \int_0^{z_4} \mathrm{d}z_3 \, \Gamma \left(\begin{smallmatrix} 1 & 0 & 1 \\ z_3 & 0 & 0 \end{smallmatrix} ; z_3 \right)$$

... need Fay relations

$$\Gamma \left(\begin{smallmatrix} 1 & 0 & 1 \\ z_3 & 0 & 0 \end{smallmatrix} ; z_3 \right) = 2\Gamma \left(\begin{smallmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \end{smallmatrix} ; z_3 \right) + \Gamma \left(\begin{smallmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \end{smallmatrix} ; z_3 \right) - 2\Gamma \left(\begin{smallmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{smallmatrix} ; z_3 \right) + \zeta_2 \Gamma \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} ; z_3 \right) \; .$$

Above example then integrates to

$$s_{12}s_{23} \leftrightarrow -2\omega(2,0,0,0,0) - \omega(0,2,0,0,0) + 2\omega(1,1,0,0,0) - \zeta_2 \omega(0,0,0)$$

$$= -\omega(1,0,0,0,1)$$

after using eMZV relations at length five.

After using momentum conservation for s_{ij}, first orders simplify to

$$I_{1234}(s_{ij}) = \omega(0,0,0) - 2\omega(0,1,0,0)\,(s_{12}+s_{23}) + 2\omega(0,1,1,0,0)\,\big(s_{12}^2+s_{23}^2\big)$$
$$- 2\omega(0,1,0,1,0)\,s_{12}s_{23} + \beta_5\,(s_{12}^3+2s_{12}^2s_{23}+2s_{12}s_{23}^2+s_{23}^3) + \beta_{2,3}\,s_{12}s_{23}(s_{12}+s_{23}) + \ldots$$

with shorthands [Brödel, Mafra, Matthes, OS 1412.5535]

$$\beta_5 = \frac{4}{3}\big[\omega(0,0,1,0,0,2)+\omega(0,1,1,0,1,0)-\omega(2,0,1,0,0,0)-\zeta_2\omega(0,1,0,0)\big]$$
$$\beta_{2,3} = \frac{1}{3}\omega(0,0,1,0,2,0)-\frac{3}{2}\omega(0,1,0,0,0,2)-\frac{1}{2}\omega(0,1,1,1,0,0)$$
$$-2\omega(2,0,1,0,0,0)-\frac{4}{3}\omega(0,0,1,0,0,2)-\frac{10}{3}\zeta_2\omega(0,1,0,0)\,.$$

Choice of indecomposable eMZVs requires guidance,

in particular at higher α'-order $\leftrightarrow$ weight $\leftrightarrow$ length ...

also: n-point amplitude naturally compatible with eMZV language !

III. 1 eMZV relations and indecomposables

The following MZVs are believed to be indecomposable over $\mathbb{Q}$:

weight w	0	1	2	3	4	5	6	7	8	9	10	11	12
indec. MZVs	1	$\emptyset$	ζ_2	ζ_3	$\emptyset$	ζ_5	$\emptyset$	ζ_7	$\zeta_{3,5}$	ζ_9	$\zeta_{3,7}$	ζ_{11}, $\zeta_{3,3,5}$	$\zeta_{3,9}$, $\zeta_{1,1,4,6}$

MZVs satisfy shuffle and stuffle relations, and eMZVs obey

- shuffle $\omega(n_1,\ldots,n_r)\omega(k_1,\ldots,k_s) = \omega\big((n_1,\ldots,n_r) ⧢ (k_1,\ldots,k_s)\big)$
- reflection: $\omega(n_1,n_2,\ldots,n_r) = (-1)^{n_1+n_2+\ldots+n_r}\omega(n_r,\ldots,n_2,n_1)$
- Fay rel's $\omega(n_1,n_2,\ldots) \leftrightarrow \omega(n_1{+}j,n_2{-}j,\ldots)$ such as $\omega(0,5)=\omega(2,3)$

all of which preserve the weight $\sum_{j=1}^r n_j$ of $f^{(n_j)}$ integrands.

