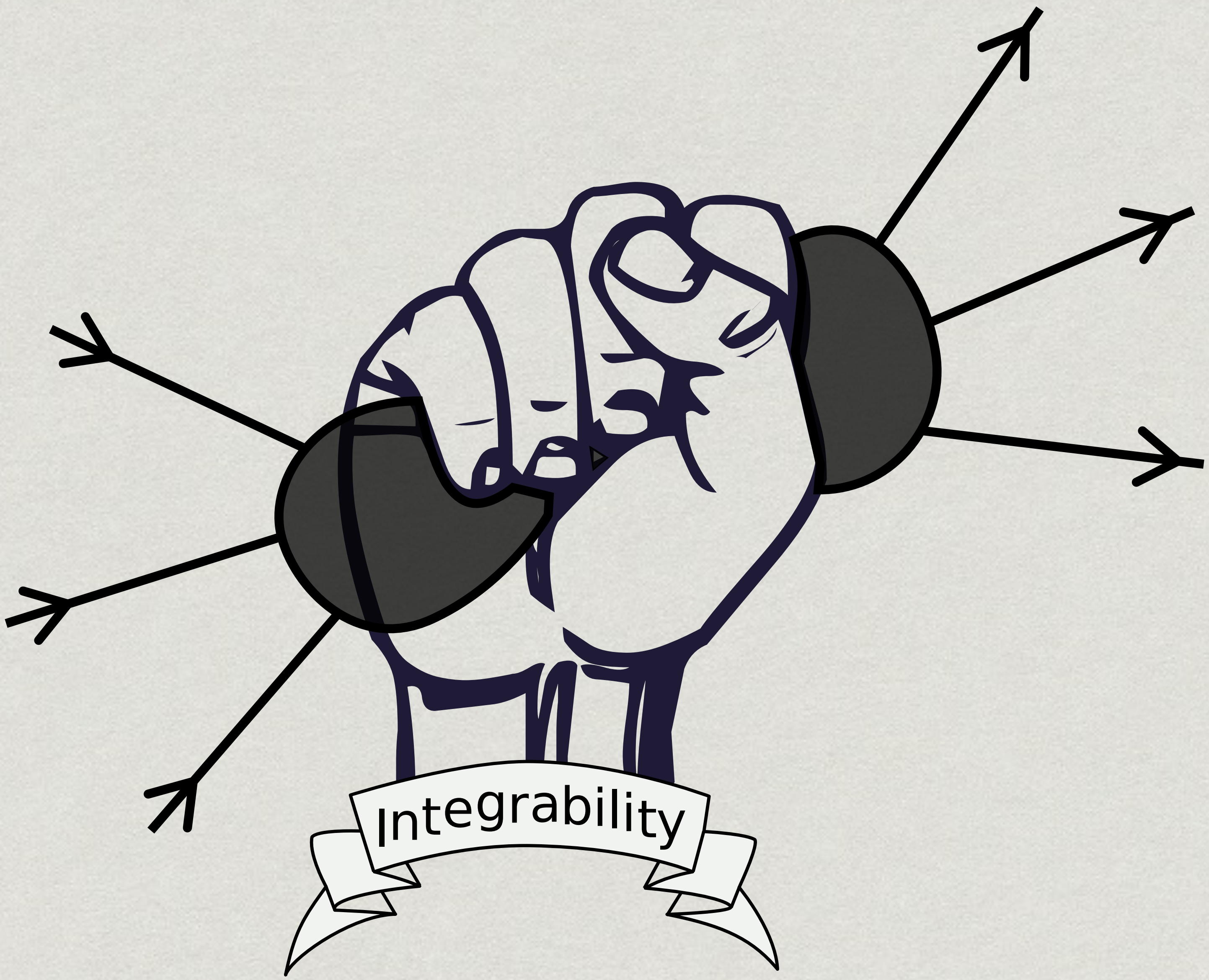
Integrability for scattering amplitudes the six point amplitude at all loops

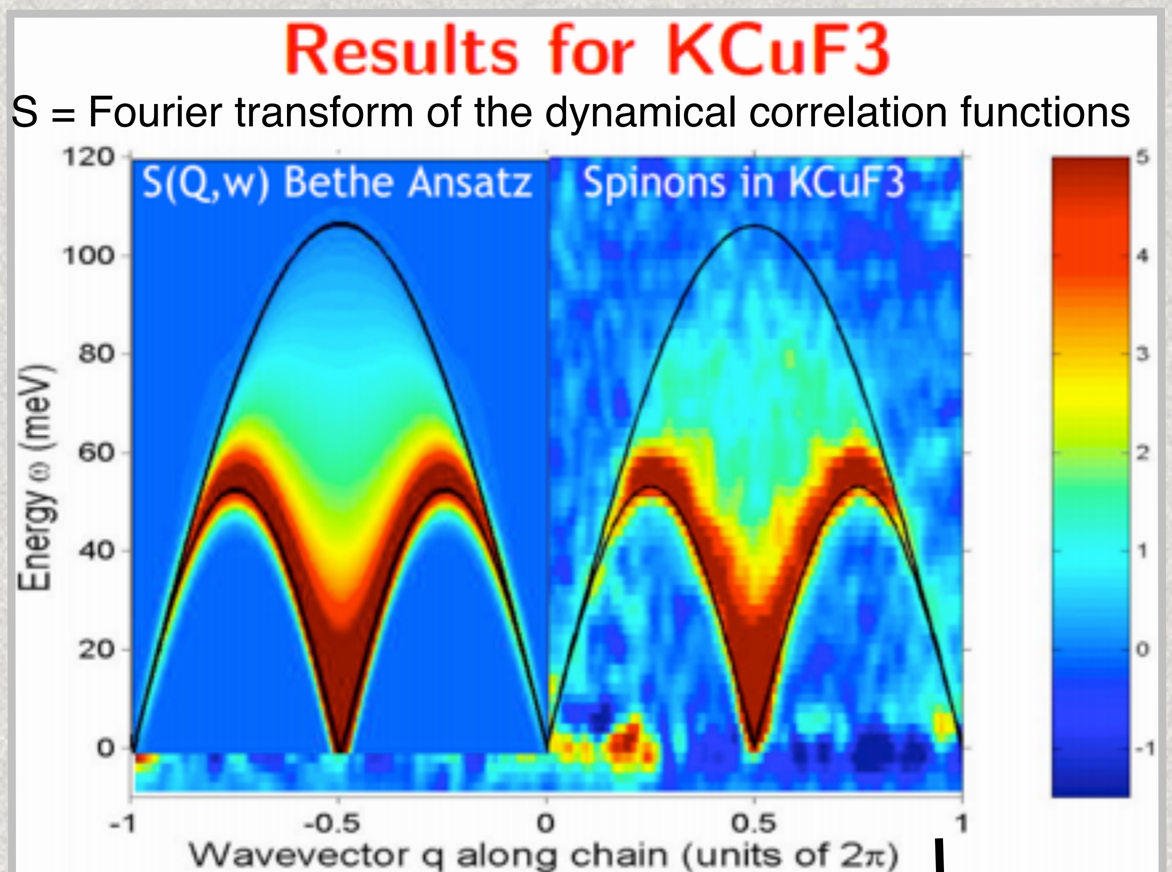


A. Sever
Tel Aviv University

Amplitudes 2015

Integrability

A spectacular miracle in 2d!



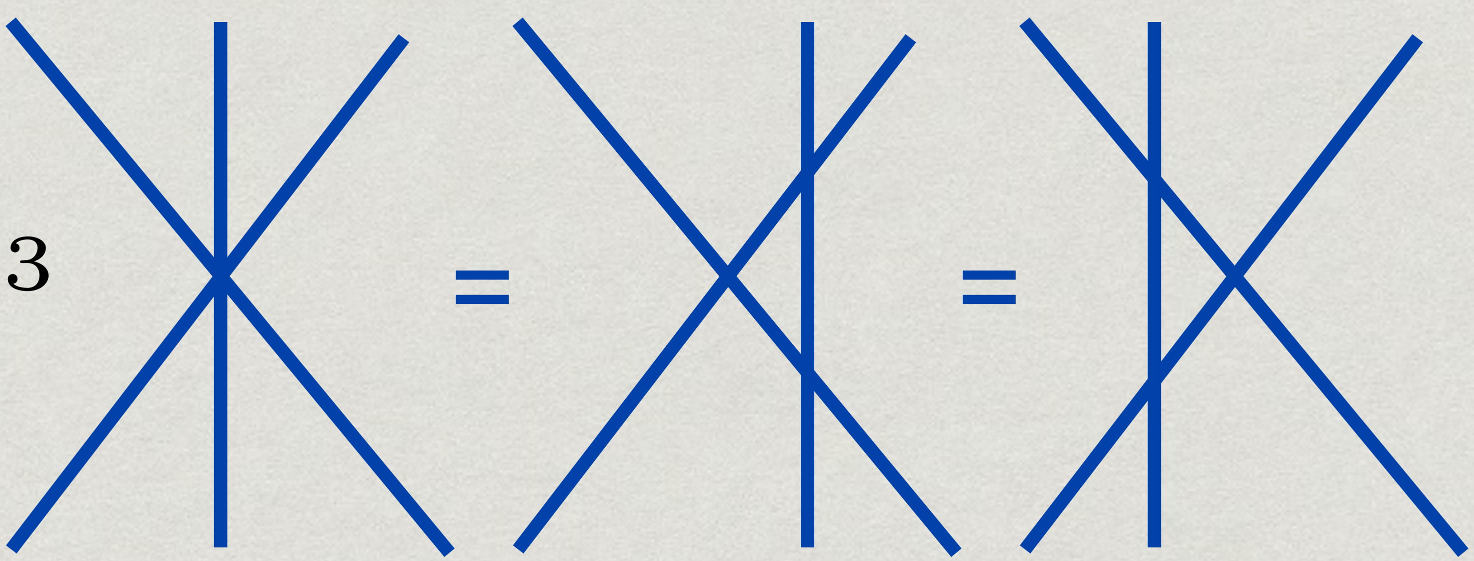
Integrability

In 2d (only) an interacting theory can be integrable = solvable

$$Q_1 = \sum p_j \ , \ Q_2 = \sum p_j^2 \ , \ \Rightarrow \ \{p_1, p_2\} = \{p'_1, p'_2\}$$

Integrability : If $\quad \exists Q_3 = \sum p_j^3 \ \Rightarrow \ \{p_1, p_2, p_3\} = \{p'_1, p'_2, p'_3\}$

$$e^{i\alpha Q_3}$$



- Factorized scattering.
- S-matrices obey YB.

Powerful. Often synonym to exact solvability.

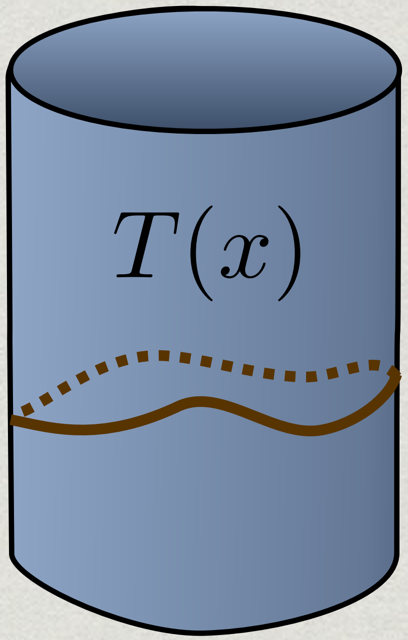
Integrability

Comments -

- One way the encode all the conserved charges is through the **transfer matrix** or **holonomy** matrix

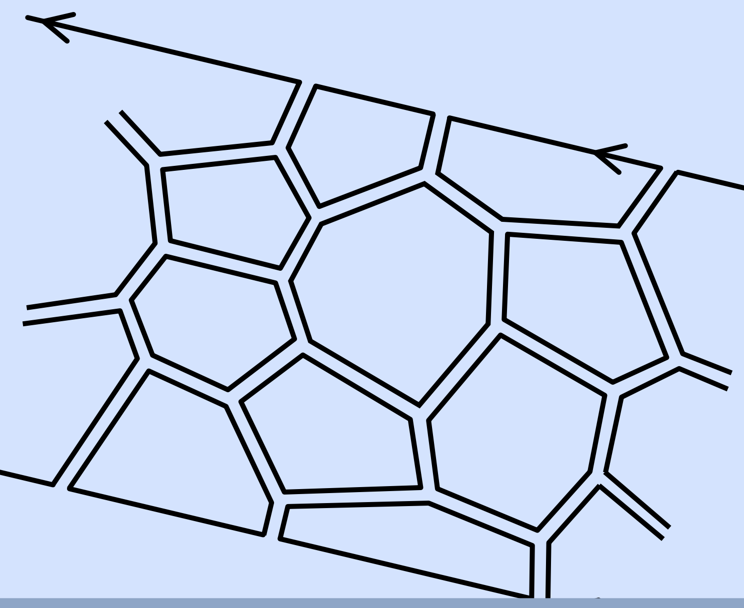
$$T(x) = \sum x^n Q_n$$

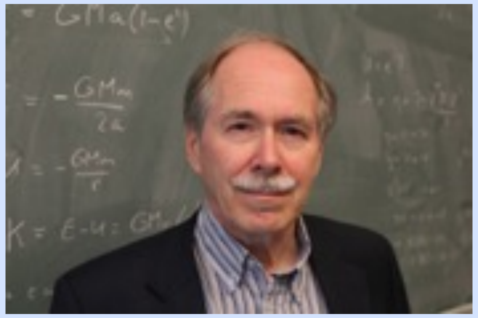
spectral parameter



- The algebra form by the conserved charges is called *Yangian*

To apply integrability to **4d** scattering amplitudes we should **map** them to a **2d** problem on top of the ’t Hooft (planar) string

