Schouten identities for the solution of systems of differential equations for Feynman integrals

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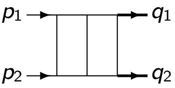
based on collaboration with *E. Remiddi*

[arXiv:1311.3342], [arXiv:15xx.xxxx]

Any Feynman Diagrams is (*after some tedious but elementary algebra!*) nothing but a collection of scalar **Feynman Integrals**

$\mathcal{I}(p_1, p_2, q_1) =$



with $\quad q_2 = p_1 + p_2 - q_1$

A (*possible*) representation in momentum space (*massless* case!)

$$\mathcal{I}(p_1, p_2, q_1) = \int \frac{\mathfrak{D}^d\, k\, \mathfrak{D}^d\, l}{k^2\, l^2\, (k-l)^2\, (k-p_1)^2\, (k-p_{12})^2\, (l-p_{12})^2\, (l-q_1)^2}$$

Typical **2-loop** Feynman Integral required for the computation of a $2 \to 2$ scattering process.

How do we (*tentatively!*) compute **analytically** such integrals?

1. Integrals are ill-defined in $d = 4 \rightarrow$ need a **regularization procedure**!
2. Use of dimensional regularization to regulate **UV** and **IR divergences**.
3. Dimensional regularisation turned out to be much MORE than just a **regularization scheme**!

$$\Downarrow$$

Dimensionally regularized Feynman integrals always converge!

This allows to derive a large number of **unexpected relations...**

- Integration by Parts, Lorentz invariance identities, Schouten Identities,...

This large set of identities makes it *simpler* to compute **Feynman integrals** in d continuous dimensions than in $d = 4$!

A general **scalar** Feynman Integral (l-loops) can be written as

$$\mathcal{I}(\sigma_1, ..., \sigma_s; \alpha_1, ..., \alpha_n) = \int \prod_{j=1}^{l} \frac{d^d k_j}{(2\pi)^d} \frac{S_1^{\sigma_1} \dots S_s^{\sigma_s}}{D_1^{\alpha_1} \dots D_n^{\alpha_n}}$$

where

$D_n = (q_n^2 + m_n^2)\,,$ are the **propagators**

$S_n = k_i \cdot p_j\,,$ are **scalar products** among internal and external momenta

This introduces the concept of **Topology**

1. **Integration By Parts Identities** (IBPs)

$$\int \prod_{j=1}^{l} \frac{d^d\, k_j}{(2\pi)^d} \left(\frac{\partial}{\partial k_j^\mu}\, v_\mu\, \frac{S_1^{\sigma_1} \dots S_s^{\sigma_s}}{D_1^{\alpha_1} \dots D_n^{\alpha_n}} \right) = 0\,, \qquad v^\mu = k_j^\mu, p_k^\mu$$

2. They generate huge systems of linear equations which relates integrals with **different powers** of numerators and denominators.

3. The integrals always belong to the same **topology**, as defined above.

The IBPs can be solved using computer algebra (Reduze2, AIR, FIRE5...)
As a result, all integrals are expressed as linear combination of a small subset of
Master Integrals

Typically we go from **thousands** to **tens** of integrals $\rightarrow$ The basis is of course **not unique**!

Dimensionally regularised Feynman Integrals fulfil differential equations!

[Kotikov '90, Remiddi '99, Gehrmann-Remiddi '00,...]

Let us take a **topology** of integrals which depend on two external invariants

$$p^2\,,\ m^2 \qquad \to x = \frac{p^2}{m^2}\,.$$

$$\mathcal{I}(p^2, m^2; \alpha_1\,, ..., \alpha_n) = \int \prod_{j=1}^{l} \frac{d^d\,k_j}{(2\pi)^d}\,\frac{1}{D_1^{\alpha_1}\ldots D_n^{\alpha_n}}\,, \qquad \text{(same with scalar products)}$$

Assume IBPs reduce all integrals into this topology to **N Master Integrals** $m_i(p^2; d)$, with $i = 1, ..., N$.