$\longrightarrow$ Which eMZVs remain indecomposable w.r.t. $\mathbb{Q}$[MZV]?

However, $f^{(0)}$ @ zero weight $\Rightarrow\ \exists\ \infty$ eMZVs $\omega(n,\underbrace{0,\ldots,0}_{\text{any number}})$ @ weight n

$\Rightarrow$ organize relations by length r:

length $r=1$: only constant eMZVs:
$$\omega(n)=\begin{cases}-2\,\zeta_n & : \ n \text{ even}\\ 0 & : \ n \text{ odd}\end{cases}$$

length $r=2$: shuffle and reflection reduce even-weight eMZVs to $r=1$:

$$\omega(n_1,n_2)\,\Big|_{n_1+n_2 \text{ even}}=\begin{cases}2\,\zeta_{n_1}\,\zeta_{n_2} & : \ n_1,n_2 \text{ both even}\\ 0 & : \ n_1,n_2 \text{ both odd}\end{cases}$$

odd-weight $\omega(n_1,n_2)$ depend on τ and can be reduced to $\omega(0,2p-1)$:

$$\omega(n_1,n_2)\,\Big|_{n_1+n_2 \text{ odd}}=(-1)^{n_1}\omega(0,n_1+n_2)+2\delta_{n_1,1}\zeta_{n_2}\omega(0,1)-2\delta_{n_2,1}\zeta_{n_1}\omega(0,1)$$
$$+\,2\left\{\sum_{p=1}^{\lceil\frac{1}{2}(n_2-3)\rceil}\binom{n_1+n_2-2p-2}{n_1-1}\zeta_{n_1+n_2-2p-1}\omega(0,2p+1)-(n_1\leftrightarrow n_2)\right\}$$

any length r: eMZVs with $(-1)^{\text{length}} = (-1)^{\text{weight}}$ reducible to lower length

length $r=3$: $\lceil \frac{w}{6} \rceil$ indecomposable $\omega(n_1,n_2,n_3)$ @ even $w=\sum_{j=1}^{3} n_j$:

w	2	4	6	8	10	12	14
indec. eMZVs at $r=3$	$\omega(0,0,2)$	$\omega(0,0,4)$	$\omega(0,0,6)$	$\omega(0,0,8)$ $\omega(0,3,5)$	$\omega(0,0,10)$ $\omega(0,3,7)$	$\omega(0,0,12)$ $\omega(0,3,9)$	$\omega(0,0,14)$ $\omega(0,3,11)$ $\omega(0,5,9)$

at $w=8$, for instance, shuffle, reflection & Fay imply

$$\begin{aligned}\omega(0,6,2) = &-\frac{21}{2}\zeta_8 + 2\,\omega(0,3)\,\omega(0,5) - 14\,\zeta_6\,\omega(0,0,2)\\ &- 6\,\zeta_4\,\omega(0,0,4) - \frac{9}{2}\omega(0,0,8) - \frac{2}{5}\omega(0,3,5)\end{aligned}$$

Further eMZV relations available @ `https://tools.aei.mpg.de/emzv` any length r: eMZVs with $(-1)^{\text{length}} = (-1)^{\text{weight}}$ reducible to lower length

length $r = 3$: $\lceil \frac{w}{6} \rceil$ indecomposable $\omega(n_1, n_2, n_3)$ @ even $w = \sum_{j=1}^{3} n_j$:

w	2	4	6	8	10	12	14
indec. eMZVs at $r=3$	$\omega(0,0,2)$	$\omega(0,0,4)$	$\omega(0,0,6)$	$\omega(0,0,8)$ $\omega(0,3,5)$	$\omega(0,0,10)$ $\omega(0,3,7)$	$\omega(0,0,12)$ $\omega(0,3,9)$	$\omega(0,0,14)$ $\omega(0,3,11)$ $\omega(0,5,9)$