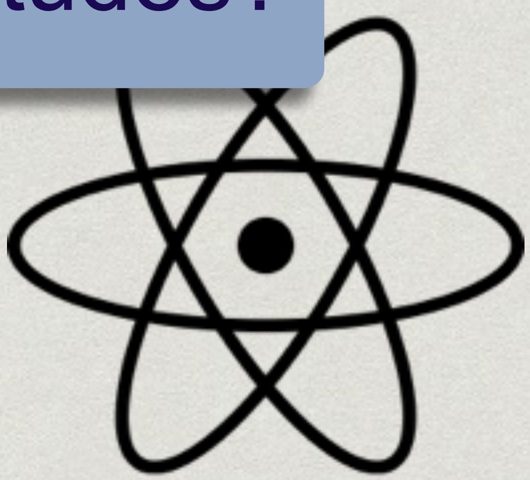


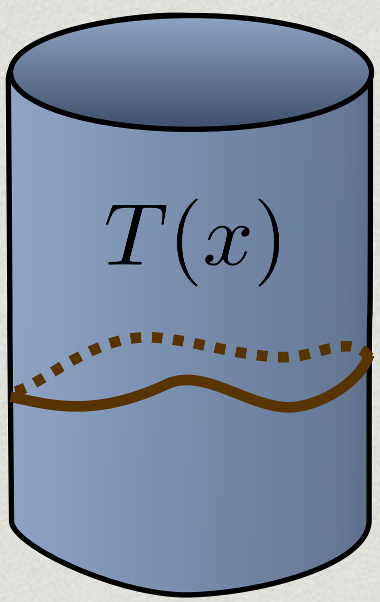


‘t Hooft $1/N_c$ expansion

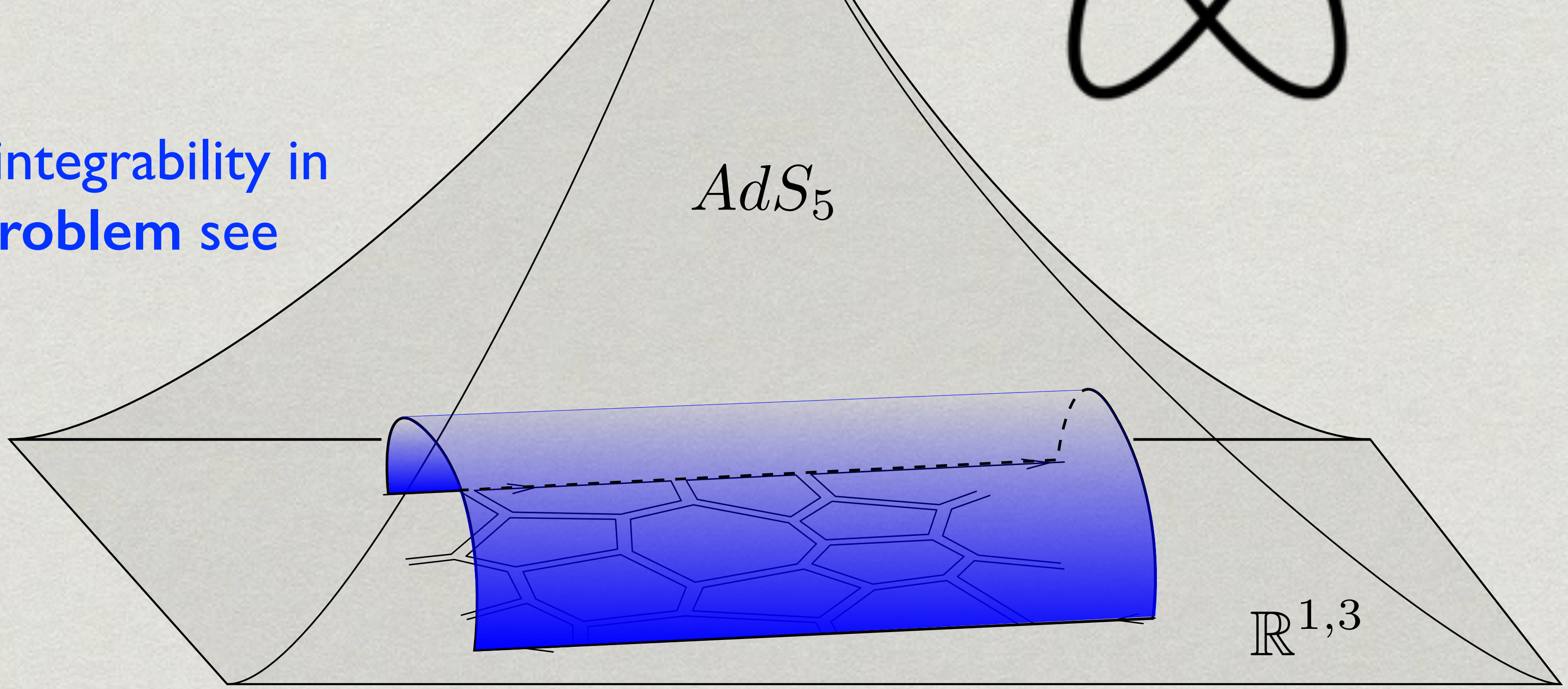
How to do this for scattering amplitudes?

The holographic ’t Hooft string of “$\mathcal{N}$**=4 SYM**” turns out to be **integrable**!

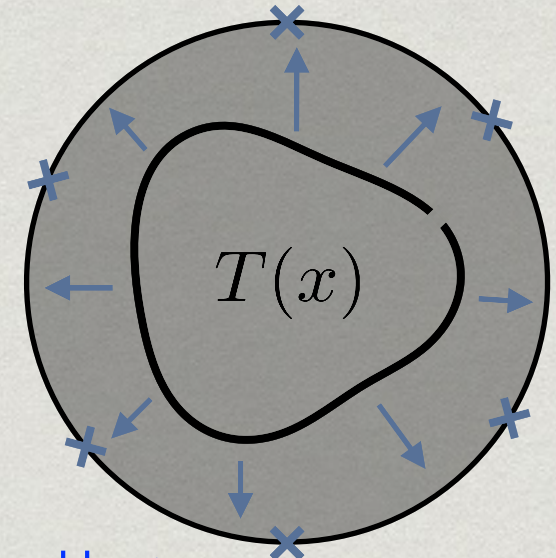




For a review of integrability in the **spectrum problem** see [Beisert et al]

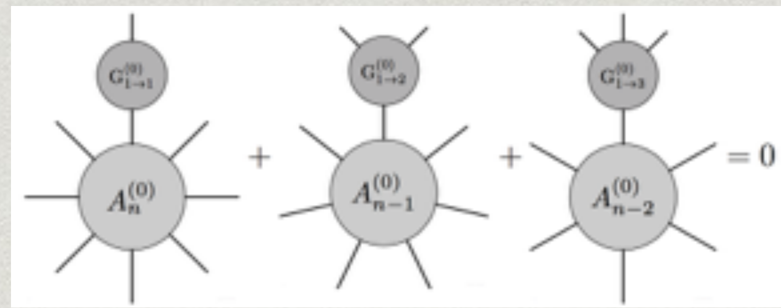


4d scattering amplitude $\leftrightarrow$ 2d integrability map



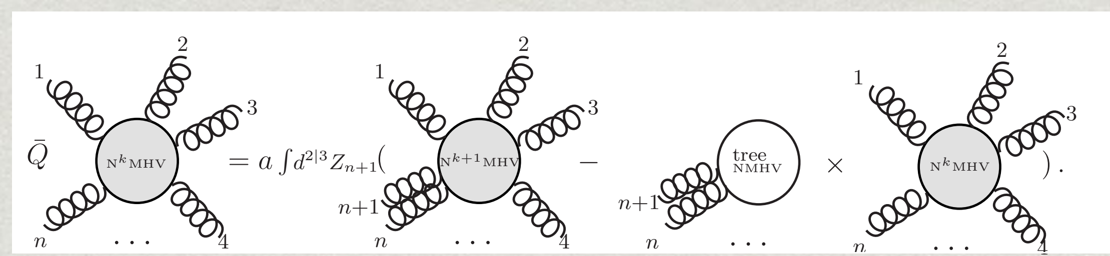
- **Symmetries**

  Corrected generators



[Drummond, Henn, Plefka], [Bargheer, Beisert, Galleas, Loebbert, McLoughlin], [A.S, Vieira], [Beisert, Henn, McLoughlin, Plefka]

$\bar{Q}$ - equation



[Caron-Huot, He], [Bullimore, Skinner]

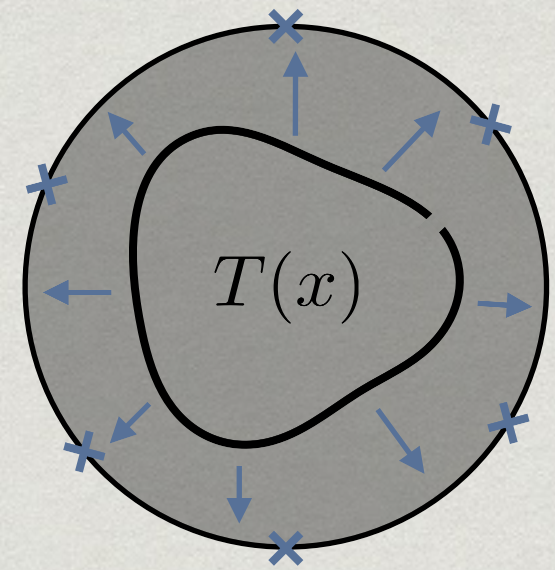
$\Rightarrow$ There is a Yangian symmetry for scattering amplitudes ($psu(2,2|4)$)

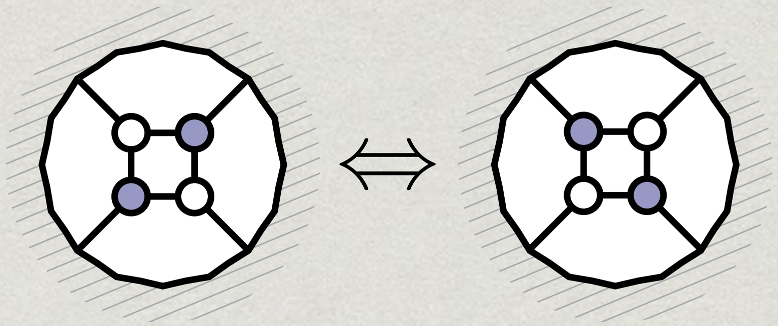
4d scattering amplitude $\leftrightarrow$ 2d integrability map

- **Symmetries**

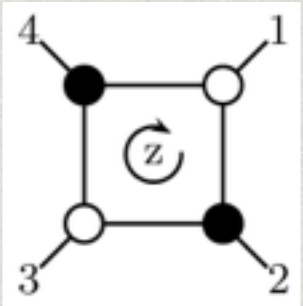
  Corrected generators
  $\bar{Q}$ - equation

- **On-shell diagrams and Deformed on-shell diagrams**





{Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Hodges, Trnka}



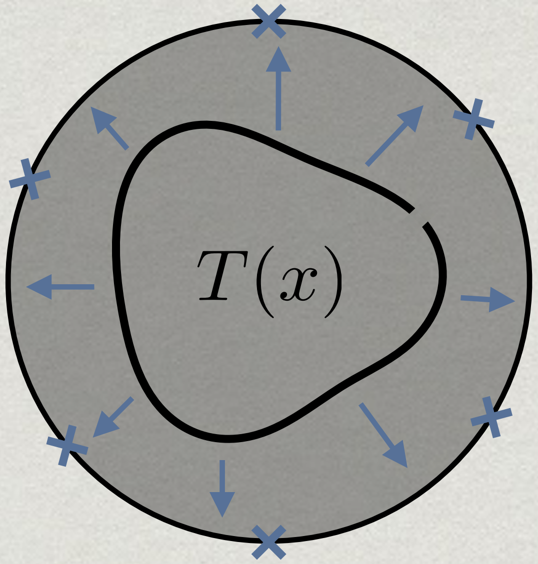
{Ferro, Lukowski, Meneghelli, Plefka, Staudacher, Kanning}, [Chicherin, Derkachov, Kirschner], [Beisert, Broedel, Rosso], [Frassek, Kanning, Ko, Staudacher]

