Twistors, Strings & Twistor Strings

a review, with recent developments

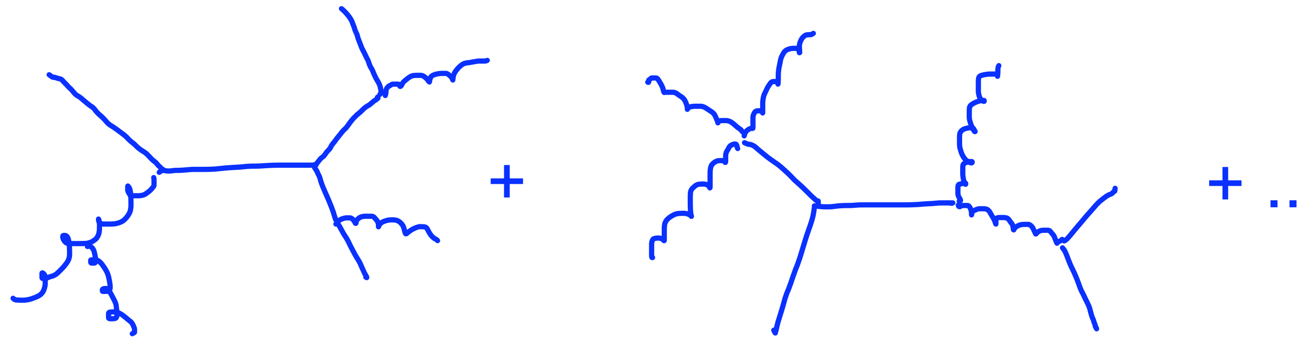
David Skinner, DAMTP

- Spaces of null rays and the scattering equations
  - ambitwistor strings & CHY formulae
  - curved backgrounds & the Einstein equations
  - a new approach to loop amplitudes
- Twistor strings in four dimensions
  - the scattering equations and spinor helicity
  - a new formula for all sEYM trees
- Relation to string theory
  - twistor origin of the superstring
  - KLT contours and localization

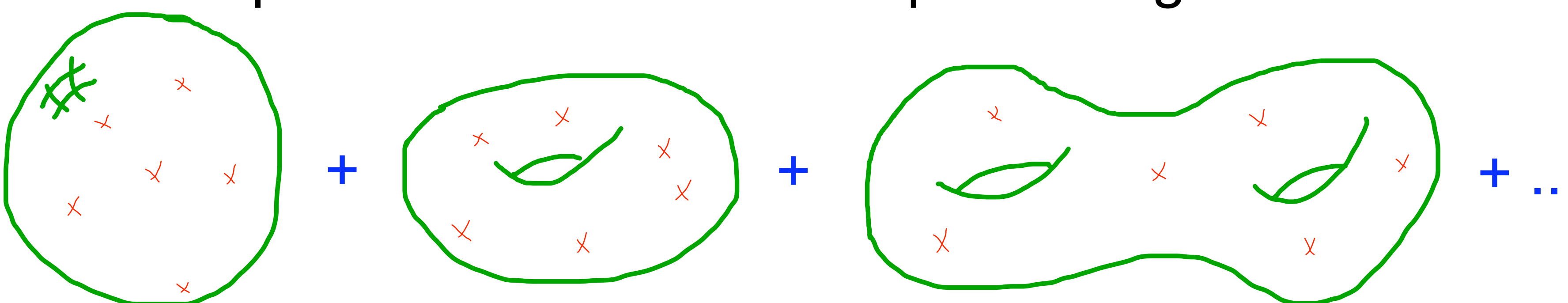
Null rays & the scattering equations

Even at tree level, Feynman diagrams for multi-particle processes present a formidable problem in graph combinatorics

$$\sum_{\text{graphs}} \left(\cdots \right) =$$



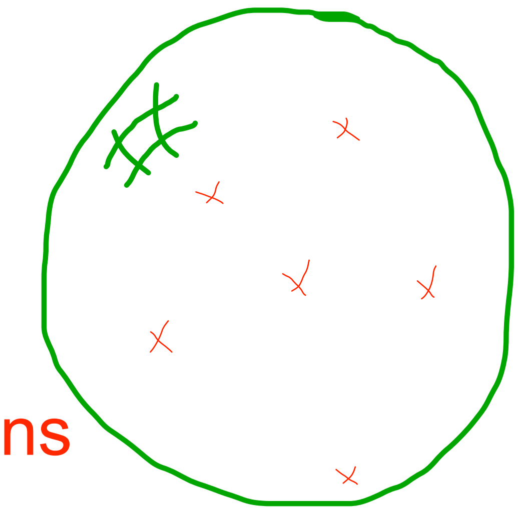
In string theory, the combinatoric problem is trivial. Instead one has to perform difficult moduli space integrals



$$= \sum_{g \geq 0} \int_{\mathcal{M}_{g,n}} \left(\cdots \right)$$

Cachazo, He & Yuan have found that, in a wide range of massless theories, tree amplitudes may be computed as

$$\sum_{z_i \,|\, S_j(z_k)=0} F(k_i, \epsilon_i, z_i)$$



- auxiliary data z_i solve the scattering equations

$$S_i(z_j) \equiv \sum_{j\neq i} \frac{k_i \cdot k_j}{z_{ij}} = 0 \qquad i = 1, \ldots, n-3$$