length $r = 4$: conjecture $\left\lfloor \frac{1}{2} + \frac{1}{48}(w+5)^2 \right\rfloor$ indecomp's @ odd $w = \sum_{j=1}^{4} n_j$:

w	1	3	5	7	9
indec. eMZVs at $r=4$	$\omega(0,0,1,0)$	$\omega(0,0,0,3)$	$\omega(0,0,0,5)$ $\omega(0,0,2,3)$	$\omega(0,0,0,7)$ $\omega(0,0,2,5)$ $\omega(0,0,4,3)$	$\omega(0,0,0,9)$, $\omega(0,0,4,5)$ $\omega(0,0,2,7)$, $\omega(0,1,3,5)$

summary @ length $r \leq 7$: indecomposable eMZVs @ $(-1)^{\text{weight}} \neq (-1)^{\text{length}}$

r \ w	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
2	1		1		1		1		1		1		1		1		1		1		1		1
3		1		1		1		2		2		2		3		3		3		4		4	
4	1		1		2		3		4		5		7		8		10		12		14		16
5		1		2		4		6		9		13		17		23		30		37		47	
6	1		2		4		8		13		22		31		45		?		?		?		?
7		1		4		8		16		29		48		?		?		?		?		?	

- strong evidence that shuffle, reflection & Fay $\Rightarrow$ **any** eMZV relation
- will derive these numbers from combinatorial principles

[Brödel, Matthes, OS 1507.02254]

III. 2 Direct computation of eMZVs

eMZVs satisfy ODE w.r.t. modular parameter $q = e^{2\pi i\tau}$: [Enriquez 1301.3042]

$$-4\pi^2 q \frac{\mathrm{d}}{\mathrm{d}q}\, \mathrm{eMZV}\begin{pmatrix} \text{weight } w \\ \text{length } \ell \end{pmatrix} \sim \mathrm{G}_k(q) \times \mathrm{eMZV}\begin{pmatrix} \text{weight } w{-}k{+}1 \\ \text{length } \ell{-}1 \end{pmatrix}$$

recursive in length, generating *Eisenstein series* $\mathrm{G}_k(q)$ in each step:

$$\mathrm{G}_k(q) = \begin{cases} 0 & : \ k \text{ odd} \\ 2\zeta_k + \frac{2(-1)^{k/2}(2\pi)^k}{(k-1)!} \sum_{m,n=1}^{\infty} m^{k-1} q^{mn} & : \ k \text{ even} \end{cases}$$

Supplement the above ODE in q by boundary values: [Enriquez 1301.3042]

$$\lim_{q\to 0} \mathrm{eMZV}\begin{pmatrix} \text{weight } w \\ \text{length } \ell \end{pmatrix} \sim (2\pi i)^{\pm k} \times \mathrm{MZVs}\big(\text{weight } w \mp k\big)$$

generating series for eMZV degenerates to Drinfeld associator

III. 2 Direct computation of eMZVs

eMZVs satisfy ODE w.r.t. modular parameter $q = e^{2\pi i\tau}$: **[Enriquez 1301.3042]**

$$-4\pi^2 q \frac{\mathrm{d}}{\mathrm{d}q} \, \mathrm{eMZV}\begin{pmatrix} \text{weight } w \\ \text{length } \ell \end{pmatrix} \sim \mathrm{G}_k(q) \times \mathrm{eMZV}\begin{pmatrix} \text{weight } w{-}k{+}1 \\ \text{length } \ell{-}1 \end{pmatrix}$$

recursively get q-expansion from *truncated Eisenstein series* G^0_k:

$$\mathrm{G}^0_k(q) \sim \begin{cases} -1 & : \ k=0 \\ \sum_{m,n=1}^{\infty} m^{k-1} q^{mn} & : \ k = 2,4,6,\ldots \end{cases}$$