1) All integrals depend on $x = p^2/m^2$ only.

2) **Differentiation** w.r.t to an external invariant:

$$p^2 = p_\mu\, p^\mu \qquad \rightarrow \qquad \frac{\partial}{\partial\, p^2} = \frac{1}{2\, p^2}\, \left(p^\mu \frac{\partial}{\partial p^\mu} \right)$$

$$\frac{\partial}{\partial\, p^2} \int \prod_{j=1}^{l} \frac{d^d\, k_j}{(2\pi)^d} \frac{1}{D_1^{\alpha_1} \dots D_n^{\alpha_n}} = \int \prod_{j=1}^{l} \frac{d^d\, k_j}{(2\pi)^d} \frac{1}{2\, p^2} \left(p^\mu \frac{\partial}{\partial p^\mu} \right) \frac{1}{D_1^{\alpha_1} \dots D_n^{\alpha_n}}$$

3) With IBPs integrals on r.h.s. can be **reduced** again to the MIs

$$\frac{\partial}{\partial\, p^2} m_i(d; p^2) = \sum_{j=1}^{N} c_{ij}(d; p^2)\, m_j(d; p^2)\,.$$

System of N coupled differential equations for $m_i(d; p^2)$

How does this help in practise?

What if $\mathbf{N = 1}$? (*There is only 1 MI!*)

If there is only 1 master integral the situation is *in principle* trivial:

$$\frac{\partial}{\partial p^2} m(d;p^2) = c(d;p^2)\, m(d;p^2)$$

First order linear equation, can be solved by **quadrature**

$$m(d;p^2) = C_0 \exp\left(\int_0^{p^2} dt\, c(d;t) \right)$$

What if $\mathbf{N} > \mathbf{1}$? (*Life is not that easy anymore!*)

If the system is coupled, it corresponds to a N-**th order** differential equation for any of the MIs. No general strategy for a solution is known.

$\Downarrow$

Observations

1. We are **free** of choosing our **basis of MIs**!
2. We are interested in the expansion for $d \to 4$!

$\Downarrow$

Changing the basis can **simplify** the structure of the differential equations!

At least in the limit for $d \to 4$!

We can very often find a basis of MIs where the equations become **triangular** as $d \to 4$ [Gehrmann, Remiddi '00, '01]

$$\frac{\partial}{\partial p^2} \left(\begin{array}{c} m_1 \\ \dots \\ m_N \end{array} \right) = \left(\begin{array}{cccc} c_{11}^{(0)} & c_{12}^{(0)} & \dots & c_{1N}^{(0)} \\ 0 & c_{22}^{(0)} & \dots & c_{2N}^{(0)} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & c_{NN}^{(0)} \end{array} \right) \left(\begin{array}{c} m_1 \\ \dots \\ m_N \end{array} \right) + \mathcal{O}(d-4)$$

This reduces, *in principle*, the problem to the case $N = 1$

$$\Downarrow$$

In order to obtain expansion in $(d-4)$ we must perform **many repeated integrations by quadrature**!

We can do better $\rightarrow$ **Canonical Basis** [A.Kotikov '10, **J.Henn** '13]
(see talk by J.Henn)

Suppose we are able to find a **basis of Master Intergrals** such that the system of differential equations takes the following form:

$$\frac{\partial}{\partial p^2}\begin{pmatrix} m_1 \\ \dots \\ m_N \end{pmatrix} = (d-4)\begin{pmatrix} c_{11}(p^2) & \dots & c_{1N}(p^2) \\ c_{21}(p^2) & \dots & c_{2N}(p^2) \\ \dots & \dots & \dots \\ c_{N1}(p^2) & \dots & c_{NN}(p^2) \end{pmatrix}\begin{pmatrix} m_1 \\ \dots \\ m_N \end{pmatrix}$$

Such that:

a) The dependence from the kinematics is **factorised** from d.