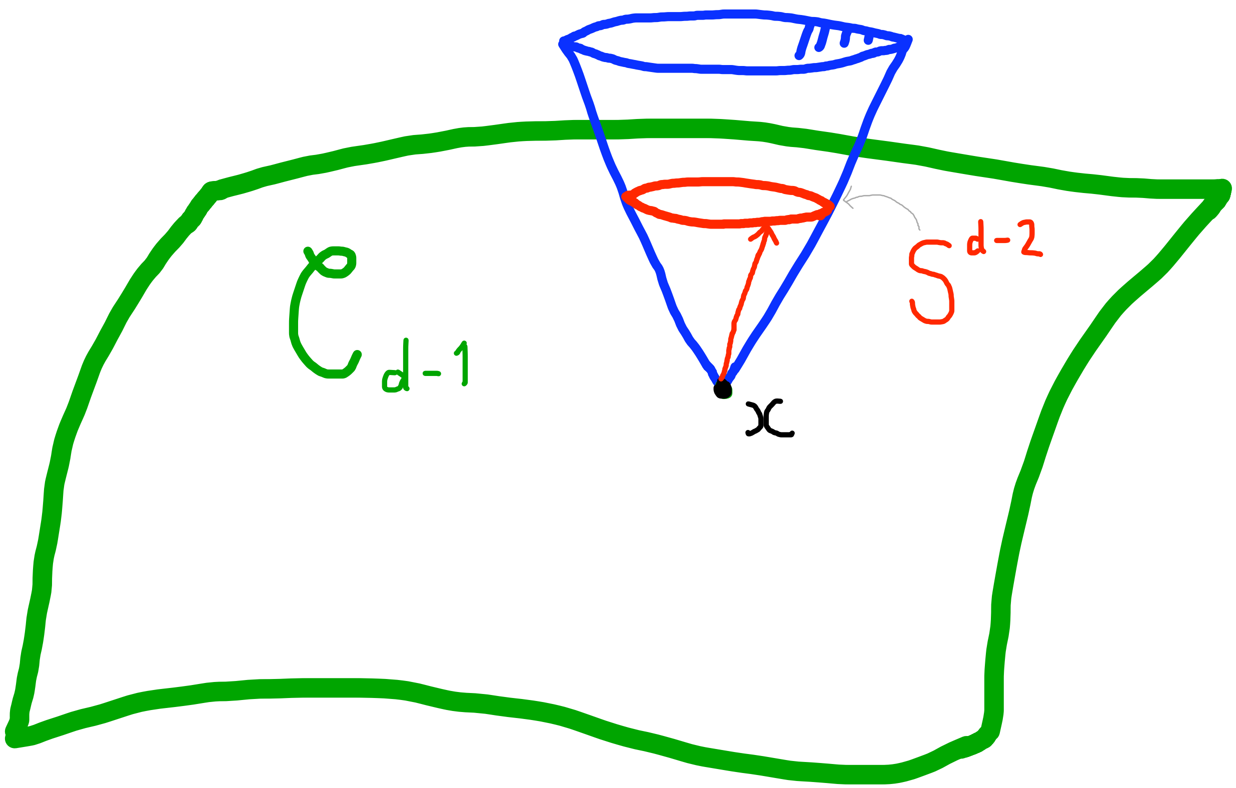
- much recent progress on evaluating $F(k_i, \epsilon_i, z_i)$'s without knowing soln explicitly Cachazo, Gomez; Baadsgaard, Bjerrum-Bohr, Bourjaily, Damgaard

- have arisen before in various contexts

Fairlie, Roberts; Gross, Mende; Gaudin; Roiban, Spradlin, Volovich; Witten

In any space-time, of dimension d, we can define the *space of null rays*. This space is often called "(projective) ambitwistor space" $\mathbb{PA}$.

- $\mathbb{PA}$ has dimension $2d-3$
- smooth provided space-time is globally hyperbolic



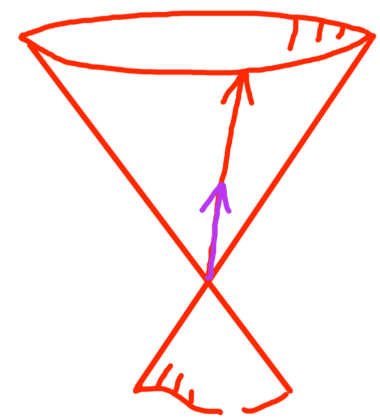
$$S^{d-2} \to \mathbb{PA}$$
$$\downarrow$$
$$\mathcal{C}_{d-1}$$

There are many different, but equivalent ways to describe this space. Most important for us is the following:

- The cotangent bundle T^*M to space-time is naturally symplectic, with symplectic form $\omega = dp_\mu \wedge dx^\mu$

- Take a Hamiltonian $H = \frac{1}{2}(p, p)$. The associated Hamiltonian vector generates flow along geodesics with tangent $g^{-1}(p, \ )$.

Restricting to the constant energy surface $H=0$ and quotienting by the flow, we obtain the space of scaled null geodesics. It's again symplectic.



- To get to $\mathbb{PA}$ we quotient by the scale of p. We get a contact manifold.

$$\theta = p_\mu dx^\mu \qquad \text{with } p \text{ null}$$

Le Brun

It's easy to construct a theory that describes maps $\Sigma \to \mathbb{PA}$.

For flat space-time,

$$S = \int_\Sigma P_\mu \bar\partial X^\mu + \frac{e}{2} P^2$$

- The action makes sense if P_μ has conformal weight (1,0), in which case e must be a Beltrami differential of weight (-1,1).

$$\delta X^\mu = \alpha P^\mu \qquad \delta e = -\bar\partial\alpha$$

The theory looks straightforward to quantize, using the BRST procedure to handle this gauge freedom.

- If there are no vertex operators, we can fix the gauge $e=0$ and get a free ghost action $S^{gh} = \int_\Sigma \tilde b \bar\partial \tilde c$

However simple it may appear, as it stands this model is inconsistent.