$\Rightarrow$ eMZVs are $\mathbb{Q}[\mathrm{MZV}]$ linear combination of iterated Eisenstein integrals

$$\gamma(k_1,k_2,\ldots,k_r) \equiv \frac{1}{(4\pi^2)^r} \int_{0\leq q_i \leq q_{i+1} \leq q} \mathrm{G}^0_{k_1}(q_1)\frac{\mathrm{d}q_1}{q_1} \, \mathrm{G}^0_{k_2}(q_2)\frac{\mathrm{d}q_2}{q_2} \, \cdots \, \mathrm{G}^0_{k_r}(q_r)\frac{\mathrm{d}q_r}{q_r}$$

special cases of iterated Shimura integrals **[Manin 2005; Brown 2014]**

q-expansion of Eisenstein integrals easy to compute

$$\int_0^q (q')^N \, \frac{\mathrm{d}q'}{q'} \ = \ \frac{q^N}{N}$$

and turns out to resemble MZVs as nested sums

$$\gamma(k_1, 0^{p_1-1}, k_2, 0^{p_2-1}, \ldots, k_r, 0^{p_r-1}) \sim \sum_{0<n_1<n_2<\cdots<n_r} \frac{\sigma_{k_1-1}(n_1)\sigma_{k_2-1}(n_2-n_1)\ldots\sigma_{k_r-1}(n_r-n_{r-1})q^{n_r}}{n_1^{p_1} n_2^{p_2}\ldots n_r^{p_r}}$$

$$\zeta(1, 0^{p_1-1}, 1, 0^{p_2-1}, \ldots, 1, 0^{p_r-1}) \sim \sum_{0<n_1<n_2<\cdots<n_r} \frac{1}{n_1^{p_1} n_2^{p_2}\ldots n_r^{p_r}} \ = \ \zeta_{p_1,p_2,\ldots,p_r}$$

Divisor sums $\sigma_k(n) \equiv \sum_{d|n} d^k$ obstruct analogues of stuffle-relations

$\Rightarrow$ no relations among $\gamma(\ldots)$ beyond shuffle $\gamma(W_1)\gamma(W_2) = \gamma(W_1 ⧢ W_2)$

Length-two example: indecomposable eMZVs $\omega(0,k)$ @ $k = 3, 5, 7, \ldots$

$$\omega(0,k) = \ k\gamma(k+1) \ = \ \text{rational} \times \pi^{k-1} \sum_{m,n=1}^{\infty} \frac{m^{k-1}}{n} q^{mn}$$

III. 3 Back to counting indecomposable eMZVs

Naively: indecomposable eMZVs $\leftrightarrow$ shuffle-independent Eisenstein int's?

$$\text{eMZV}\begin{pmatrix} \text{weight } \sum_j k_j - r \\ \text{length } r-1 \end{pmatrix} \overset{?}{\longleftrightarrow} \frac{\{\gamma(k_1, k_2, \ldots, k_r) \text{ @ } k_j = 0, 2, 4, \ldots\}}{\text{shuffle-relations } \gamma(W_1)\gamma(W_2) = \gamma(W_1 ⧢ W_2)}$$

Correct at length two $\omega(0, k) \leftrightarrow \gamma(k+1)$, but $\exists$ problems at length three:

$$\text{e.g.} \quad \underbrace{\omega(0,0,12),\ \omega(0,3,9)}_{2 \times \text{indecomposable}} \quad \leftrightarrow \quad \underbrace{\gamma(0,14),\ \gamma(2,12),\ \gamma(4,10),\ \gamma(6,8)}_{4 \times \text{shuffle-independent}}$$

Additional selection rules arise from derivation algebra $\{\epsilon_0, \epsilon_2, \epsilon_4, \ldots\}$

"dual" to Eisenstein series $\mathrm{G}_0, \mathrm{G}_2, \mathrm{G}_4, \ldots$ subject to relations

$$[\epsilon_2, \epsilon_k] = 0 \ , \qquad [\epsilon_{10}, \epsilon_4] - 3\,[\epsilon_8, \epsilon_6] = 0 \ , \qquad \text{etc.}$$

Explains above mismatch "$2 \times \omega(\ldots)$ vs. $4 \times \gamma(\ldots)$"