b) The functions $c_{jk}(p^2)$ must be in d-log form, i.e. $\int^{p^2} dt\, c_{jk}(t) \propto \ln\left(f(p^2)\right)$

If the differential equations are in **canonical form** we can trivially integrate them **order-by-order in** $(d-4)$

$$\Downarrow$$

a) Solution can be expressed in terms of **Multiple polylogarithms** (MPLs) [Remiddi,Vermaseren; Gehrmann,Remiddi; Goncharov]

$$G(a_1, a_2, \ldots, a_n; x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n; x)\,, \qquad G(\underbrace{0, \ldots, 0}_{n}; x) = \frac{1}{n!} \ln^n x\,.$$

(See talk by E. Panzer)

b) No rational factors, and **uniform transcendental weight!**

Compact and "beautiful" results!

How to find **such basis**?

1. Henn's criteria [J.Henn '13]

2. Magnus expansion for linear dependence on ϵ
   [Argeri, Di Vita, Mastrolia, Mirabella, Schlenk, Schubert, Tancredi '14]

3. R. Lee algorithm if there is **only one non-trivial ratio**. [R. Lee '14]

x) Can we know a priori if with these methods, we will get anywhere?

xx) What is the **reason behind** the possibility of finding such a basis?

$$\Downarrow$$

1. If the DE can be put in triangular form $\rightarrow$ **canonical basis** is *often* obtainable with limited effort [Gehrmann, Manteuffel, Tancredi, Weihs '13].

2. All **known cases** where DE can be put in triangular form, can be integrated in terms of multiple polylogaritms.

3. **Central question**: how do we know whether the equations can be put in triangular form as $d \rightarrow 4$?

A *possible* path to an answer

1) Now very **naively** take two coupled diff. equations

$$\frac{\partial}{\partial p^2} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$$

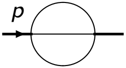
2) What if we could say $m_2 = f(p^2)\, m_1 + \mathcal{O}(d-n)$?

Then we would naively expect that

$$\frac{\partial}{\partial\, p^2} \left(m_2 - f(p^2) m_1 \right) = \mathcal{O}(d-n)!$$

Idea developed for the first time in [Remiddi, Tancredi '13] for the **two-loop sunrise** with **three different masses**.

Let's look at something simpler (but somehow more interesting!)



$$= \mathcal{I}(p^2, m^2; \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$$

$$= \int \mathfrak{D}^d k \mathfrak{D}^d l \frac{(k \cdot p)^{\alpha_4} (l \cdot p)^{\alpha_5}}{(k^2)^{\alpha_1} (l^2)^{\alpha_2} ((k - l + p)^2 - m^2)^{\alpha_3}}$$

1. Using **IBPs** we can reduce all these integrals to only **2 Master Integrals**.

$$S_1 = \mathcal{I}(p^2, m^2; 1, 1, 1, 0, 0)\,, \qquad S_2 = \mathcal{I}(p^2, m^2; 1, 1, 2, 0, 0)\,.$$

2. Derive now DE for these two integrals, we find:

$$\frac{d}{d\,p^2} S_1 = \frac{(d-3)}{p^2} S_1 - \frac{m^2}{p^2} S_2$$

$$\frac{d}{d\,p^2} S_2 = \frac{(d-3)(3d-8)}{2\,p^2\,(p^2 - m^2)} S_1 + \left(\frac{2(d-3)}{p^2 - m^2} - \frac{(3\,d-8)}{2\,p^2} \right) S_2$$

We know that they can be put in canonical form as $d \to 4$, and therefore also simply decoupled. Where does the **right basis** come from?