- The problem lies with space-time diffeomorphisms:

$$X^\mu \to f^\mu(X) \qquad P_\mu \longrightarrow \frac{\partial f^\nu}{\partial X^\mu} P_\nu$$

the second transformation requires regularization.

- The same issue can be seen in the path integral. If we perturb around a classical solution $X(z) = X_0 \in M$ then we obtain a chiral determinant

$$\frac{1}{\det(\bar\partial_{X^* TM})} = \int DP\, Dx\; e^{-\int p \bar\partial x}$$

and again this is not Diff(M) invariant.

The simplest way to cancel this anomaly is to add d complex, or 2d real fermions:

$$\int_\Sigma \psi_{1\mu} \bar{\partial} \psi_1^\mu + \psi_{2\mu} \bar{\partial} \psi_2^\mu$$

– These fermions lead to new chiral currents

$$G_1 = P_\mu \psi_1^\mu \qquad G_2 = P_\mu \psi_2^\mu$$

which are also gauged.

$$Q = \oint c\,T + \tilde{c}\,H + \gamma_1 G_1 + \gamma_2 G_2$$

All anomalies vanish provided d = 10.

The simplest (NS-NS) vertex operators are

$$V = c\tilde{c}\,\delta^2(\gamma)\,\epsilon_{\mu\nu}\,\Psi_1^\mu\,\Psi_2^\nu\,e^{ik\cdot X}$$

and describe a graviton, dilation & B-field just as in the RNS string.

$$[Q, V] = 0 \quad\Rightarrow\quad \begin{cases} k^2 = 0 \\ \epsilon_{\mu\nu} k^\mu = \epsilon_{\mu\nu} k^\nu = 0 \end{cases}$$

However, these conditions come from double contractions with H and G, rather than T.

There are no massive (or tachyonic) states in the spectrum, because X(z) X(w) ~ 0.

Integrating out X, in the presence of vertex operators we learn

$$\bar{\partial} P = -\sum_{i=1}^{n} k_i \, \delta^2(z - z_i) \qquad\qquad P(z) = \sum_{i=1}^{n} \frac{k_i \, dz}{z_i - z}$$

so P^2 is a meromorphic quadratic differential with simple poles at the vertex operators:

$$P^2(z) = \sum_{i,j} \frac{k_i \cdot k_j \, dz^2}{(z - z_i)(z - z_j)}$$

- At genus zero, any such differential vanishes identically if it has fewer than 4 poles.

$$\mathrm{Res}_i \, P^2(z) = 0 \quad \text{for } i = 1, \ldots, n-3 \quad \Rightarrow \quad P^2(z) = 0$$

- These scattering equations arise from the moduli of the gauge field e on a marked curve.

Scattering eqs. $\Longleftrightarrow$ Map to $\mathbb{PA}$

Performing the path integral leads to the formulae for tree amplitudes discovered by Cachazo, He & Yuan:

fermion correlators

$$M^{g,B,\phi}(\{k_i, \epsilon_i, \tilde{\epsilon}_i\}) = \sum_{z_i^* \mid P^2(z)=0} \frac{\mathrm{Pf}'(k_i, \epsilon_i, z_i)\,\mathrm{Pf}'(k_i, \tilde{\epsilon}_i, z_i)}{\mathrm{Jac}(k_i, z_i)}$$

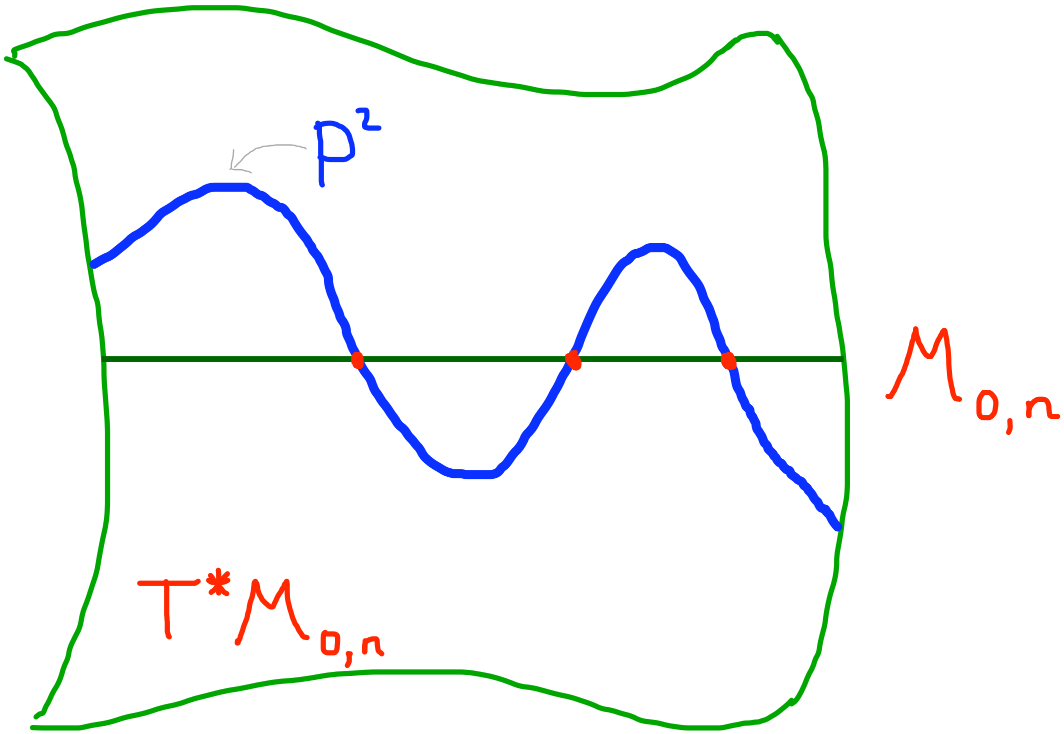
sum over (n-3)!
soln of Sc.Eq.