Generating series of eMZVs: elliptic associator [Enriquez 1301.3042]

$$A(q) \equiv \sum_{r\geq 0}(-1)^r \sum_{n_1,n_2,\ldots,n_r\geq 0} \omega(n_1,n_2,\ldots,n_r)\, \mathrm{ad}_x^{n_r}(y)\ldots \mathrm{ad}_x^{n_2}(y)\mathrm{ad}_x^{n_1}(y)$$

Derivations $\epsilon_0, \epsilon_2, \epsilon_4, \ldots$ act on non-commutative variables x, y via

$$\epsilon_{2n}(x) = (\mathrm{ad}_x)^{2n}(y) \ , \qquad \epsilon_0(y) = 0$$

$$\epsilon_{2n}(y) = [y, (\mathrm{ad}_x)^{2n-1}(y)] + \sum_{1\leq j<n} (-1)^j [(\mathrm{ad}_x)^j(y), (\mathrm{ad}_x)^{2n-1-j}(y)] \ @\ n > 0$$

satisfy various commutator relations and enter the associator's ODE

$$q\frac{\mathrm{d}}{\mathrm{d}q}A(q) = \frac{1}{4\pi^2}\left(\sum_{n=0}^{\infty}(2n-1)\mathrm{G}_{2n}(q)\epsilon_{2n}\right)A(q)\,.$$

$\Rightarrow$ more precise version of "ϵ_{2n} are dual to G_{2n}"

$\Rightarrow$ commutator relations among ϵ_{2n} impose selection rules on $\gamma(\ldots)$

Hence, more accurate picture is

$$\mathrm{eMZV}\binom{\text{weight } \sum_j k_j - r}{\text{length } r-1} \longleftrightarrow \frac{\{\gamma(k_1,k_2,\ldots,k_r) \text{ @ } k_j = 0,2,4,6,\ldots\}}{\{\text{shuffle-relations}\} \times \{\epsilon_k\text{-relations}\}}$$

For instance, $[\epsilon_2, \epsilon_k] = 0$ obstructs non-trivial appearance of $k_j = 2$ and

$$\omega(0,n) \leftrightarrow \gamma(n+1) \, , \qquad \omega(0,3,5) \leftrightarrow \gamma(4,6)$$

$$\omega(0,0,n) \leftrightarrow \gamma(n+2,0) \, , \qquad \omega(0,3,7) \leftrightarrow \gamma(4,8)$$