1. Schouten Identities say that in $d = 2$ dimensions there can be only 2 vectors that are **linearly independent**.
2. The Sunrise depends on 3 vectors $\to k, l, p$!
3. In $d = 2 \to \epsilon(k,l,p) = k^\mu \, l^\nu \, p^\rho \, \epsilon_{\mu\nu\rho} = 0\,.$
4. Now **square it** and **contract** the epsilon tensors in $d = 3$
$$\epsilon^2(k,l,p) = k^2 l^2 p^2 - k^2 (l\cdot p)^2 - l^2(k\cdot p)^2 - p^2(k\cdot l)^2 + 2(k\cdot l)(l\cdot p)(k\cdot p)$$
5. Does not depend on d anymore! Build a d-**dimensional polynomial**
$$P(d;k,l,p) = k^2 l^2 p^2 - k^2 (l\cdot p)^2 - l^2(k\cdot p)^2 - p^2(k\cdot l)^2 + 2(k\cdot l)(l\cdot p)(k\cdot p)$$

   where now
$$P(1;k,l,p) = P(2;k,l,p) = 0\,.$$

What do I do with this polynomial that is zero in $d = 2$?

1. As $k, l \to \infty$, in the **UV limit**, $P(d; k, l, p) \approx k^2\, l^2$
2. As $k, l \to 0$, in the **IR soft limit**, $P(d; k, l, p) \approx k^2\, l^2 \to 0$
3. As $k, l \parallel p$, with $p^2 = 0$, in the **IR collinear limit** $P(d; k, l, p) \to 0$.

We know *UV* and *IR* properties of these polynomials $\to$ they can *partly* cure **IR divergences**!

We can use it to find **relations** between the **first order(s)** of the **master integrals** as $d \to 2$ (or $d \to 2\,n$ in case)

1) The two masters S_1 and S_2 have at most a double pole

$$S_j(d;p^2) = \frac{1}{(d-2)^2}\, S_j^{(-2)}(p^2) + \frac{1}{(d-2)}\, S_j^{(-1)}(p^2) + S_j^{(0)}(p^2) + \dots$$

2) Consider, for example, the quantity

$$Z(d;1,1,1) = \int \mathfrak{D}^d\, k \mathfrak{D}^d\, l\, \frac{P(d;k,l,p)}{(k^2)\,(l^2)\,((k-l+p)^2 - m^2)}$$

3) One can show that $Z(d;1,1,1)$ has only a single pole in $(d-2)$!

$$Z(d;1,1,1) = \mathcal{O}\left(\frac{1}{d-2}\right)$$

4) Use IBPs to **reduce** $Z(d;1,1,1)$ to MIs,

$$Z(d;1,1,1) = C_1(d;p^2)\, S_1(d;p^2) + C_2(d;p^2)\, S_2(d;p^2)\,,$$

5) **Expand** this relation for $d \to 2$ (and keeping only double pole!)

$$\mathcal{O}\left(\frac{1}{d-2}\right) = \frac{1}{(d-2)^2}\,\frac{(m^2)^2\,p^2}{6}\,\left(S_1^{(-2)}(p^2) - (p^2-m^2)S_2^{(-2)}(p^2)\right) + \mathcal{O}\left(\frac{1}{d-2}\right)$$

6) For **consistency** we find precisely what we were looking for

$$S_1^{(-2)}(p^2) = (p^2-m^2)S_2^{(-2)}(p^2)$$

This relation can be verified explicitly numerically (or analytically)!

What happens if I start from different $Z(d; n_1, n_2, n_3)$?

- Starting from $Z(d;2,1,2) = Z(d;1,2,2)$ and with the same argument

$$\mathcal{O}\left(\frac{1}{d-2}\right) = -\frac{1}{(d-2)^2}\,\frac{m^2}{12}\,\left(S_1^{(-2)}(p^2) - (p^2-m^2)S_2^{(-2)}(p^2)\right) + \mathcal{O}\left(\frac{1}{d-2}\right)$$

- Starting from $Z(d;2,2,1)$ again same argument

$$\mathcal{O}\left(\frac{1}{d-2}\right) = \frac{1}{(d-2)^2}\,\frac{m^2}{6}\,\left(S_1^{(-2)}(p^2) - (p^2-m^2)S_2^{(-2)}(p^2)\right) + \mathcal{O}\left(\frac{1}{d-2}\right)$$

We find either this relation, or the trivial identity $0 = 0$ (no information...).