Hessian

- CHY have discovered analogous formulae for trees in a wide range of massless theories (see Cachazo's talk)
  worldsheet origins known / being developed
  Ohmori; Casali, Geyer, Mason, Monteiro, Roehrig

- Generalization to allow masses given by Naculich
  Can be seen as KK reduction of massless case

The CHY formula is just what we'd expect to get as the output of a localization calculation



– meromorphic quad. diffs. are cotangent vectors to the moduli space of marked curves

The moduli space of pairs (e,J) in ambitwistor strings is $T^*\mathcal{M}_{g,n}$
Ohmori has used this to understand the model using localization.

– Amplitude cries out for interpretation as some form of index!

Since the amplitudes are just those of gravity, the Einstein eqs must be the exact consistency requirement for the theory to exist on a curved background

- no α' / higher derivative corrections

$$S = \int_\Sigma P_\mu \bar\partial X^\mu + \bar\Psi_\mu \bar\partial \Psi^\mu + \bar\Psi_\mu \Gamma^\mu_{\nu\lambda} \bar\partial X^\nu \Psi^\lambda$$

$$= \int_\Sigma \Pi_\mu \bar\partial X^\mu + \bar\Psi_\mu \bar\partial \Psi^\mu \qquad \left(\Pi_\mu \equiv P_\mu + \bar\Psi_\nu \Gamma^\nu_{\mu\lambda} \Psi^\lambda\right)$$

- the currents become (up to derivative terms) Adamo, Casali, DS

$$G = \Psi^\mu \Pi_\mu \qquad H = g^{\mu\nu}\left(\Pi_\mu - \Gamma^\kappa_{\mu\lambda}\bar\Psi_\kappa \Psi^\lambda\right)\left(\Pi_\nu - \Gamma^\rho_{\nu\sigma}\bar\Psi_\rho \Psi^\sigma\right)$$

$$+ \frac{1}{2} R_{\mu\nu}{}^{\kappa\lambda} \Psi^\mu \Psi^\nu \bar\Psi_\kappa \bar\Psi_\lambda$$

$$\bar G = g^{\mu\nu} \bar\Psi_\mu \left(\Pi_\nu - \Gamma^\kappa_{\nu\lambda} \bar\Psi_\kappa \Psi^\lambda\right)$$

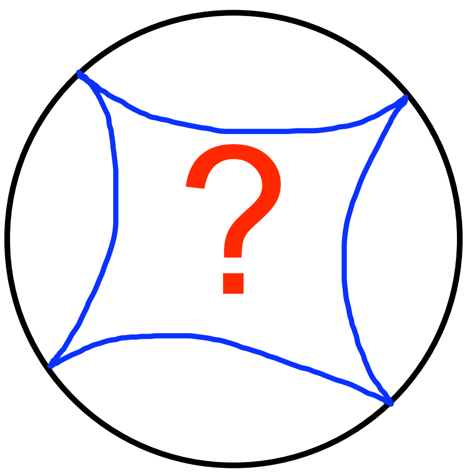
Including also the B-field and dilaton, one finds the current OPEs

$$G(z)\,G(w) \sim \frac{\Psi^4\, dH}{z-w} \qquad \bar{G}(z)\,\bar{G}(w) \sim \frac{\text{Bianchi id}}{z-w}$$

$$G(z)\,\bar{G}(w) \sim \frac{\text{dilaton eom}}{(z-w)^3} + \frac{\text{Einstein + B-field eom}}{(z-w)^2} + \frac{H}{z-w}$$

- these agree with the flat space algebra iff the expected field eqns and curvature identities hold

- field equations arise as anomalies in gauge symmetry reducing target to $\mathbb{PA}$, not w/s beta function



Striking that still have exactly free OPE for basic fields.

- Could there be a scattering equation formula for SG amplitudes in AdS?

Loop amplitudes

What can we say at higher genus?

- still have $\bar{\partial} P = -\sum_{i=1}^{n} k_i \delta^2(z - z_i)$ but now have holomorphic differentials:

zero modes = loop momenta

$$P_\mu(z) = \sum_{a=1}^{g} \ell_\mu^{(a)} \omega_a(z) \text{ + meromorphic part}$$

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$$P_\mu(z) = \sum_{a=1}^{g} \ell_\mu^a \omega_a(z) \text{ + meromorphic part}$$

- to live in ambitwistor space, we again need $P^2(z) = 0$

$\mathrm{Res}_i(P^2) = 0$ for i = 1,...,n $\qquad P^2(z_r) = 0 \quad$ r = 1, ..., 3g-3

- Pfaffians built from (known) free fermion correlators at higher genus. (Also various partition functions / theta constants.)