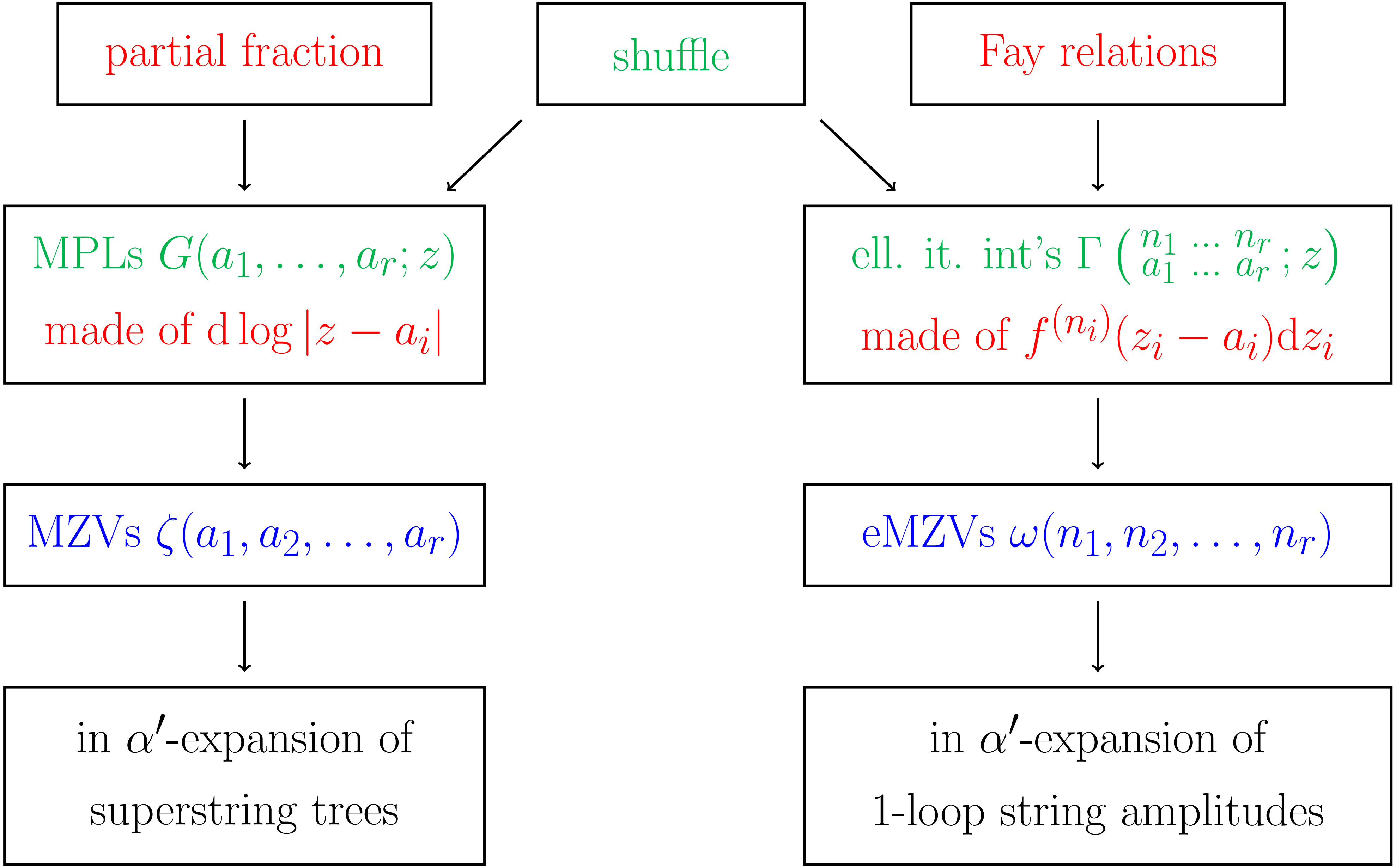
$$\left.\begin{array}{l} \omega(0,3,9) \\ \\ \omega(0,5,7) \end{array}\right\} \begin{array}{c} \leftrightarrow \quad 81\gamma(10,4) + 35\gamma(8,6) \\ \\ \text{selected by } [\epsilon_{10},\epsilon_4] - 3[\epsilon_8,\epsilon_6] = 0 \end{array}$$