Note that this is a relation between the first orders of the series expansion of the two masters!

Now introduce **new Master Integral** defined in d dimensions:

$$\mathcal{I}_1(d;p^2) = S_1(d;p^2) - (p^2 - m^2) S_2(d;p^2)\,,$$

Take as new basis

$$\mathcal{I}_1(d;p^2) = S_1(d;p^2) - (p^2 - m^2) S_2(d;p^2)\,, \quad \mathcal{I}_2(d;p^2) = S_1(d;p^2)$$

The equations become **triangular**!!!

$$\frac{d\mathcal{I}_1}{dp^2} = (d-2)\left[\mathcal{I}_1\left(\frac{2}{p^2-m^2} - \frac{3}{2\,p^2}\right) - \mathcal{I}_2\left(\frac{2}{p^2-m^2}\right)\right] + (d-2)^2\,\mathcal{I}_2\,\frac{3}{2\,p^2}$$

$$\frac{d\mathcal{I}_2}{dp^2} = (d-2)\frac{\mathcal{I}_2}{p^2} + \mathcal{I}_1\left(\frac{1}{p^2-m^2} - \frac{1}{p^2}\right) - \mathcal{I}_2\left(\frac{1}{p^2-m^2}\right)$$

If we make the masters start at the same order in the expansion

$$\mathcal{I}_1(d;p^2) = S_1(d;p^2) - (p^2 - m^2) S_2(d;p^2)\,, \quad \mathcal{I}_2(d;p^2) = (d-2)\, S_1(d;p^2)$$

The equations become

$$\frac{d\mathcal{I}_1}{dp^2} = (d-2)\left[\left(\frac{2}{p^2-m^2} - \frac{3}{2\,p^2}\right)\mathcal{I}_1 + \frac{3}{2\,p^2}\mathcal{I}_2\right] - \left(\frac{2}{p^2-m^2}\right)\mathcal{I}_2$$

$$\frac{d\mathcal{I}_2}{dp^2} = (d-2)\left[\left(\frac{1}{p^2-m^2} - \frac{1}{p^2}\right)\mathcal{I}_1 + \frac{1}{p^2}\mathcal{I}_2\right] - \left(\frac{1}{p^2-m^2}\right)\mathcal{I}_2$$

REMARK: Problem is solved also for $d \approx 4$! It is enough to shift the basis from $d \to d-2$, for example with Tarasov-Lee identities [Tarasov '97, Lee '10]

This can be attempted in principle for **any topology**

One is tempted to say:

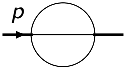
Given a sector with N coupled MIs, if I can find n relations as $d \to 2, 4, 6, \dots$
then I can **decouple** at most n masters as $d \to 2, 4, 6, \dots$

1) Do reduction and identify MIs
2) Use Schouten identities to study relations between first order of the series expansion of the MIs. $\to$ **it works for many less trivial examples!**

Problems and obscure points:

a) Schouten identities can become **tedious to implement**
b) I need **enough momenta**! $\to$ for sunrise I can do it only in $d = 2$! But if this is true, then a relation **must exist** in $d = 4$, since I know I can decouple equations also in $d = 4$ (And even find a canonical basis!)
c) More importantly, would **any relation** among the poles do the job? Finding one is trivial if poles are just **numbers** or **rational functions**!