4d scattering amplitude $\leftrightarrow$ 2d integrability map

- **Symmetries**

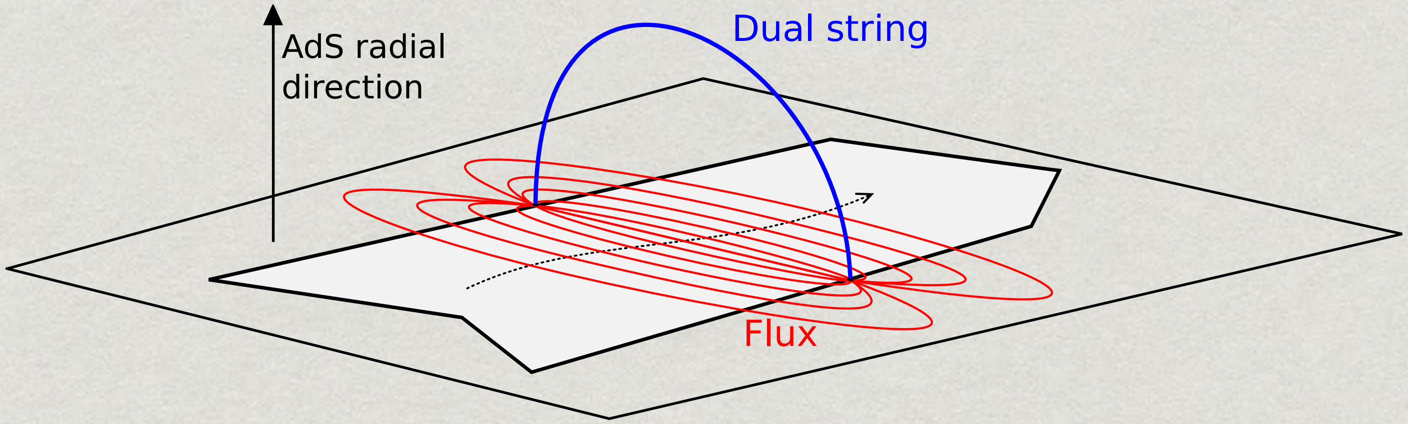
  Corrected generators

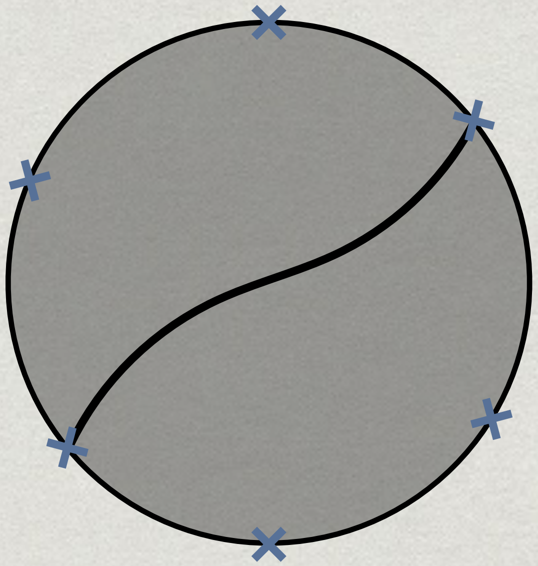
  $\bar{Q}$ - equation



- **On-shell diagrams and Deformed on-shell diagrams**

- **Pentagon operator product expansion**





[Alday, Gaiotto, Maldacena, A.S,Vieira], [Basso, A.S, Vieira]
{Basso, Belitsky, Caetano, Caron Huot, Cordova, A.S, Vieira, Wang}

Today

Main message

We are now entering to a new era where **physical juice** can be extracted from **finite coupling** solutions that otherwise would be very hard / impossible to get.

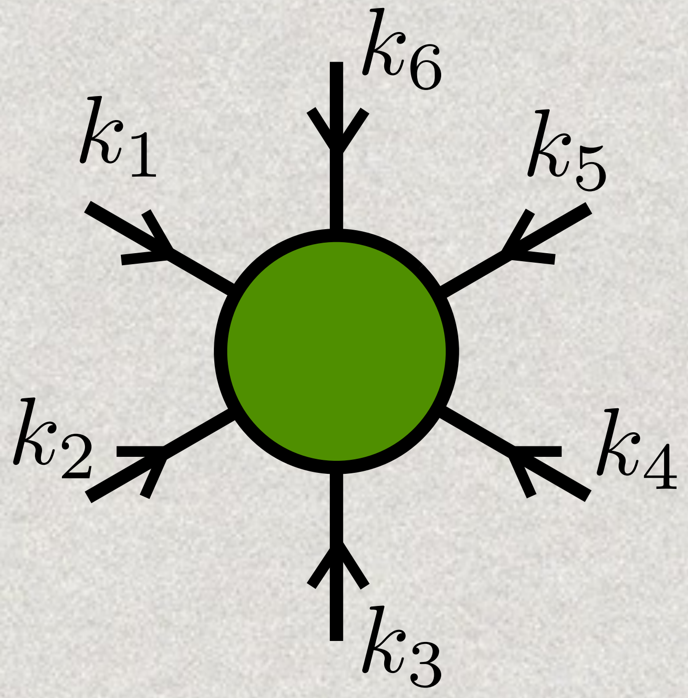
Few examples in scattering amplitudes

- Do amplitudes factories at finite coupling?
- Does BFKL holds at finite coupling (beyond resuming leading logs)?
- Does the minimal area formula hold for scattering amplitudes at strong coupling?
- How helicity amplitudes that are not related by SUSY differ from each other?

4d scattering amplitude $\leftrightarrow$ 2d integrability map

4d description

2d description

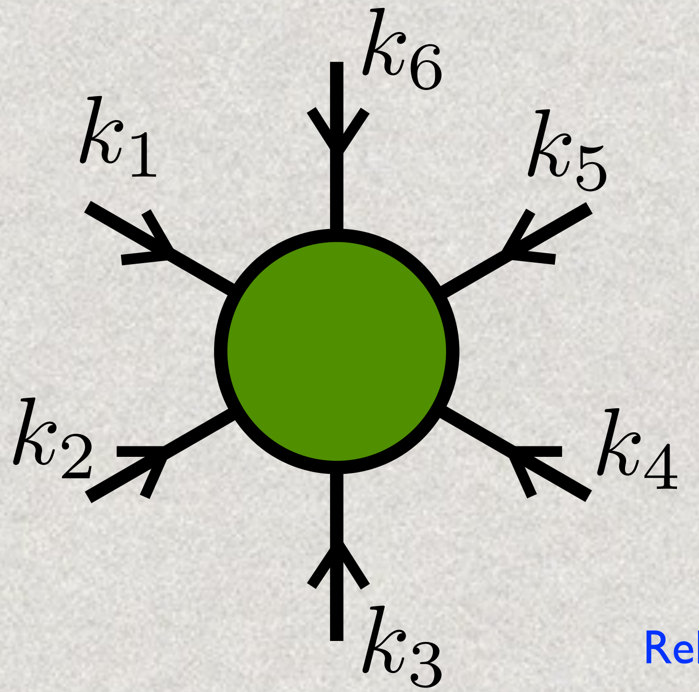


$=$

?

4d scattering amplitude $\leftrightarrow$ 2d integrability map

4d description



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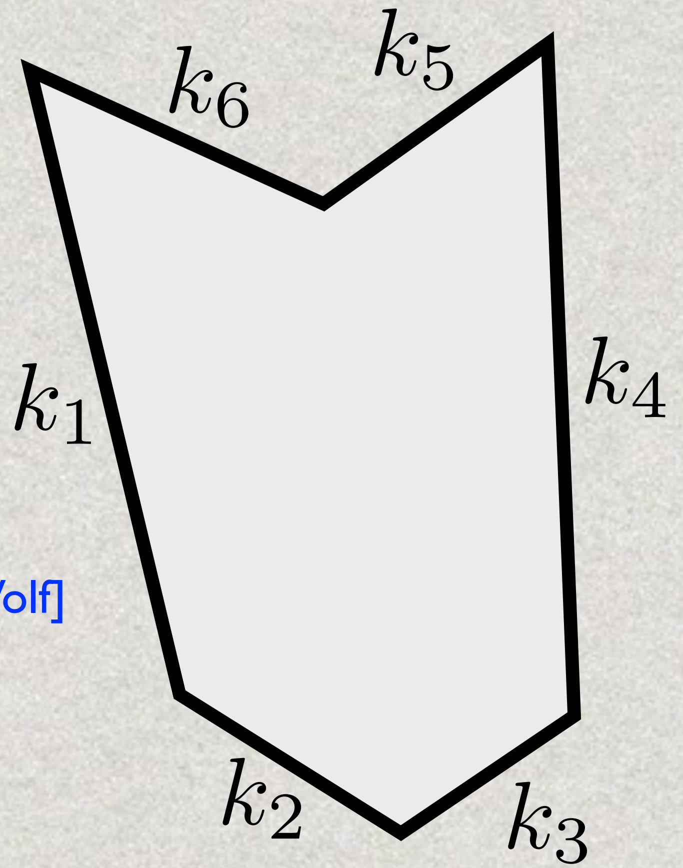
[Alday, Maldacena]
[Brandhuber, Heslop, Travaglini]
[Drummond, Henn, Korchemsky, Sokatchev]
Relation to integrability — [Beisert, Ricci, Tseytlin, Wolf]

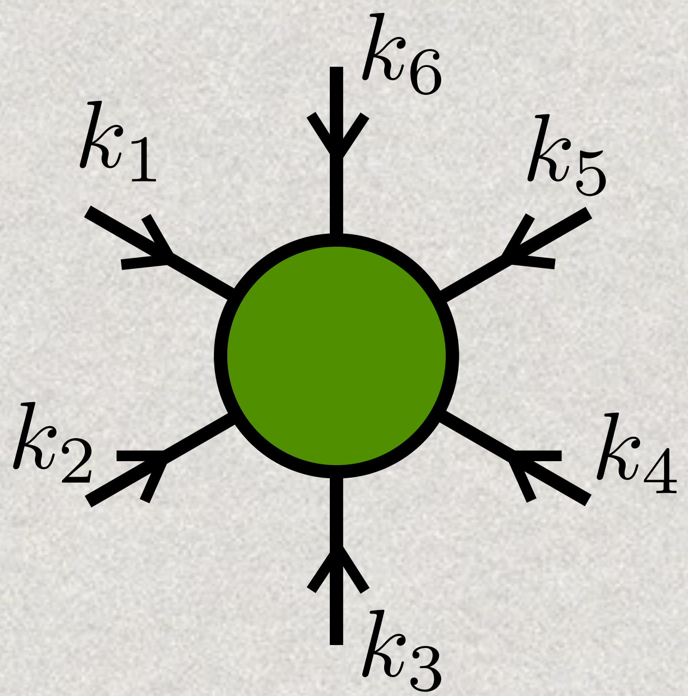
Another 4d description



Polygon Wilson loop

4d scattering amplitude $\leftrightarrow$ 2d integrability map

4d description



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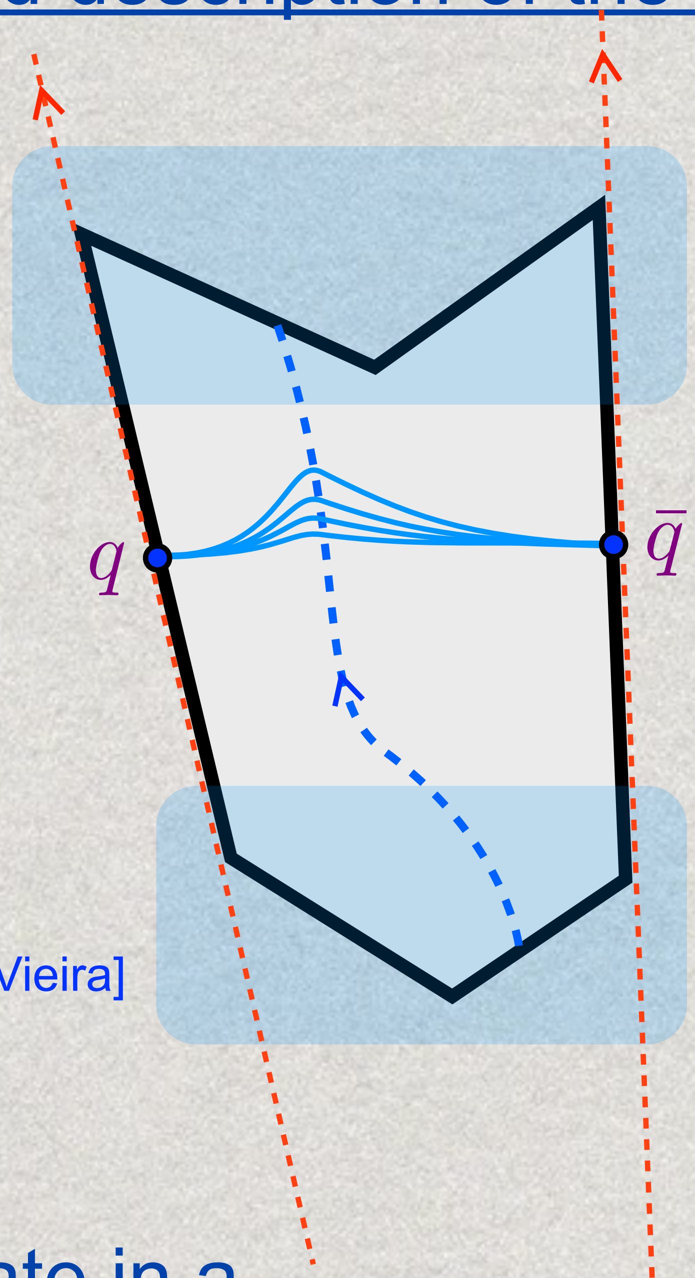
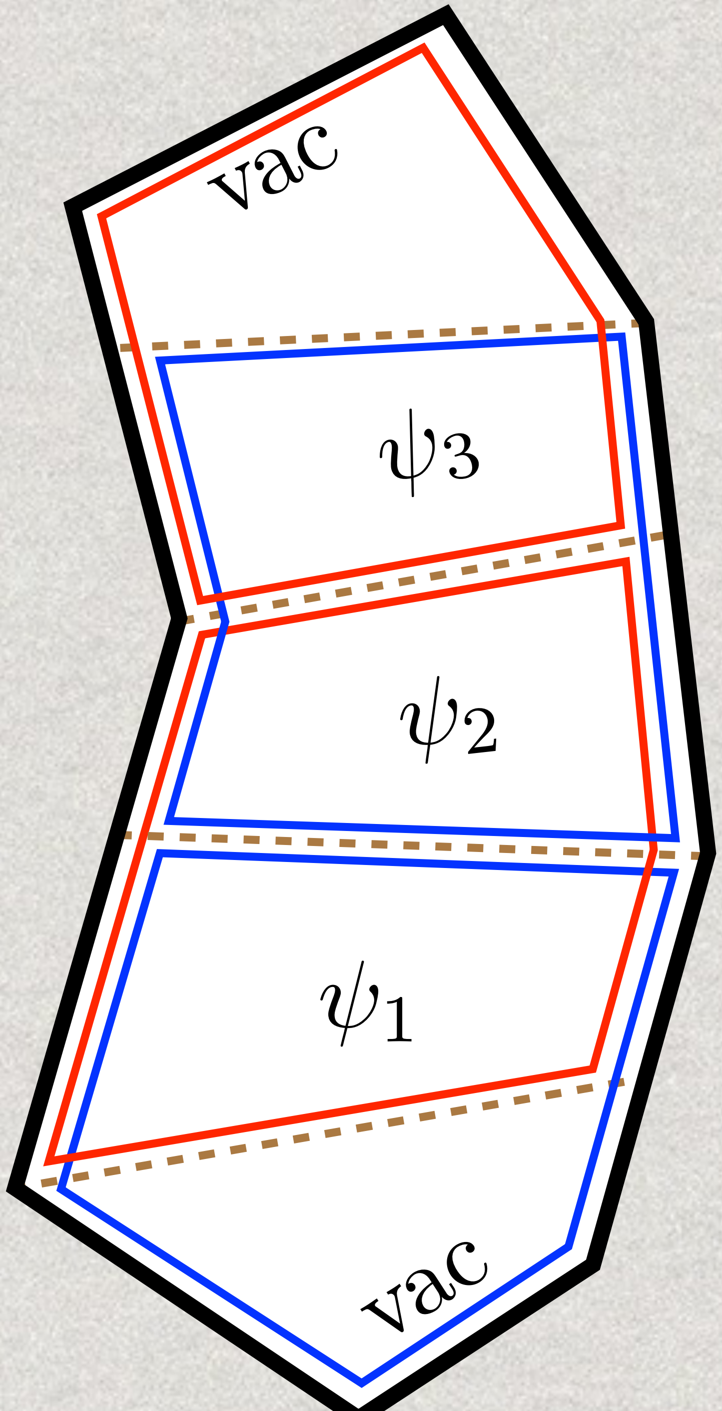
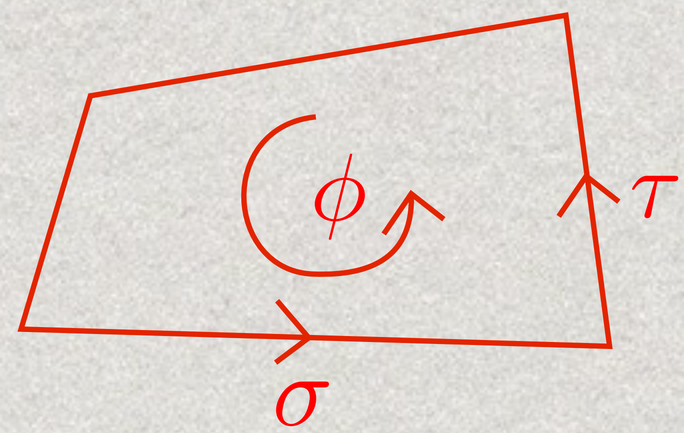
2d description of the WL