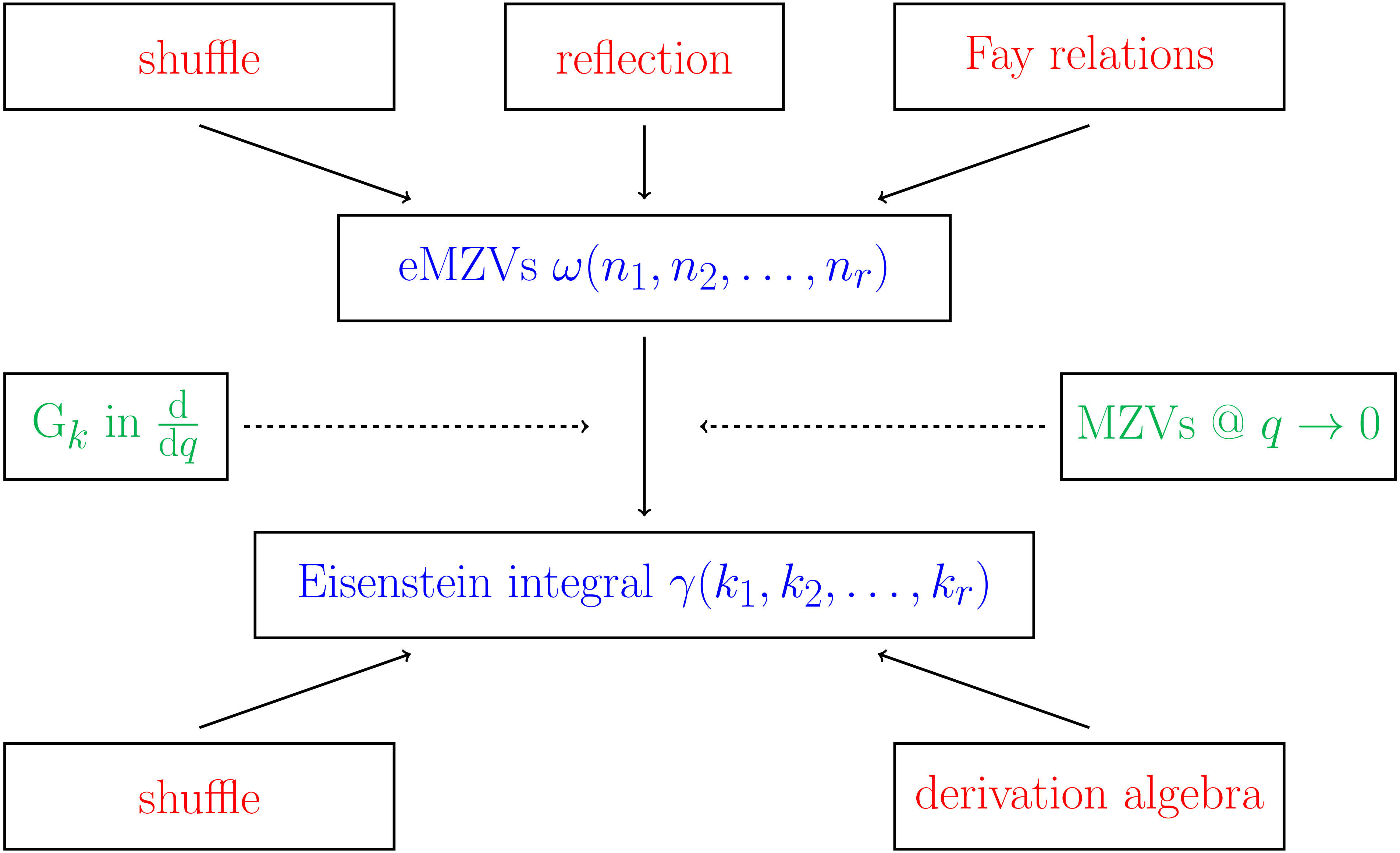
Cusp forms control ϵ_k-relations at various weights and “depths”, e.g.

$$\begin{aligned} 0 \; = \; & 80\,[\epsilon_{12},[\epsilon_4,\epsilon_0]] + 16\,[\epsilon_4,[\epsilon_{12},\epsilon_0]] - 250\,[\epsilon_{10},[\epsilon_6,\epsilon_0]] \\ & -125\,[\epsilon_6,[\epsilon_{10},\epsilon_0]] + 280\,[\epsilon_8,[\epsilon_8,\epsilon_0]] - 462\,[\epsilon_4,[\epsilon_4,\epsilon_8]] - 1725\,[\epsilon_6,[\epsilon_6,\epsilon_4]] \end{aligned}$$

Perfect agreement with counting of indecomposable eMZVs!