The resulting proposal for "the 1-loop integrand in SG" passed various checks at g = 1:

- factorization gives rational expression — Adamo, Casali, DS
- correct $t^8 \tilde{t}^8 R^4$ tensor structure at n = 4 — Casali, Tourkine

but felt a long way away from the rational function we want

The conjecture has just now been proved ($n \leqslant 5$, $g = 1$) in a remarkable paper by Geyer, Mason, Monteiro & Tourkine

- at $g = 1$, worldsheet path integral gives

$$\int \underbrace{d^d\ell \, \frac{dq}{q}}_{\text{moduli}} \overbrace{\bar{\delta}(P^2(z_0)) \prod_{i=1}^{n-1} \bar{\delta}(\mathrm{Res}_i \, P^2(z))}^{\text{scattering eqns}} \overbrace{\left(\sum_{\text{spin str.}} Z^{(1)}(z_i) \, Z^{(2)}(z_i) \right)}^{\text{fermion correlators}}$$

$(q = e^{2\pi i \tau})$

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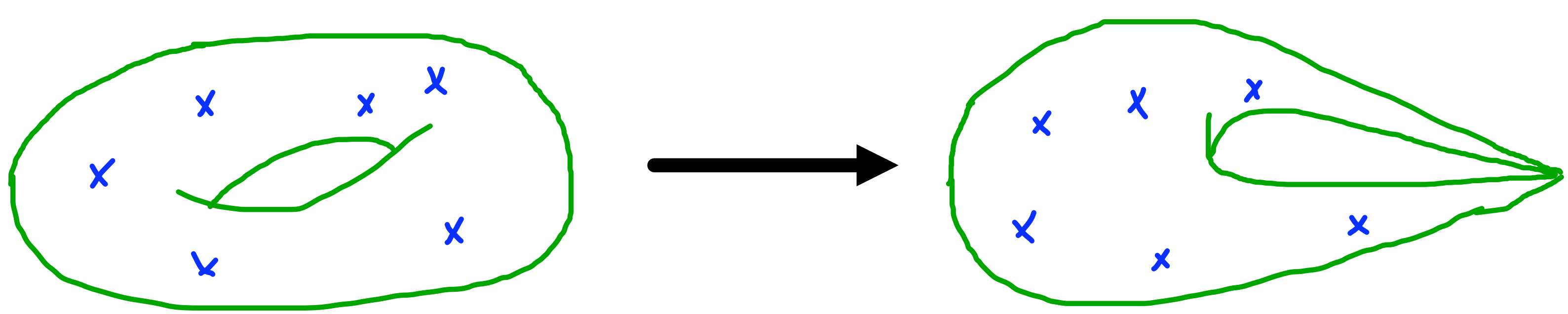
$$= - \int d^d\ell \, dq \, \bar{\delta}(q) \, \frac{1}{P^2(z_0)} \prod_{i=1}^{n-1} \bar{\delta}(\mathrm{Res}_i \, P^2(z)) \left(\sum_{\text{spin str.}} Z^{(1)}(z_i) \, Z^{(2)}(z_i) \right)$$



What is $P^2(z)$?

remaining sc. eqs. & $q \to 0$
limit kill these terms

$$P^2(z_0) = \ell^2 dz^2 \ + \text{meromorphic} \longrightarrow \ell^2$$

- general g = 1 formula reduces to

$$M^{(n,1)} = \int \frac{d^d \ell}{\ell^2} \prod_{i=1}^{n-1} \bar{\delta}\left(\mathrm{Res}_i\, P^2(z)\right) \left(\sum_{\text{spin str}} Z^{(1)}(z_i)\, Z^{(2)}(z_i) \right) \Bigg|_{q=0}$$

becomes rational
function on w/s

- remaining g=1 scattering equations simplify to become

Geyer, Mason, Monteiro, Tourkine

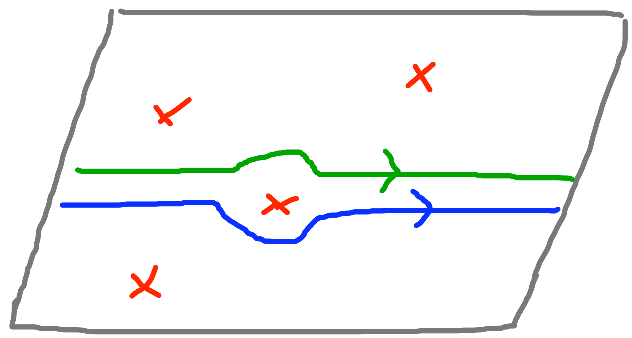
$$0 = \frac{k_i \cdot \ell}{z - z_0} - \frac{k_i \cdot \ell}{z - z_\infty} - \sum_{j \neq i} \frac{K_i \cdot K_j}{z_{ij}}$$

with off-shell loop momentum ℓ

- for n = 4, fermion correlators just give $t^8\tilde{t}^8R^4$ tensor and remaining sc. eqs. can be solved explicitly: Green, Schwarz; Casali, Tourkine

$$\mathcal{M}^{(4,1)} = t^8\tilde{t}^8R^4 \int d^d\ell \; \frac{1}{\ell^2}\sum_{\sigma\in S_4} \frac{1}{\ell\cdot K_{\sigma_1}\left(\ell\cdot(K_{\sigma_1}+K_{\sigma_2})+K_{\sigma_1}\cdot K_{\sigma_2}\right)\ell\cdot K_{\sigma_4}}$$

There's (of course!) an ambiguity in the definition of ℓ



Which propagator does it represent?