[Alday, Gaiotto, Maldacena, A.S, Vieira]

Excitation are created at the **bottom**, propagate in a **1+1 dim flux tube** and is absorbed at the **top**





geometry

$$= \sum_{\psi_i} \left[\prod_i e^{-E_i \tau_i + i p_i \sigma_i + i m_i \phi_i} \right] P(0|\psi_1) P(\psi_1|\psi_2) P(\psi_2|\psi_3) P(\psi_3|0)$$

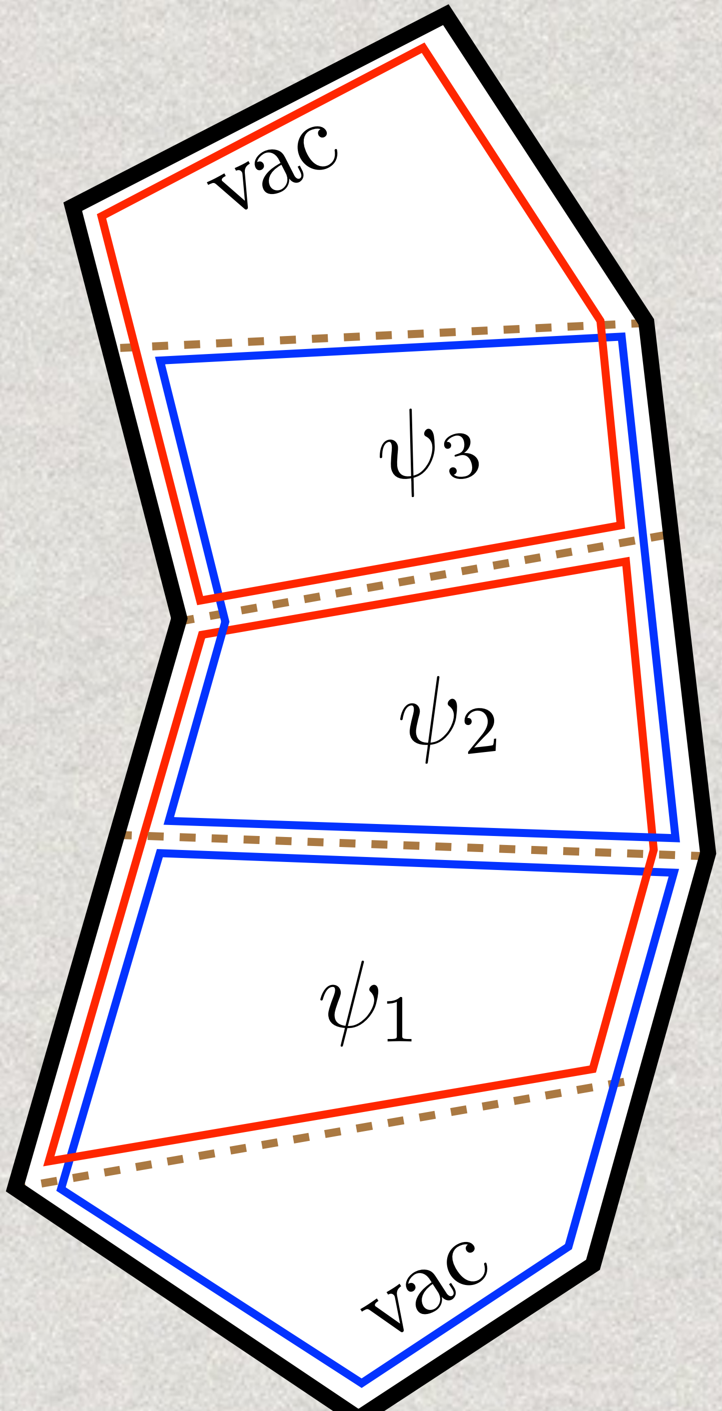
angular momentum

momentum

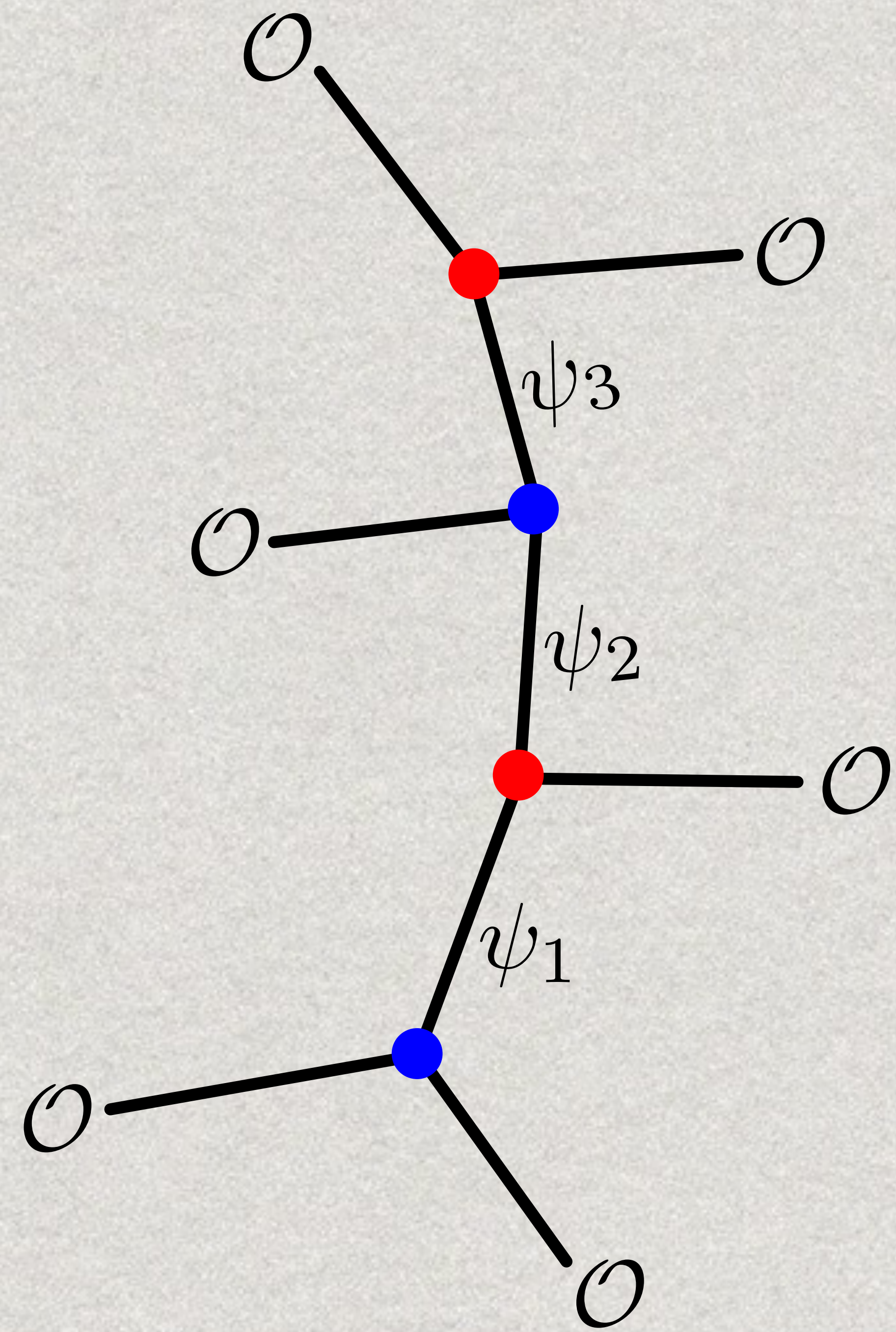
energy

pentagon transition

The decomposition



$$E \leftrightarrow \Delta$$
$$P(\psi|\varphi) \leftrightarrow C_{123}$$



OPE for Wilson loops

OPE for correlation functions

So far only we used conformal symmetry. Integrability comes in now.

Flux tube states $\longleftrightarrow$ **Large spin operators**

$$\mathcal{O} = \text{tr}\,(Z\,DDDD\ldots DDDD\,\overset{p_1}{\overrightarrow{F}}\,DDDD\ldots DDDD\,\overset{p_2}{\overrightarrow{F}}\,DDDD\ldots DDDD\,Z)$$

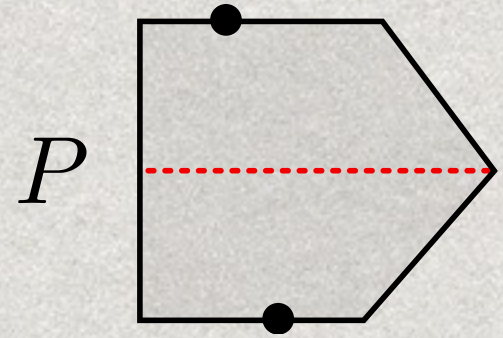
$\Longrightarrow$ Exact spectrum $E(p)$
[Basso]

Scattering phases $S(p_1, p_2)$
[Basso, Rej; Fioravanti, Piscaglia, Rossi; Basso, A.S, Vieira]

Finding the Pentagons is the most interesting part

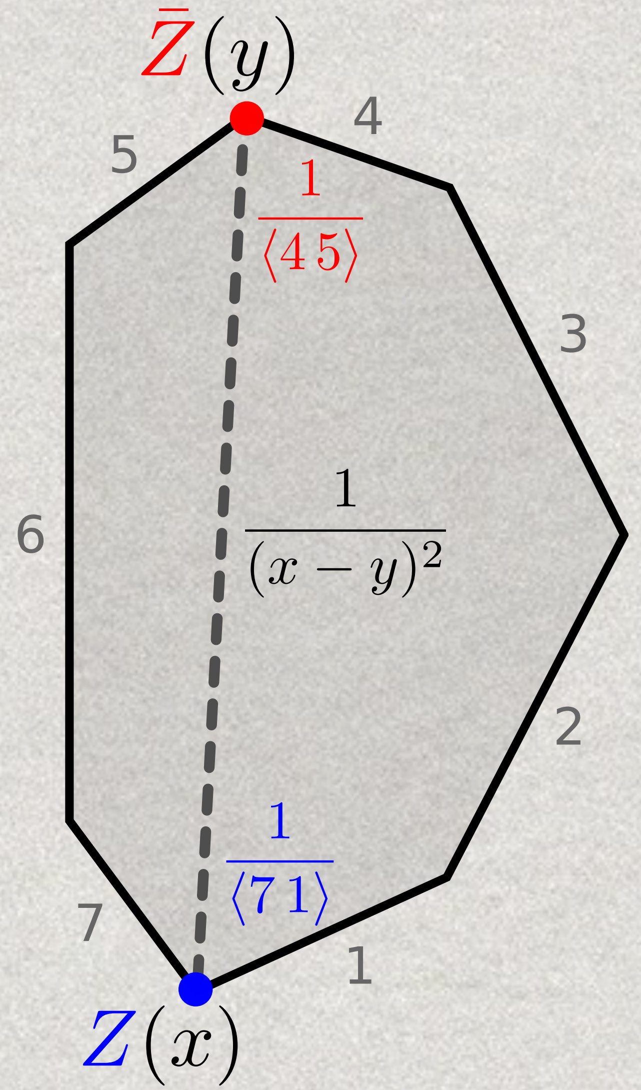
$$\mathcal{W} = \sum_{\psi_i}\left[\prod_{j=1}^{n-5} e^{-E_j\tau_j + ip_j\sigma_j + im_j\phi_j}\right] P(0|\psi_1)P(\psi_1|\psi_2)\ldots P(\psi_{n-6}|\psi_{n-5})P(\psi_{n-5}|0)$$