Summary


partial fraction
shuffle
Fay relations
MPLs $G(a_1, \ldots, a_r; z)$
made of $\mathrm{d}\log|z - a_i|$
ell. it. int's $\Gamma\left(\begin{smallmatrix} n_1 & \ldots & n_r \\ a_1 & \ldots & a_r \end{smallmatrix}; z\right)$
made of $f^{(n_i)}(z_i - a_i)\mathrm{d}z_i$
MZVs $\zeta(a_1, a_2, \ldots, a_r)$
eMZVs $\omega(n_1, n_2, \ldots, n_r)$
in α'-expansion of
superstring trees
in α'-expansion of
1-loop string amplitudes

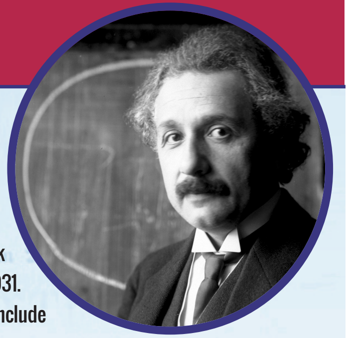


Various eMZV relations available at `https://tools.aei.mpg.de/emzv`

A Century of General Relativity

November 30 - December 2, 2015

Harnack House Berlin



The year 2015 marks the 100th anniversary of Einstein's field equations. To celebrate this event, the **Max Planck Institute for Gravitational Physics** (or Albert Einstein Institute) will host a conference during the week of November 30, 2015, exactly one hundred years after the publication of Einstein's paper. The conference will take place in the recently renovated Harnack House in Berlin, where Albert Einstein regularly lectured between 1915 and 1931. On December 3-5 the **Max Planck Institute for the History of Science** will conclude the celebratory events with a workshop on the history of Einstein's theory.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Speakers:

Eric Adelberger University of Washington, Seattle
Abhay Ashtekar Penn State University, University Park
Zvi Bern University of California, Los Angeles
Thibault Damour IHES, Bures-sur-Yvette
Reinhard Genzel Max Planck Institute for Extraterrestial Physics, Munich
Andrea Ghez University of California, Los Angeles
David Gross Kavli Institute for Theoretical Physics, Santa Barbara
Hanoch Gutfreund Hebrew University, Jerusalem
Ted Jacobson University of Maryland, College Park
Sergiu Klainerman Princeton University, Princeton
Joseph Polchinski Kavli Institute for Theoretical Physics, Santa Barbara
Frans Pretorius Princeton University, Princeton
Harvey Reall DAMTP, Cambridge
David Spergel Princeton University, Princeton
Ingrid Stairs University of British Columbia, Vancouver
Paul Steinhardt Princeton University, Princeton
Rai Weiss Massachusetts Institute of Technology, Cambridge

Scientific Organization Committee:
Bruce Allen, Alessandra Buonanno, Karsten Danzmann, Hermann Nicolai (Chair), Bernard Schutz
Max Planck Institute for Gravitational Physics (Albert Einstein Institute)
For more information and program details please visit: www.einsteinconference2015.org

www.einsteinconference2015.org

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MAX PLANCK INSTITUTE FOR THE HISTORY OF SCIENCE

One-loop string amplitudes with more external legs

For n external legs, **[Tsuchiya, Montag, Stieberger, Taylor, etc.]**

$$A^{\text{1-loop}}_{\text{string}}(1,2,\ldots,n) = \int_0^\infty \mathrm{d}t \int_{0=z_1\leq z_2\leq\ldots\leq z_n\leq 1} \mathrm{d}z_2 \,\ldots\, \mathrm{d}z_n \ \exp\Big(\sum_{i<j}^{n} s_{ij} P(z_i - z_j, it)\Big)$$

$$\times \sum \big(\text{monomial in } f^{(k_j)} \text{ @ weight } \sum_j k_j = n-4\big) \times \big(\text{kinematic factors}\big)$$

e.g. for $n=5$ points, the second line becomes **[OS, Mafra 1203.6215]**

$$f^{(1)}(z_2 - z_3) \times s_{23}s_{45}\big(s_{34}A^{\text{tree}}_{\text{YM}}(1,2,3,4,5) - s_{24}A^{\text{tree}}_{\text{YM}}(1,3,2,4,5)\big)$$

$$+ \ 5 \text{ permutations } (23 \leftrightarrow 24, 25, 34, 35, 45)$$

$\Rightarrow$ n-point amplitude naturally compatible with eMZV language !