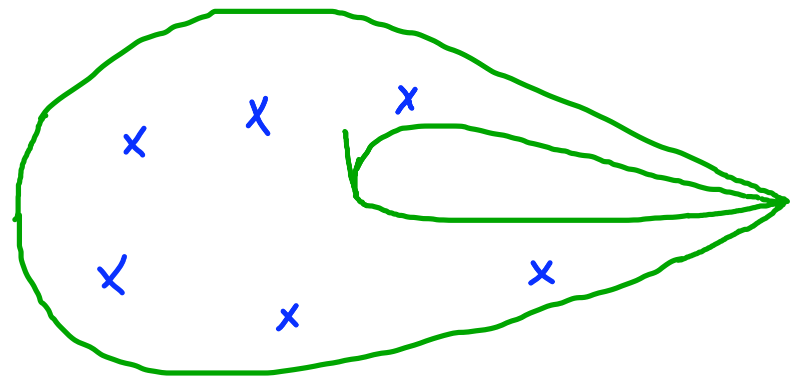
$$\ell \to \ell + K_i$$

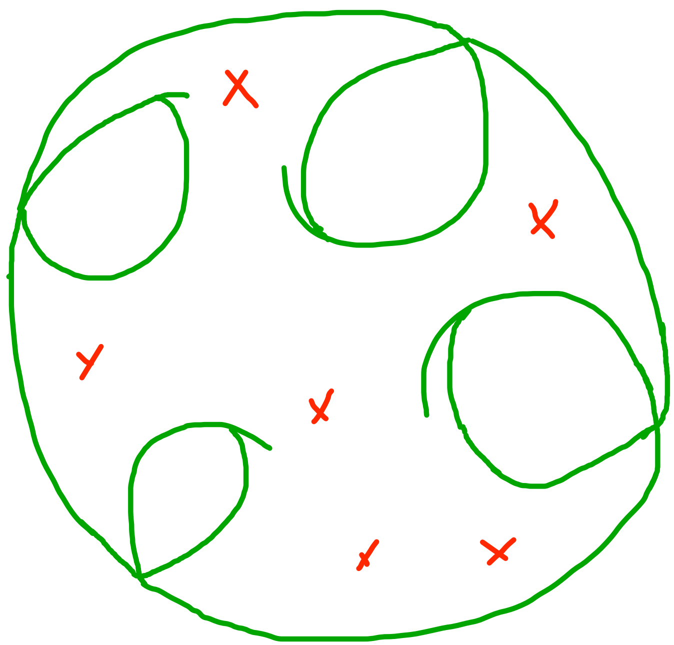
Exploiting this freedom in a smart way, GMMT find

$$= t^8\tilde{t}^8R^4 \sum \text{boxes}$$

- checked numerically for n = 5

What's so striking about GMMT's derivation is that by localizing to q = 0, the story ends up being no more complicated than for trees!





– GMMT also give natural conjectures for multi-loop amplitudes, based on Riemann spheres with g double points

– they give further expressions for loops in theories even where consistent worldsheet model unknown (e.g. YM)

Twistor strings in 4d

In four dimensions, instead of gauging the constraint $P^2(z)=0$, we can solve it using spinors:

$$P_{\alpha\dot{\alpha}}(z) = \tilde{\lambda}_{\dot{\alpha}}(z)\,\lambda_\alpha(z)$$

- The action becomes

$$\int \tilde{\lambda}_{\dot{\alpha}}\lambda_\alpha\bar{\partial}X^{\dot{\alpha}\alpha} = \int \tilde{\lambda}_{\dot{\alpha}}\bar{D}(X^{\alpha\dot{\alpha}}\lambda_\alpha) - \tilde{\lambda}_{\dot{\alpha}}X^{\alpha\dot{\alpha}}\bar{D}\lambda_\alpha$$

$$= \int \tilde{\lambda}_{\dot{\alpha}}\bar{D}\mu^{\dot{\alpha}} + \tilde{\mu}^\alpha\bar{D}\lambda_\alpha = \int W_a\bar{D}Z^a$$

- P(z) was a meromorphic section of the worldsheet canonical bundle. There's no obvious way to split this between the two spinors, so we just set

Witten

$$\lambda \in \mathcal{L}\,, \quad \tilde{\lambda} \in \tilde{\mathcal{L}}\,, \quad \mathcal{L}\otimes\tilde{\mathcal{L}} \cong K_\Sigma(z_1,\dots,z_n)$$

- There are also fermions $\bar{\rho}_a\,,\ \rho^a \in K_\Sigma^{1/2}$

- The BRST operator is

$$Q = \oint n\, W\cdot Z + \gamma_1 W\cdot\rho + \gamma_2 [W\bar\rho] + n_{11}[\bar\rho,\bar\rho] + n_{12}\bar\rho\cdot\rho + \tilde\gamma_1 Z\cdot\bar\rho + \tilde\gamma_2 \langle Z\rho\rangle + n_{22}\langle\rho,\rho\rangle$$ + ghost terms

and treats $\mathcal{L}$ and $\tilde{\mathcal{L}}$ symmetrically.

- All anomalies cancel iff $\mathcal{N}=8$

Summing over the choices of $\mathcal{L}, \tilde{\mathcal{L}}$ amounts to summing over the degree of the worldsheet GL(1) gauge field.

- Only one degree contributes. Which one is fixed by the choice representation for the external wavefunctions.

Witten, Berkovits, DS

all in twistor (Z) representation $\rightarrow \mathcal{L} \cong \mathcal{O}(n_- - 1)$

all in dual twistor (W) representation $\rightarrow \tilde{\mathcal{L}} \cong \mathcal{O}(n_+ - 1)$

Geyer, Lipstein, Mason, Monteiro

+ve hel. in Z, -ve in W $\rightarrow \begin{cases} \mathcal{L} \cong K^{1/2}(z_k) & k \in -ve \\ \tilde{\mathcal{L}} \cong K^{1/2}(z_i) & i \in +ve \end{cases}$

Computing worldsheet correlators in these models leads to manifestly supersymmetric formulae in 4d on-shell superspace