The single particle pentagon transitions



Scalar **tree level** example

$$\mathcal{R}^{(7145)}_{\text{tree}} =$$

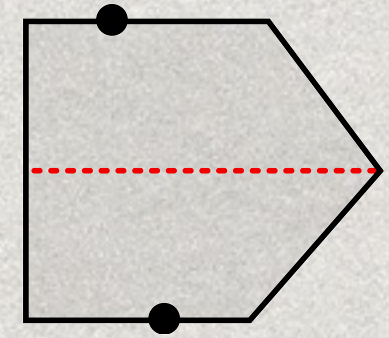


$$= \frac{1}{\langle 7\, 1\rangle\, (x-y)^2\, \langle 4\, 5\rangle}$$

[Mason, Skinner], [Caron-Huot]

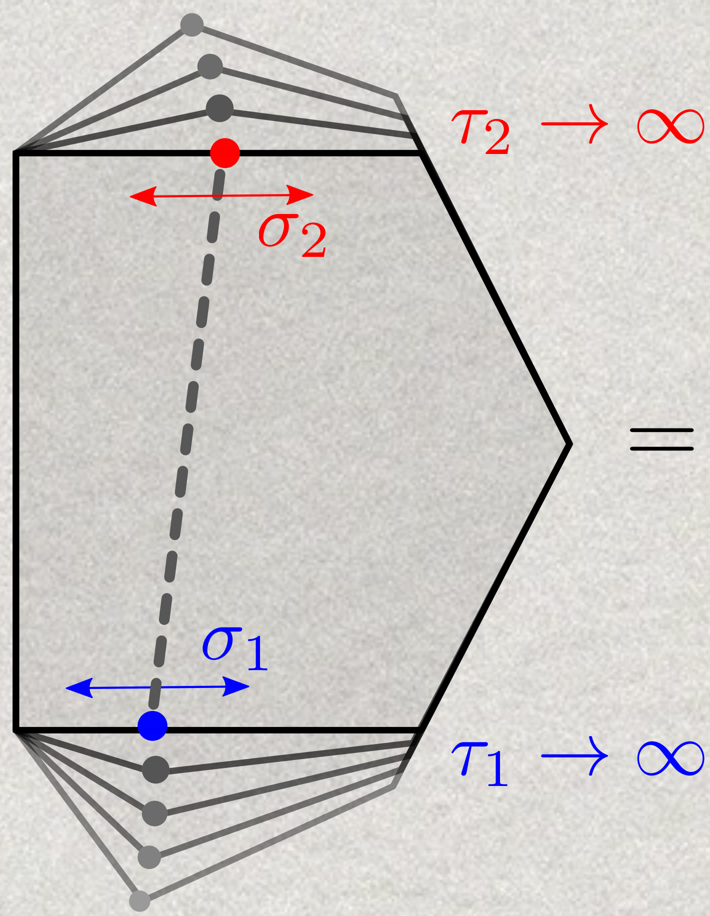
The single particle pentagon transitions

P



Scalar **tree level** example

$$\mathcal{R}^{(7145)}_{\text{tree}} =$$

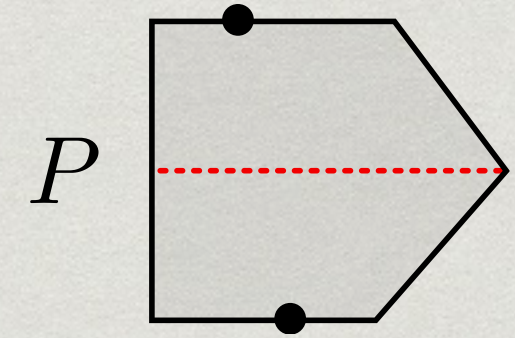


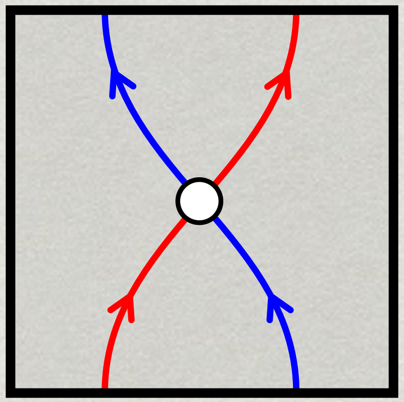
$$= \frac{e^{-\tau_1-\tau_2}}{e^{\sigma_1-\sigma_2}+e^{\sigma_2-\sigma_1}+e^{\sigma_1+\sigma_2}} + \dots$$

This looked familiar…

$$= \int \frac{dp_1 dp_2}{16\pi^2} e^{-ip_1\sigma_1+ip_2\sigma_2} \Gamma\left(\frac{1}{2}-\frac{ip_1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{ip_2}{2}\right)\Gamma\left(\frac{ip_1}{2}-\frac{ip_2}{2}\right)$$

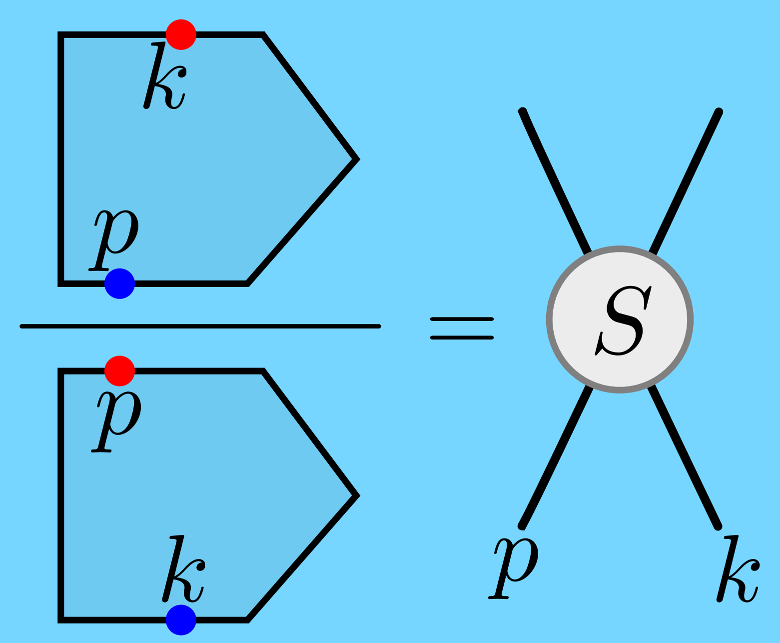
The single particle pentagon transitions



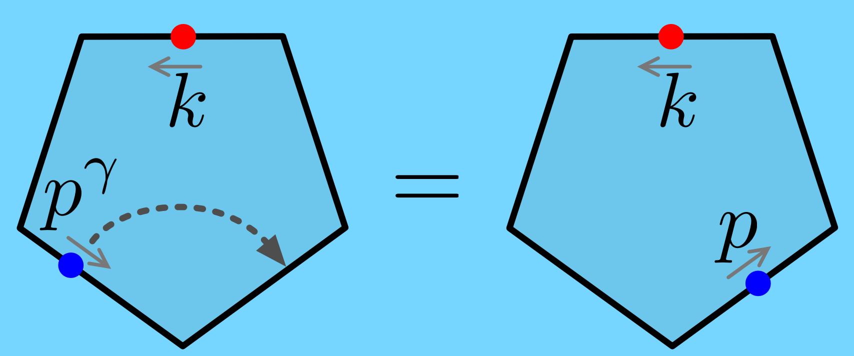
$$S(p_1,p_2) = $$  $$= \frac{\Gamma\left(\frac{1}{2}-\frac{ip_1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{ip_2}{2}\right)\Gamma\left(\frac{ip_1}{2}-\frac{ip_2}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{ip_2}{2}\right)\Gamma\left(\frac{1}{2}+\frac{ip_1}{2}\right)\Gamma\left(\frac{ip_2}{2}-\frac{ip_1}{2}\right)}$$

Axiom: also at finite coupling

$$\frac{P(p|k)}{P(k|p)} = $$



Axiom II: moving around



$$P(p^\gamma|k) = P(k|p)$$

The Solution

$$P(p|k) \propto \sqrt{\frac{S(p,k)}{S(p,k^\gamma)}}$$

[Basso,AS,Vieira]

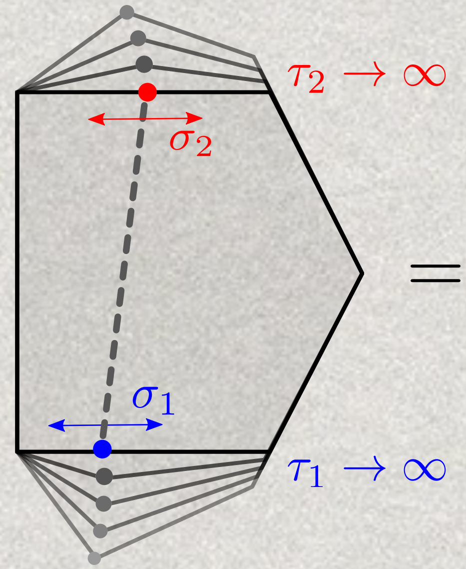
Space-time S-matrices $\leftrightarrow$ Flux tube S-matrices

Multi particle

$$P(\{p_i\}|\{k_j\}) = \frac{\prod\limits_{i,j} P(p_i|k_j)}{\prod\limits_{i>j} P(p_i|p_j) \prod\limits_{i<j} P(k_i|k_j)}$$ x (Group theory matrix part)

Checks — Weak coupling

One loop



$$= e^{-\tau_1-\tau_2} f(\sigma_1,\sigma_2) + \dots$$

Data

$$f(\sigma_1,\sigma_2) = \log\left(1+e^{2\sigma_1}\right)\log\left(1+e^{2\sigma_2}\right) - \log\left[\frac{e^{2\sigma_1}(1+e^{2\sigma_2})}{e^{2\sigma_1}+e^{2\sigma_2}+e^{2\sigma_1+2\sigma_2}}\right]\log\left[\frac{e^{2\sigma_2}(1+e^{2\sigma_1})}{e^{2\sigma_1}+e^{2\sigma_2}+e^{2\sigma_1+2\sigma_2}}\right]$$
$$+\left[\mathrm{Li}_2\left(\frac{e^{2\sigma_1}}{e^{2\sigma_1}+e^{2\sigma_2}+e^{2\sigma_1+2\sigma_2}}\right) + \mathrm{Li}_2\left(\frac{e^{2\sigma_1}}{1+e^{2\sigma_1}}\right) + \sigma_1 \leftrightarrow \sigma_2\right] - \frac{\pi^2}{6}$$

[Bern, Del Duca, Dixon, Kosower]

Integrability

$$P(u|v)_{1\ \text{loop}} = \frac{\Gamma(iu-iv)}{\Gamma(\frac12+iu)\Gamma(\frac12-iv)}\left[\frac{\pi^2}{3} - \psi_1(\tfrac12-iu) - \psi_1(\tfrac12+iv)\right.$$
$$+ H_{iu-\frac12}H_{iv-\frac12} + H_{-iu-\frac12}H_{-iv-\frac12} + H_{-iu-\frac12}H_{iv-\frac12} - H_{iu-\frac12}H_{-iv-\frac12}$$
$$\left. - H_{iu-\frac12}H_{iu-\frac12} - H_{-iu-\frac12}H_{-iu-\frac12} - H_{iv-\frac12}H_{iv-\frac12} - H_{-iv-\frac12}H_{-iv-\frac12}\right]$$

where
$$\psi(z) = \partial_z \log\Gamma(z)$$
$$H_z = \psi(z+1) - \psi(1)$$
$$\psi_1(z) = \partial_z \psi(z)$$

Checks — Weak coupling

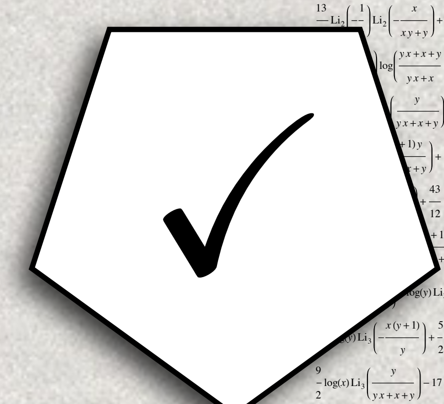
Two loop

Data [Caron-Huot, He]

$$-3\,\mathrm{Li}_{2,2}\left(-x,-\frac{y}{y(1+x)}\right)+2\,\mathrm{Li}_{3,1}\left(-x,-\frac{y}{xy+x}\right)+$$

$$x=e^{-2\sigma_1}$$

$$y=e^{-2\sigma_2}$$



Higher loops is just `shift + enter`

[Basso, AS, Vieira]

Checks — Weak coupling

Two lo

The Hexagon Function Program

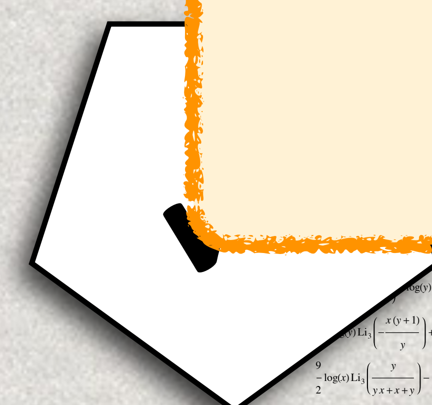
$$\mathcal{W} = a_1 f^{(1)}_{\text{hex}}(\sigma,\tau,\phi) + a_2 f^{(2)}_{\text{hex}}(\sigma,\tau,\phi) + a_3 f^{(3)}_{\text{hex}}(\sigma,\tau,\phi) + \dots$$

where the functions are a base of so-called iterated integrals of a certain degree (we can think of them as fancy generalizations of logarithms and polylogarithms). To fix the constants one can then "simply" expand the ansatz and compare with the OPE. Then it feeds back into the OPE as a very powerful self-consistency check, *both* of the Hexagon ansatz and of the integrability based conjectures.