Start from the last point, example **two-loop massive sunrise**



$$= \mathcal{I}(p^2, m^2; \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$$

$$= \int \mathfrak{D}^d k \mathfrak{D}^d l \frac{(k \cdot p)^{\alpha_4} (l \cdot p)^{\alpha_5}}{(k^2 - m^2)^{\alpha_1} (l^2 - m^2)^{\alpha_2} ((k - l + p)^2 - m^2)^{\alpha_3}}$$

1. Using **IBPs** we we find again only **2 Master Integrals**

$$S_1 = \mathcal{I}(p^2, m^2; 1, 1, 1, 0, 0)\,, \qquad S_2 = \mathcal{I}(p^2, m^2; 1, 1, 2, 0, 0)\,.$$

2. If we try same game with Schouten identities we **do not** find any further relation in $d = 2$. Equations cannot be decoupled! **Elliptic polylogaritms** and so on... (See C. Bogner's Talk)

Let's look at the two masters in $d \approx 4$

$$S_1(d; p^2, m^2) = \frac{1}{(d-4)^2}\left(\frac{3\,m^2}{8}\right) + \mathcal{O}\left(\frac{1}{(d-4)}\right)$$
$$S_2(d; p^2, m^2) = \frac{1}{(d-4)^2}\left(\frac{1}{8}\right) + \mathcal{O}\left(\frac{1}{(d-4)}\right)$$

Such that **of course** we have

$$S_1^{(-2)}(p^2, m^2) - 3\,m^2\,S_2^{(-2)}(p^2, m^2) = 0\,.$$

What happens if we use as new masters:

$$\mathcal{I}_1 = S_1\,, \qquad \mathcal{I}_2 = S_1 - 3\,m^2\,S_2 \quad ?$$

The differential equations **do not decouple** in $d = 4$! What's going wrong?

The differential equations know only about the **IBPs**!

a) If the differential equations triangularise for a given basis this information has to be **inside the IBPs**!

b) **Schouten identities** are just a tool to extract information from the IBPs. Can we read it directly from the IBPs?

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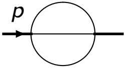
Can we then bypass the **Schoutens** completely and read this from the IBPs?

Go back to **easy sunrise** with only one mass



$$= \mathcal{I}(p^2, m^2; \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$$

$$= \int \mathfrak{D}^d k \mathfrak{D}^d l \frac{(k \cdot p)^{\alpha_4} (l \cdot p)^{\alpha_5}}{(k^2)^{\alpha_1} (l^2)^{\alpha_2} ((k - l + p)^2 - m^2)^{\alpha_3}}$$

$$S_1 = \mathcal{I}(d; 1, 1, 1, 0, 0), \qquad S_2 = \mathcal{I}(d; 1, 1, 2, 0, 0).$$

1. The two masters have at most a double pole $1/(d-2)^2$
2. Every integral in this topology can at most develop a double pole!
3. Generate the IBPs and expand them in series for $d \approx 2$!
4. Obtain a **chained system** of IBPs, which order by order, do not depend on d (one less variable, easier to solve algebraically!)

Now let's try and solve the systems of IBPs **bottom up** in $(d-2)$

a) From the first system of IBPs we find at once

$$\mathcal{I}^{(-2)}(d \to 2; 1,1,2,0,0) = \frac{1}{(p^2-m^2)}\mathcal{I}^{(-2)}(d \to 2; 1,1,1,0,0)$$

which is precisely the relation found with the Schouten!

b) Going further with the expansion this does not happen anymore!
At order $1/(d-2)$ **both MIs are needed**!

Can we go further?

a) For sunrise we can only use Schouten in $d = 2$ and then we must shift the result to $d = 4$ in order to get decoupled equations in $d = 4$.

b) What happens if we do the same exercise with IBPs in $d = 4$? Repeat same analysis

1. The two masters have at most a double pole $1/(d-4)^2$
2. Every integral in this topology can at most develop a double pole!

c) Expand IBPs in $d \approx 4$ and again solve **the first** of the chained systems.

d) Again only 1 MI is sufficient to describe the first pole at $d = 4$:

$$\mathcal{I}^{(-2)}(d \to 4; 1,1,2,0,0) = \frac{1}{m^2}\mathcal{I}^{(-2)}(d \to 4; 1,1,1,0,0)$$