Cachazo, DS; Geyer, Lipstein, Mason, Monteiro

$$\mathcal{M}^{\mathcal{N}=8}_{0,n} = \int d^n z \, \frac{\det{}' \Phi \, \det{}' \tilde{\Phi}}{\mathrm{vol}\,(GL(2))} \underbrace{\prod_{i\in +} \bar{\delta}(\langle \lambda(z_i)\, i\rangle) \prod_{k\in -} \bar{\delta}([\tilde{\lambda}(z_k)\, k])}_{\text{4d refinement of scattering eqs}} \, S(\eta_i, \tilde{\eta}_k)$$

$i, j \in +$

$$\tilde{\Phi}_{ij} = \begin{cases} \dfrac{[ij]}{z_{ij}} & i \neq j \\ -\sum_{j\neq i} \tilde{\Phi}_{ij} & \text{else} \end{cases}$$

$k, l \in -$

$$\Phi_{kl} = \begin{cases} \dfrac{\langle kl\rangle}{z_{kl}} & k \neq l \\ -\sum_{l\neq k} \Phi_{kl} & \text{else} \end{cases}$$

$$S(\tilde{\eta}_i, \eta_k) = \exp\left(\sum_{\substack{i\in + \\ k\in -}} \frac{\tilde{\eta}_i \cdot \eta_k}{z_{ik}} \right)$$

A modification of this formula gives all trees in sEYM in 4d:

Adamo, Casali, Roehrig, DS; Cachazo, He, Yuan

$$\int d^n z \, \frac{\det'\Phi \, \det'\tilde{\Phi}}{\mathrm{vol}\,(GL(2))} \prod_{\alpha \in \text{traces}} PT_\alpha \underbrace{\prod_{i\in +} \bar{\delta}(\langle \lambda(z_i)\, i\rangle) \prod_{k\in -} \bar{\delta}([\tilde{\lambda}(z_k)\, k])}_{\text{4d refinement of scattering eqs}} \, \delta(\eta_i, \tilde{\eta}_k)$$

$i, j \in +$

$$\tilde{\Phi}_{ij} = \begin{cases} \dfrac{[ij]}{z_{ij}} & i \neq j \\ -\sum_{j\neq i} \tilde{\Phi}_{ij} & \text{else} \end{cases}$$

$$\tilde{\Phi}_{i\alpha} = \sum_{m \in tr_\alpha \cap +} \frac{[im]}{z_{im}}$$

etc

$k, l \in -$

$$\Phi_{kl} = \begin{cases} \dfrac{\langle kl\rangle}{z_{kl}} & k \neq l \\ -\sum_{l\neq k} \Phi_{kl} & \text{else} \end{cases}$$

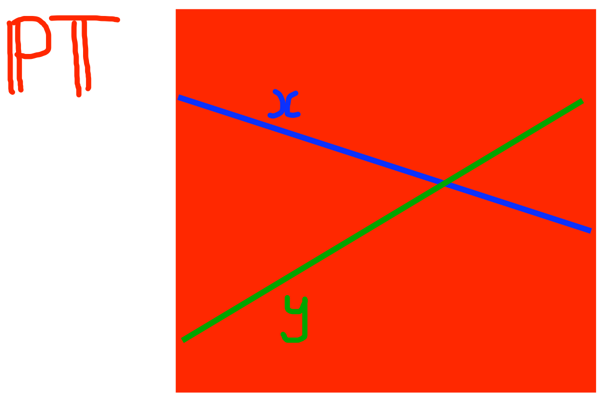
$$\Phi_{k\alpha} = \sum_{n \in tr_\alpha \cap -} \frac{\langle kn\rangle}{z_{kn}}$$

etc

An amazing property of these formulae is that the [,] and < , > terms decouple before the moduli integrals are performed.

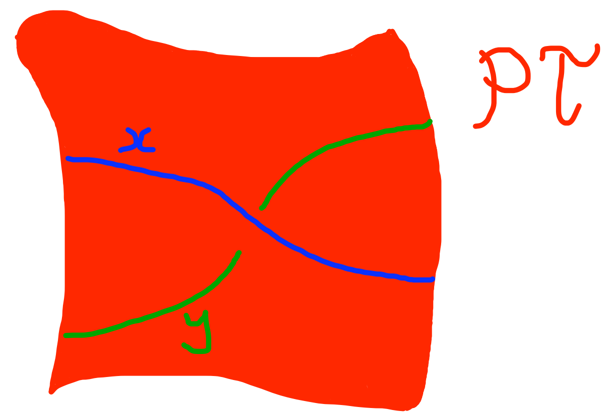
- not at all obvious in 'final answer', nor in CHY form
- reminiscent of holomorphic factorization in strings

$$S = \int_\Sigma W_I\left(\bar\partial Z^I + I^{IJ}\frac{\partial h}{\partial Z^J}\right) + \bar\rho_I\left(\bar\partial \rho^I + I^{IJ}\rho^K\frac{\partial^2 h}{\partial Z^J \partial Z^K}\right) + \cdots$$



deform complex str

$$\bar\partial \to \bar\partial + I^{IJ}\partial_I h\, \partial_J$$



Deforming either twistor / dual twistor space gives either sd / asd space-time. Amplitude seems to 'glues' these structures together.