	3 loops (symbol) [7]	4 loops (symbol) [8]
# of constants before imposing (most of) OPE	2	80
# of constants after imposing $e^{-\tau \pm i\phi}$	0	4
# of constants after imposing $e^{-2\tau \pm 2i\phi}$	✓	0
# of constants after imposing $e^{-2\tau + 0i\phi}$	✓	✓

[Dixon, Drummond, Duhr, Henn, Pennington, Von Hippel]

See — Von Hippel talk



[Basso, AS, Vieira]

Strong coupling — the Emergence of a **String** and an **holographic** direction

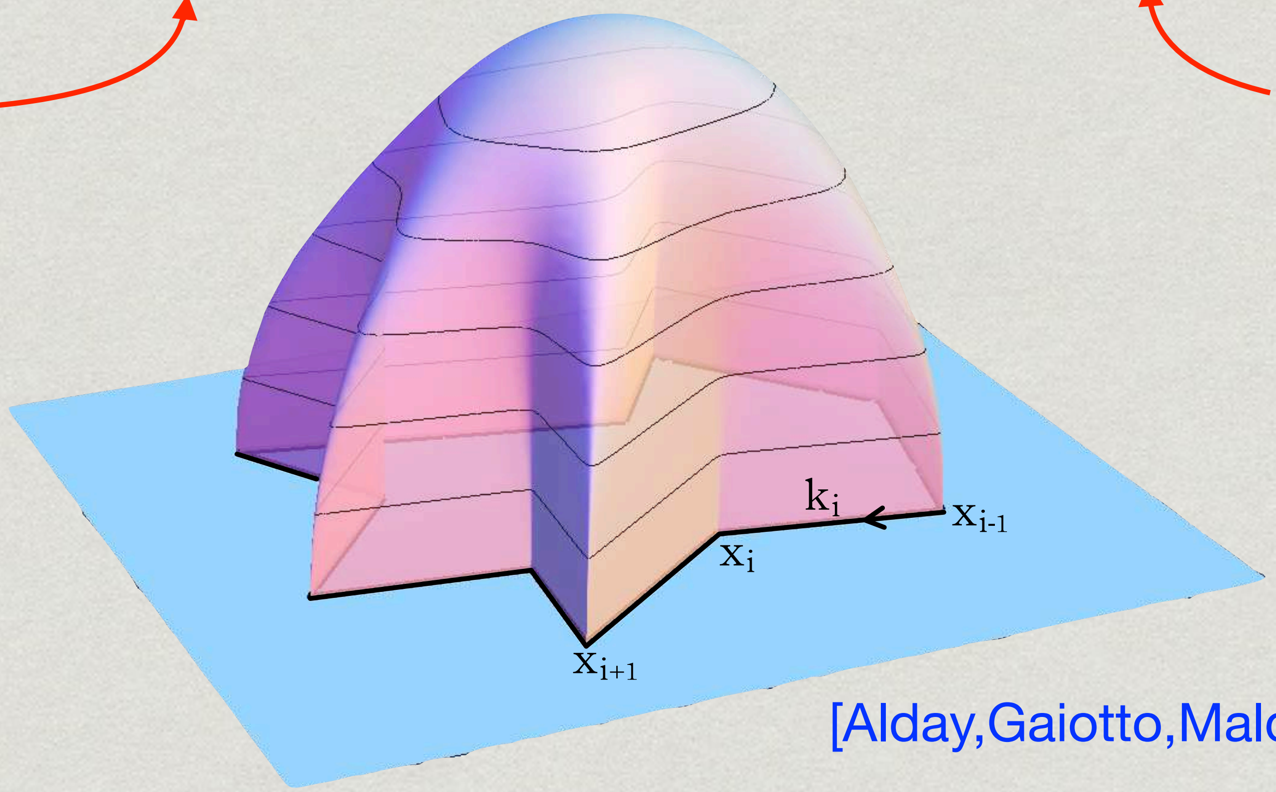
vacuum

$$\mathcal{W}_6 = 1 - \frac{\sqrt{\lambda}}{2\pi}\int d\theta \left[\frac{e^{i\phi}+e^{-i\phi}}{\pi\cosh^2(2\theta)} e^{-\sqrt{2}\tau\cosh\theta + i\sqrt{2}\sigma\sinh\theta} + \frac{1}{\pi\sinh^2(2\theta)} e^{-2\tau\cosh\theta + i2\sigma\sinh\theta}\right] + \dots$$

2 transverse modes in AdS

AdS radial mode

$$= e^{-\frac{\sqrt{\lambda}}{2} Y Y_c} =$$

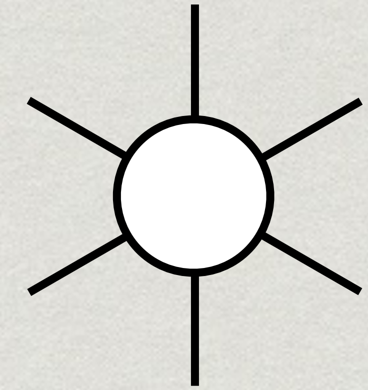


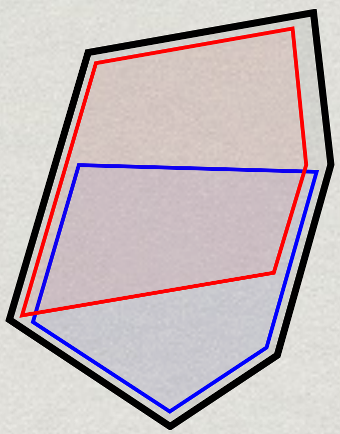
[Alday,Gaiotto,Maldacena,AS,Vieira]

Quantum gas of particles @ finite coupling $\longrightarrow$ *Purely Geometrical Problem* @ strong coupling

The **full** MHV six gluon amplitude **@ finite coupling**

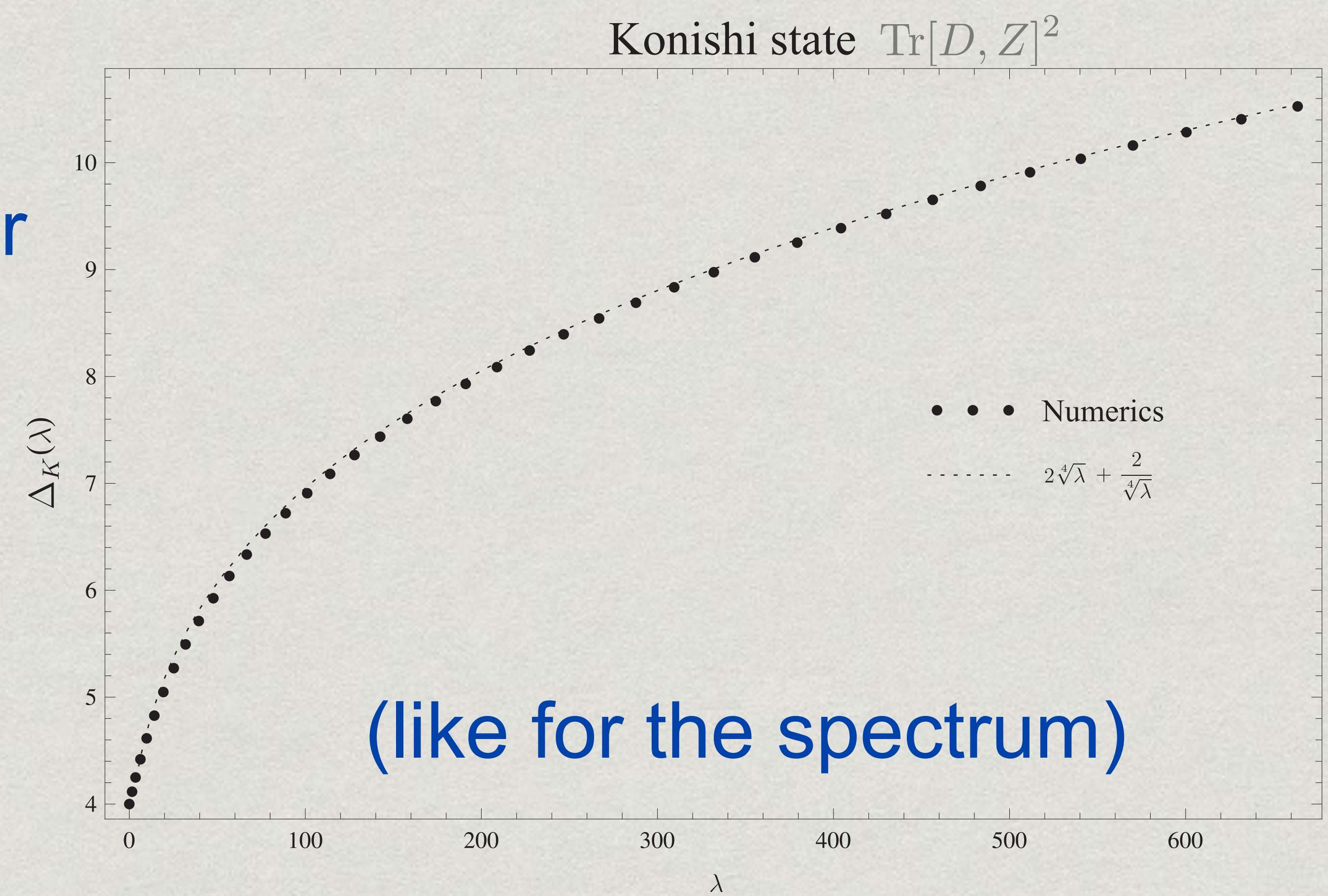
To be published very soon [Basso,AS,Vieira]



$\mathcal{W}_{\rm hex} =$  $= \sum_{\mathbf{\Psi}} \int d\mathbf{p}\,\mu(\mathbf{p})\, P_{\mathbf{\Psi}}(0|\mathbf{p}) P_{\mathbf{\Psi}}(\bar{\mathbf{p}}|0)\, h_{\mathbf{\Psi}}(\mathbf{p})\, e^{-\tau \sum E(p_i) + i\sigma \sum p_i + i\phi \sum m_i}$

Next — we will put on a computer and generate a ***plot***

[with Von Hippel]



(like for the spectrum)

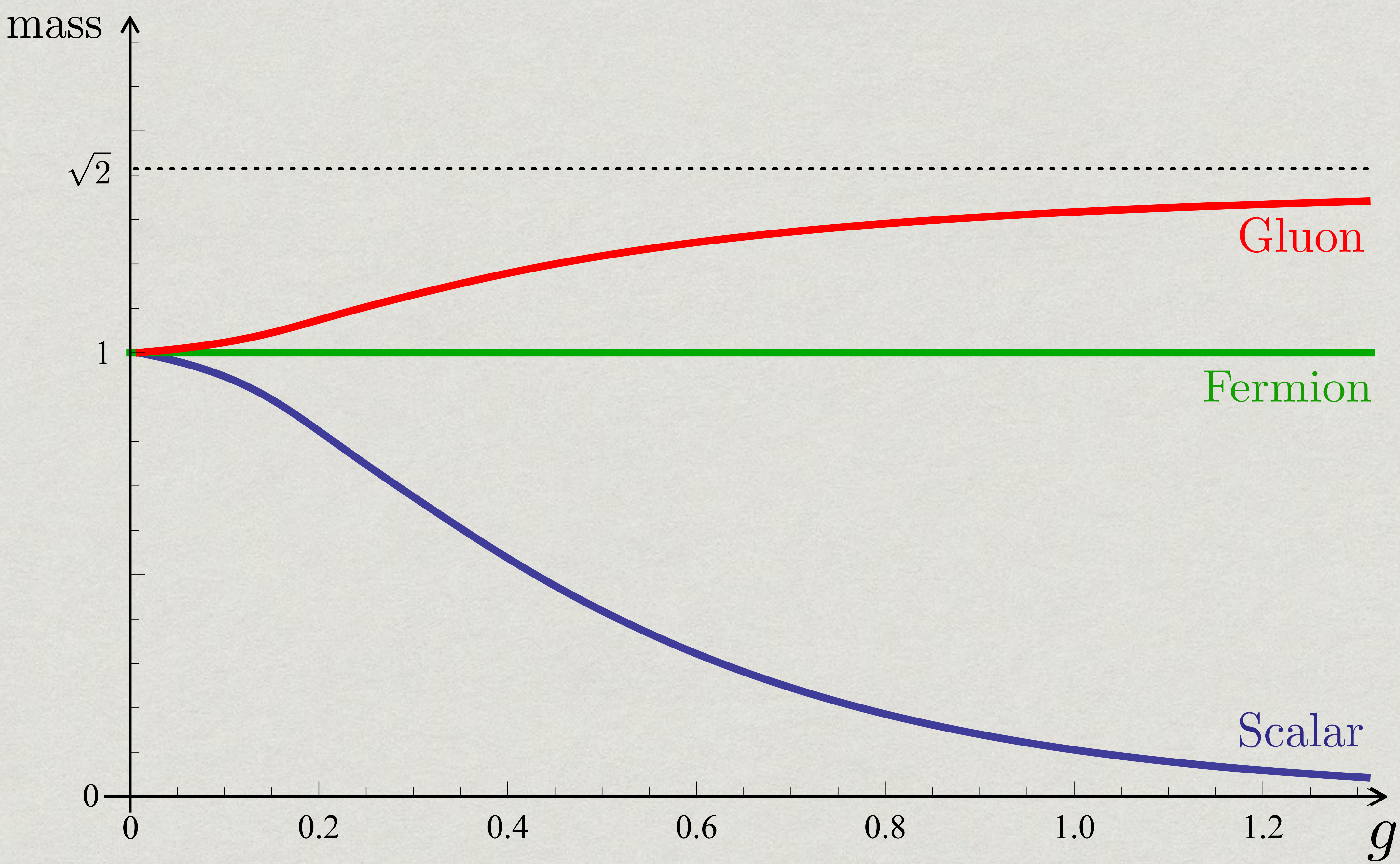
Gluon mass at finite coupling

$\mathrm{tr}(\mathrm{ZZDZZZDZZZZ}) \rightarrow \mathrm{tr}(\mathrm{ZDZDZDZD^2ZDZZDZDZ}) \rightarrow \mathrm{tr}(\mathrm{ZDDDDDDDDDDDDDDDDD\mathbf{F}DDDDDDDDDZ})$