Looking at analytical expressions, this is again a trivial relation among the poles

$$\mathcal{I}^{(-2)}(d;1,1,1,0,0) = \frac{1}{(d-4)^2}\left(\frac{m^2}{2}\right) + \mathcal{O}\left(\frac{1}{(d-4)}\right)$$

$$\mathcal{I}^{(-2)}(d;1,1,2,0,0) = \frac{1}{(d-4)^2}\left(\frac{1}{2}\right) + \mathcal{O}\left(\frac{1}{(d-4)}\right)$$

But this time it was contained inside the IBPs and in fact it can be used in order to decouple the differential equations!

a) Introduce new masters

$$\widetilde{\mathcal{I}}_1 = \mathcal{I}^{(-2)}(d;1,1,1,0,0) - m^2\mathcal{I}^{(-2)}(d;1,1,2,0,0)$$
$$\widetilde{\mathcal{I}}_2 = (d-4)\,\mathcal{I}^{(-2)}(d;1,1,1,0,0)$$

b) Differential equations become

$$\frac{d\widetilde{\mathcal{I}}_1}{dp^2} = (d-4)\left[\left(\frac{2}{p^2-m^2} - \frac{3}{2\,p^2}\right)\widetilde{\mathcal{I}}_1 + \frac{3}{2}\left(\frac{1}{p^2-m^2} - \frac{1}{p^2}\right)\widetilde{\mathcal{I}}_2\right] - \left(\frac{2}{p^2-m^2} - \frac{1}{p^2}\right)\widetilde{\mathcal{I}}_1 + \left(\frac{3}{2(p^2-m^2)} - \frac{1}{p^2}\right)\widetilde{\mathcal{I}}_1$$

$$\frac{d\widetilde{\mathcal{I}}_2}{dp^2} = \frac{(d-4)}{p^2}\left[\widetilde{\mathcal{I}}_1 + \widetilde{\mathcal{I}}_2\right]$$

again decoupled as $d \to 4$.

What about the **massive sunrise** and the relation found for the poles?

a) Play the same game, generate and solve IBPs in $d = 2$, where MIs are **finite**. One finds **No Relation**, both are needed!

b) Now play this game in $d = 4$. Expanding one finds two independent relations. The latter can be inverted giving

$$S_1^{(-2)}(d \to 4; p^2) = \frac{3}{2\, m^2}\; T^{(-2)}(d \to 4)$$
$$S_2^{(-2)}(d \to 4; p^2) = \frac{1}{2\, m^4}\; T^{(-2)}(d \to 4)\,,$$

which just gives the **poles** in terms of the **tadpole** (sub-topology)!

$$T(d) = \int \frac{\mathfrak{D}^d\, k \mathfrak{D}^d\, l}{(k^2 - m^2)(l^2 - m^2)} = \frac{(m^2)^{(d-2)}}{(d-2)^2(d-4)^2}\,.$$

Poles of MIs are fake! In other words, there exists a completely **finite** basis in $d = 2, 4, 6, \ldots$ etc, such that all poles are entirely determined by sub-topologies!

Conclusions

1) As expected, if a topology has N MIs in d dimensions, fixing the number of dimensions, $d = n$, can reduce the number of independent MIs.

2) This piece of information can be **extracted** using Schouten identities or solving IBPs in fixed number of dimensions.

3) The new relations can be used to find a **new basis of MIs**, for which the differential equations assume triangular form as $d \to n$.

4) The exercise can be repeated with much **less trivial topologies** (already done for some planar and non-planar two-loop three-point functions).

Outlook and open questions

a) If there is a relation in $d = n$, then we can decouple the diff. eq. in $d = n$. Is this **always true**?

b) Is the other way around always true? I.e. if we can **decouple** the equations, is there always such a relation?

c) This procedure seems to be suitable for **automation** at the level of IBPs solution. As Input one needs information on the poles of the integrals! (See finite basis, [Manteuffel, Panzer, Schabinger '14])

Stay tuned!