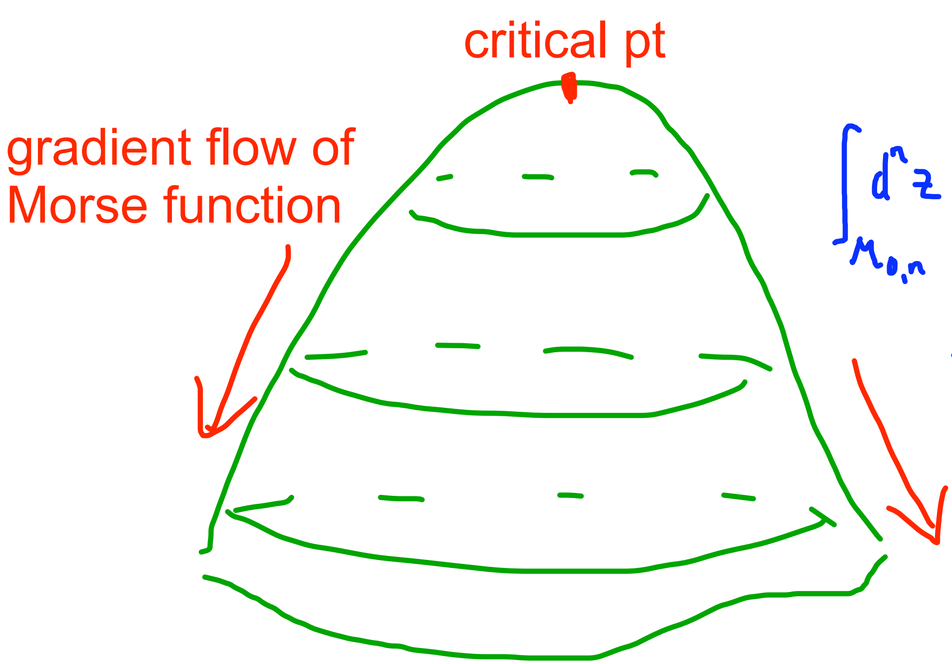
Twistors and Strings

The theories I've spoken about involve gravity, so require a UV completion. Could there be a way to 'turn on' α' ?

- conversely, obtain the ambitwistor string from the $\alpha' \to 0$ limit of strings

One approach: contours used to localize onto sc. eqs. are in same homology class as KLT contours

Witten; Ohmori; Baadsgaard, Bjerrum-Bohr, Bourjaily, Damgaard, Tourkine, Vanhove



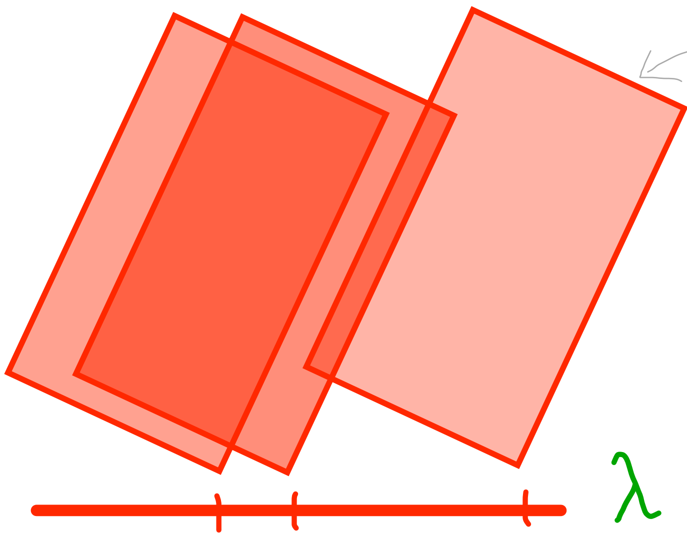
$$\int_{M_{0,n}} d^n z \, d^n \bar{z} \; F(z_i) \, \bar{F}(\bar{z}_i) = \sum_{p,q} A_{pq} \int_{\Gamma_p} d^n z \, F(z_i) \times \int_{\Gamma_q} d^n \tilde{z} \, \tilde{F}(\tilde{z}_i)$$

Another approach is suggested by Berkovits' recent work on the origin of the pure spinor string

- null directions are built into the very heart of string theory

$$T = \eta_{\mu\nu}\,\partial X^\mu \partial X^\nu \qquad \bar{T} = \eta_{\mu\nu}\,\bar{\partial} X^\mu \bar{\partial} X^\nu$$

- can we solve, rather than gauge, these constraints?



$\mu^{\dot\alpha} = x^{\dot\alpha\alpha}\lambda_\alpha$

gives identification $\mathbb{R}^4 \cong \mathbb{C}^2$

$\lambda \in \mathbb{CP}^1 \cong$ SO(4) / U(2)

- in ten dimensions, $\lambda \in$ SO(10) / U(5) is a projective pure spinor

Berkovits starts from the action

$$S = \int_\Sigma d^2\sigma \; P_\mu \partial X^\mu + \omega_\alpha \nabla_\sigma \lambda^\alpha - \tfrac{1}{2} L^\alpha (P_\mu + \partial_\sigma X_\mu) \gamma^\mu_{\alpha\beta} \lambda^\beta + K_\alpha \nabla_\sigma \lambda^\alpha$$

+ right movers

- The constraint imposed by the Lagrange multiplier L^α says that $(P_\mu + \partial_\sigma X_\mu)$ is null
- Together with the constraint imposed by K_α this implies $T = 0$ so that the action is $\mathrm{Diff}(\Sigma)$ invariant without bc ghosts
- Space-time fermionic directions θ^α emerge as ghosts for L^α

Gauge fixing → Pure spinor string

Gauge fixing → Green-Schwarz string

Gauge fixing → Twistor string?

Conclusions

The Witten & Berkovits twistor strings of a decade ago provided an intriguing & beautiful way to think about amplitudes. They inspired many new ideas, but came at high cost:

- wrong theory (conformal gravity $\rightarrow$ non-unitary)
- seemed impossible to extend to loop amplitudes
- not obvious how related to standard string theory / QFT
- just too damn hard!

Over the past couple of years, months & even days (!) these obstacles have been / are being overcome.

I think that right now is the most exciting time in twistor theory since 2003.

- lots of new results, both conceptual and practical, expected to emerge in the coming months...