BMN point-like string
[Berenstein,Maldacena,Nastase]

$(\mathrm{D_{null}})^{S\rightarrow\infty}$

GKP folded string
[Gubser,Klebanov,Polyakov]



Application

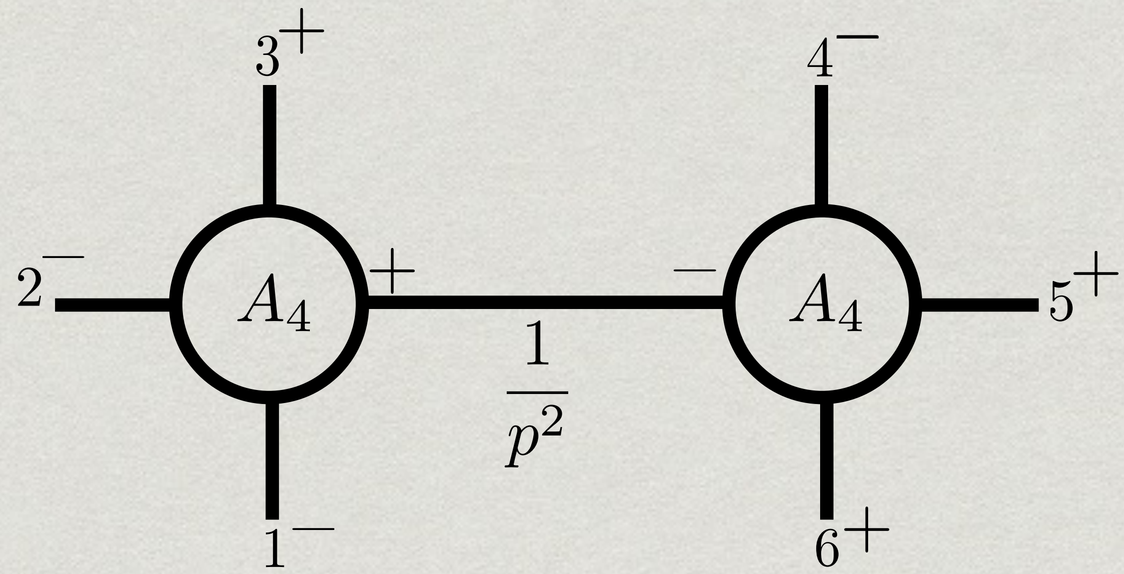
We are now entering to a new era where **physical juice** can be extracted from **finite coupling** solutions that otherwise would be very hard / impossible to get.

Application

I. Multi-particle factorization in massless gauge theory

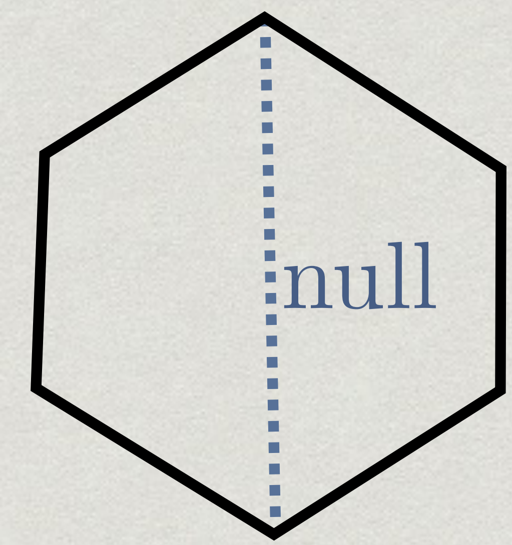
[Basso,AS,Vieira - in progress]

$A_6^{\text{NMHV}} \xrightarrow[p^2\to 0]{}$

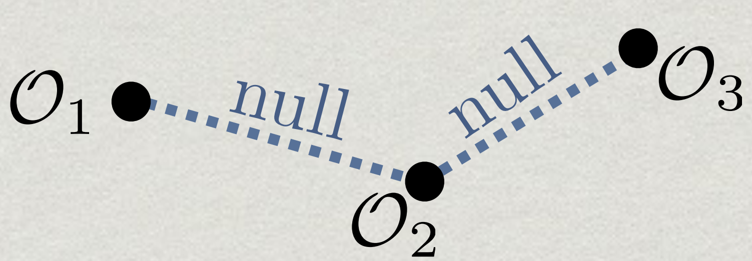


$\simeq A_4(1,2,3,p)\frac{F(p,s_{i,i+1})}{p^2}A_4(-p,4,5,6)$

Dual limit for Wilson loops



Related null limit for correlation functions



How this limit looks at finite coupling?

Application

I. Multi-particle factorization in massless gauge theory

[Basso,AS,Vieira - in progress]

$$I \equiv \int_0^\infty du\, e^{-u\,p^2 - \Gamma_{\rm cusp} \log^2 u}$$

Relevant part of the amplitude

$$\Gamma_{\rm cusp} = 4g^2 - \frac{4}{3}\pi^2 g^4 + \ldots$$

1) At weak coupling

$$I = \frac{1}{p^2} \sum_l g^{2l}\, \mathrm{Pol}_l(\log p^2)$$

Perfect match with up to 3 loops
[Dixon, von Hippel]

2) At any $g \neq 0$

$$I|_{p^2=0} = \int_0^\infty du\, e^{-\Gamma_{\rm cusp} \log^2 u} < \infty$$

No pole!

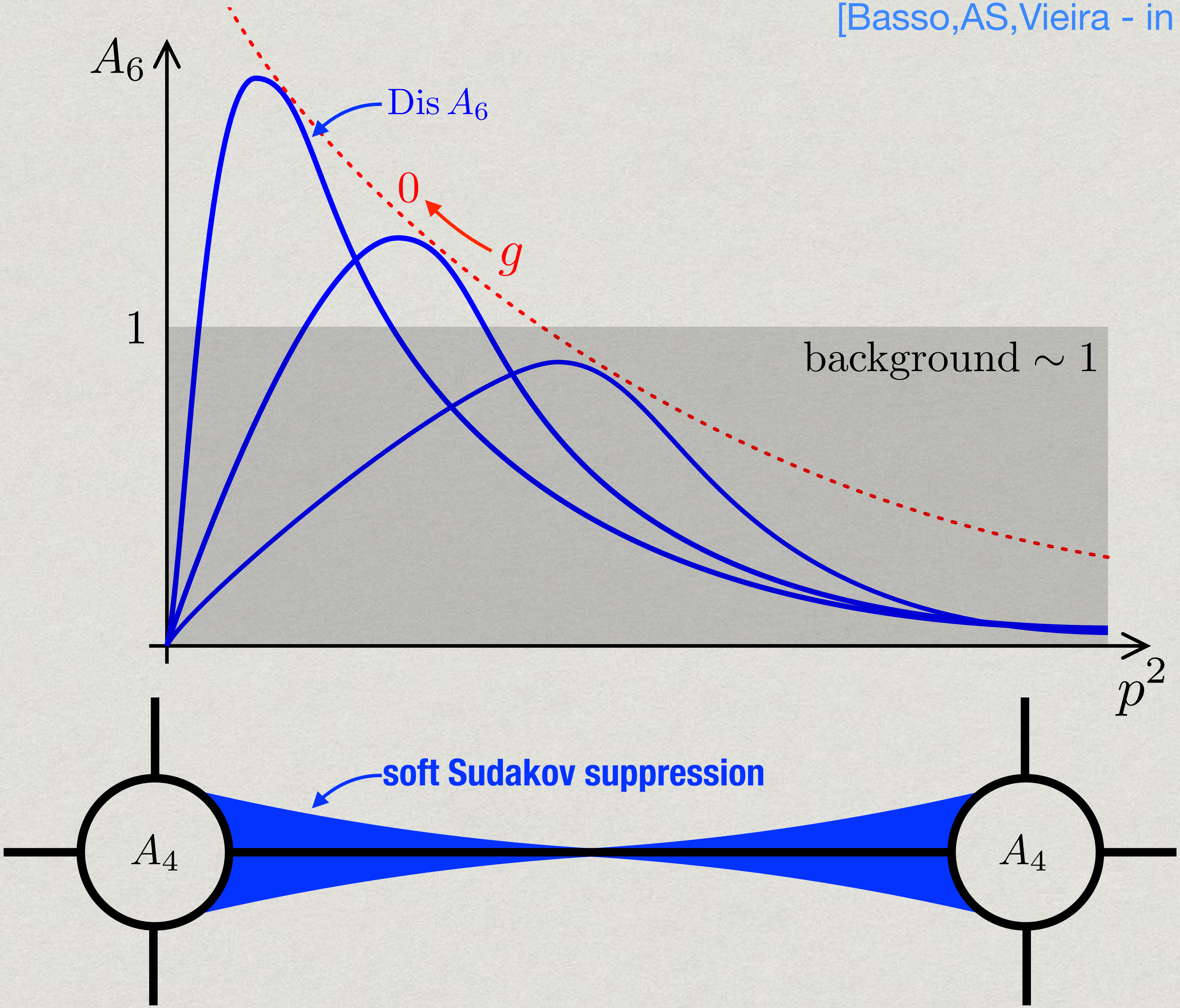
3) There is a discontinuity

$$\mathrm{Dis}\, A_6 \propto e^{-\Gamma_{\rm cusp} \log^2(p^2)} \neq 0$$

Application

I. Multi-particle factorization in massless gauge theory

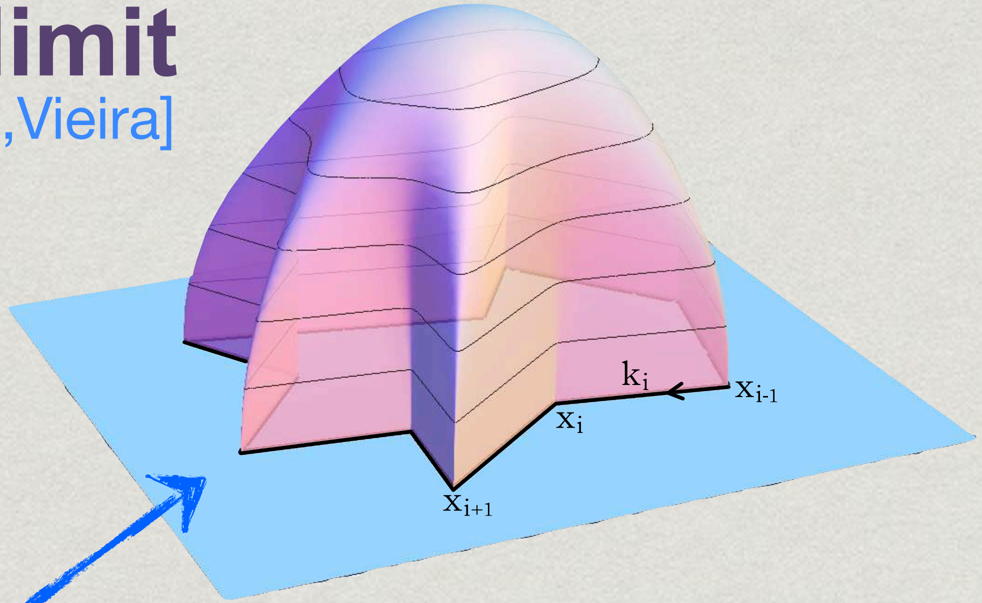
[Basso,AS,Vieira - in progress]



Application

II. Strong coupling and collinear limit

[Basso,AS,Vieira]



[Alday,Gaiotto,Maldacena,AS,Vieira]

$$A \xrightarrow[\lambda\to\infty]{} e^{-\frac{\sqrt{\lambda}}{2\pi}\mathrm{Area}} + \ldots$$

- Is that so?
- What are the leading corrections?

classical

$$\mathcal{W}_{n=6} = \underbrace{f_6\,\lambda^{-\frac{7}{288}}\,e^{\frac{\sqrt{\lambda}}{144}}}_{\text{quantum}}\,e^{-\overbrace{\frac{\sqrt{\lambda}}{2\pi}A_{n=6}}^{\text{classical}}}\left(1+O(1/\sqrt{\lambda})\right)$$

quantum

Pre-factor

$$f_6 = \frac{1.04}{(\sigma^2+\tau^2)^{1/72}} + O(e^{-\sqrt{2}\tau})$$

Application

II. Strong coupling and collinear limit

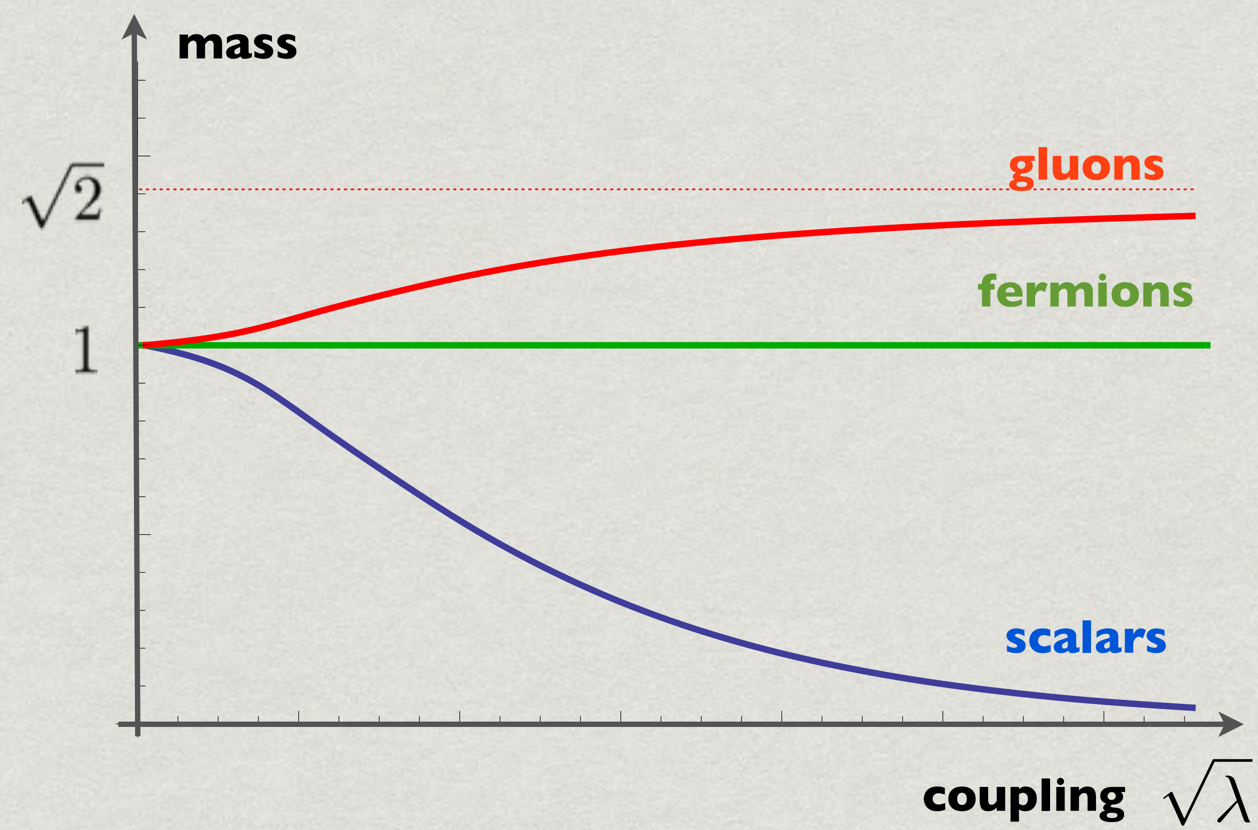
[Basso,AS,Vieira]

$$\mathcal{W}_{n=6} = f_6\, \lambda^{-\frac{7}{288}}\, e^{\frac{\sqrt{\lambda}}{144} - \frac{\sqrt{\lambda}}{2\pi} A_{n=6}} (1 + O(1/\sqrt{\lambda}))$$

Quantum fluctuation piece is of the **same order** as the classical one!

$$m = \frac{2^{1/4}}{\Gamma(5/4)} \lambda^{1/8} e^{-\frac{\sqrt{\lambda}}{4}} (1 + O(1/\sqrt{\lambda})) \ll 1$$

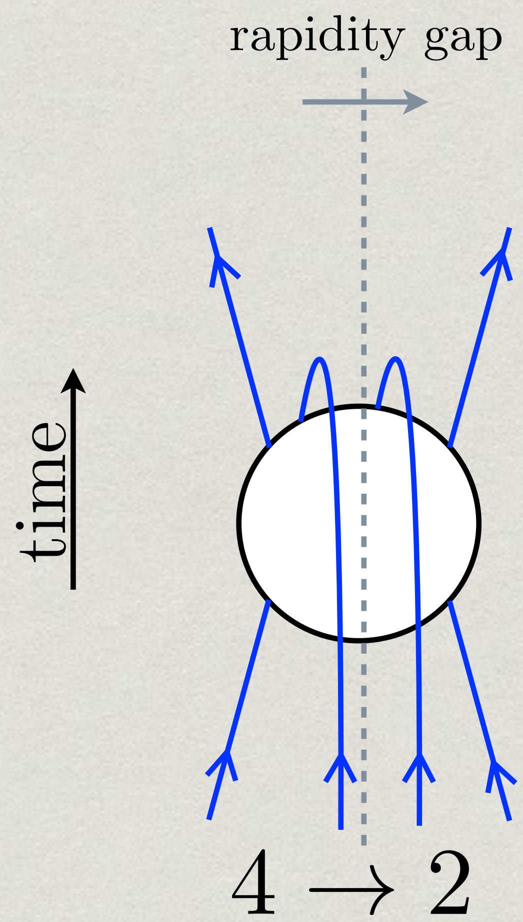
Scalar mass is *exponentially small* at strong coupling



Application

III. Adjoint BFKL (multi-Regge limit) @ finite coupling

[Caron-Huot, Basso, A.S]



"Mandelstam region"

[Bartels,Lipatov,Sabio Vera]
[Bartels,Lipatov,Prygarin]

- An expansion @ weak coupling (leading logs)
- Similar to the collinear (OPE) expansion

$\Rightarrow$

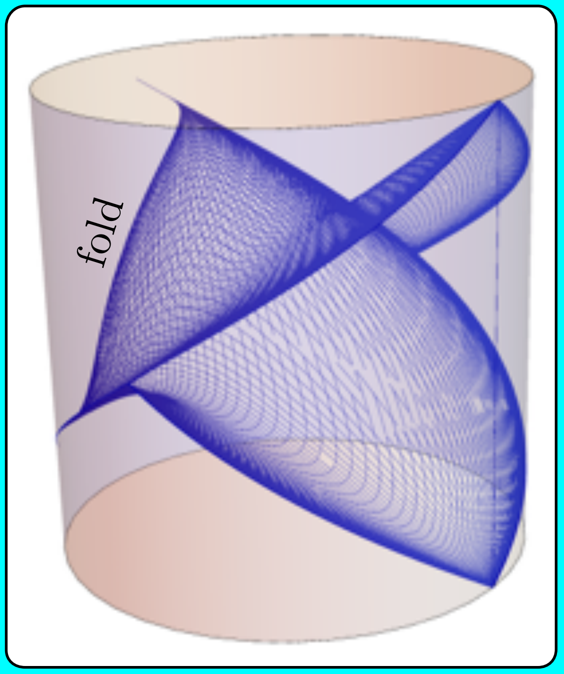
By analytic continuation from the collinear limit

- Exact solution for BFKL energies $\omega(\nu, m)$

$$-\omega(\nu,m) = 2g^2\left\{-\frac{2|m|}{\nu^2+m^2} + \psi\left(1+\frac{|m|+i\nu}{2}\right) + \psi\left(1+\frac{|m|-i\nu}{2}\right) - 2\psi(1)\right\} + O(g^4)$$

- Expansion is valid at finite coupling
- Strong coupling description

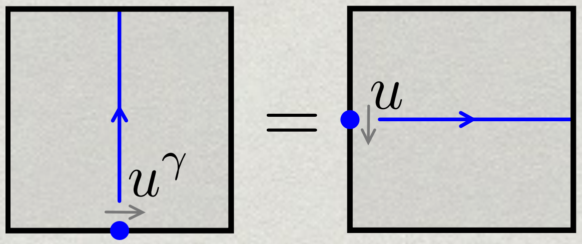
Minimal surface in AdS



Connection to other approaches

- **Symmetries**
  - In the POPE we are choosing a specific channel $\Rightarrow$ Global symmetries (Yangian) are not manifest
  - Understanding how these symmetries comes about will be very fruitful for both (may be instrumental in resuming the POPE)
  - We are now in position to start understanding **cyclicity** and **$\bar{\mathrm{Q}}$-equation**

- **Cyclicity**
  By analytic continuation of external momenta
  + mirror relation



- **$\bar{\mathrm{Q}}$-equation**

See [Belitsky] for recent check

r.h.s. - Derivatives w.r.t. external momenta $\leftrightarrow$ derivatives w.r.t. $\{\tau, \sigma, \phi\}$

l.h.s. (anomaly) - Know how to insert fermions at zero momentum (no need for amplitudes with more particles!)

[Basso, A.S. Vieira] - next week.

Connection to other approaches

- **Deformed on-shell diagrams**

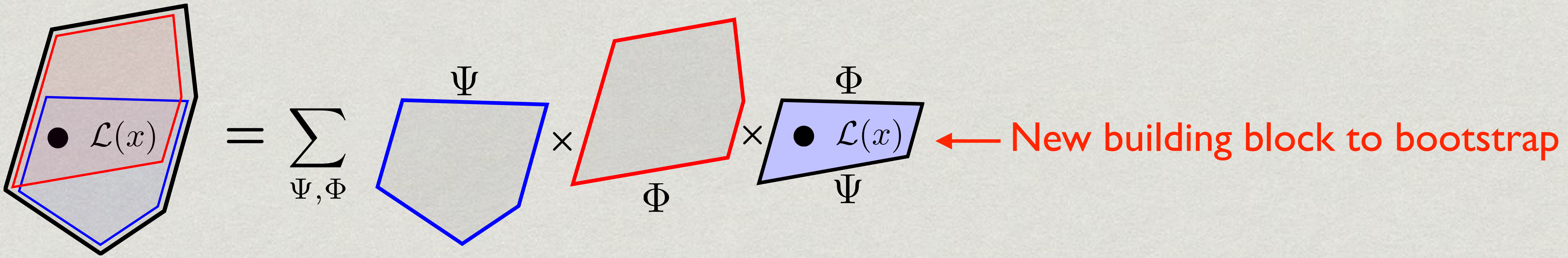
Compute a deformed amplitude (6-point NMHV) ? $\Rightarrow$ Expand in the POPE variables ? $\Rightarrow$ Deformed POPE building blocks ?

- **On-shell diagrams**

Bootstrap correlation function with local operators

$$\langle W_{\text{polygon}} \; \mathcal{L}(x) \rangle$$

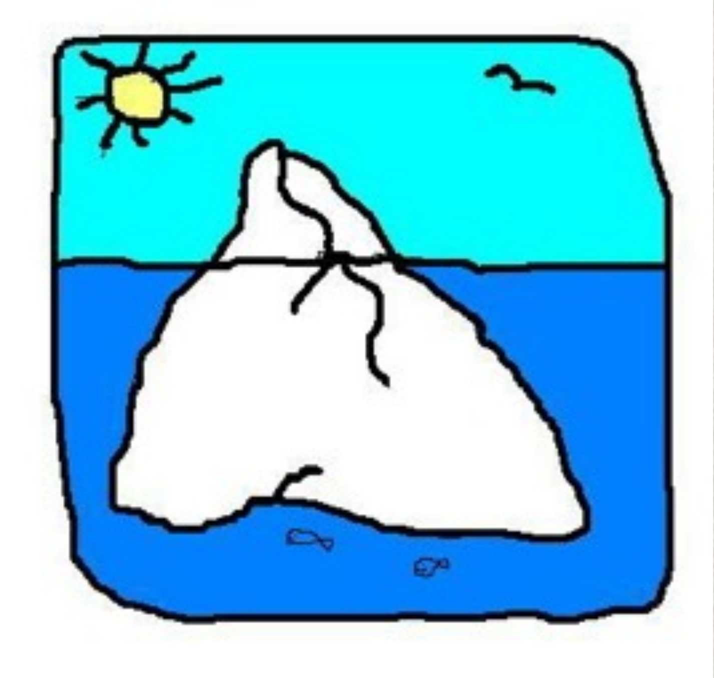
[Gaiotto, Mazac] - in progress



Direct connection when number of Lagrangian insertion equals to the loop order

Outlook

Scattering Amplitudes in =4 SYM



- Plot from weak to strong coupling
- Re-summing the OPE series at finite coupling?
- Extend to other kinematical regimes

  (BFKL - [Basso, Caro-Huot, A.S])
- Connection between the OPE and approaches that make other symmetries manifest
- Go beyond the planar limit - 1/N corrections using integrability

Applications

- General structures in scattering amplitudes
- Lessons for non-perturbative QFT.
- Lessons for large N QCD, effective QCD string descriptions.
- Test-ground for numerical QCD.
- Develop perturbative expansion around this theory?