The dilatation operator of N=4 SYM, amplitudes and Yangian symmetry

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with

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Brandhuber, Penante, GT, Young 1412.1019 [hep-th] & 1502.06626 [hep-th] + in progress
Brandhuber, Heslop, GT, Young, to appear tomorrow

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Two themes

- Calculation of one-loop dilatation operator from two-point correlators with on-shell methods
(Brandhuber, Penante, GT, Young)
    - from MHV diagrams
    - from generalised unitarity applied to two-point functions
        - single-scale problem, unitarity particularly simple
- Derive the action of the Yangian on the dilatation operator from the Yangian symmetry of amplitudes
(Brandhuber, Heslop, GT, Young)
    - substantiate the idea that there exists a Yangian symmetry in N=4 SYM with different manifestations

Dilatation operator

- General form of two-point functions of primary operators in a conformal theory:

$$\langle 0|\mathcal{O}(x_1)\bar{\mathcal{O}}(x_2)|\rangle \sim \frac{1}{((x_{12})^2)^{\Delta_0+\gamma}} \qquad x_{12} := x_1 - x_2$$

  - Δ_0 = classical dimension, γ = anomalous dimension (assume momentarily no mixing)
  - expanding in g: $\langle 0|\,\mathcal{O}(x_1)\bar{\mathcal{O}}(x_2)\,|0\rangle \sim \frac{1}{((x_{12})^2)^{\Delta_0}}(1-\gamma\log(x_{12}^2\Lambda^2))$
  - anomalous dimension extracted from log divergence
  - pole in $1/\epsilon$ in dimensional regularisation

- Definition of the dilatation operator $$H_{AB} = \mu\frac{\partial}{\partial\mu}\log Z_{AB}$$
  - Z_{AB} = renormalisation constants for the operators $\{\mathcal{O}_A\}$

The SO(6) sector

- Scalar operators:

$$\mathcal{O}_{A_1B_1,A_2B_2,\ldots,A_LB_L}(x) \;=\; \mathrm{Tr}(\phi_{A_1B_1}\phi_{A_2B_2}\cdots\phi_{A_LB_L})(x)$$

  - $A_1, \ldots, B_L = 1,\ldots,4$ fundamental R-symmetry indices

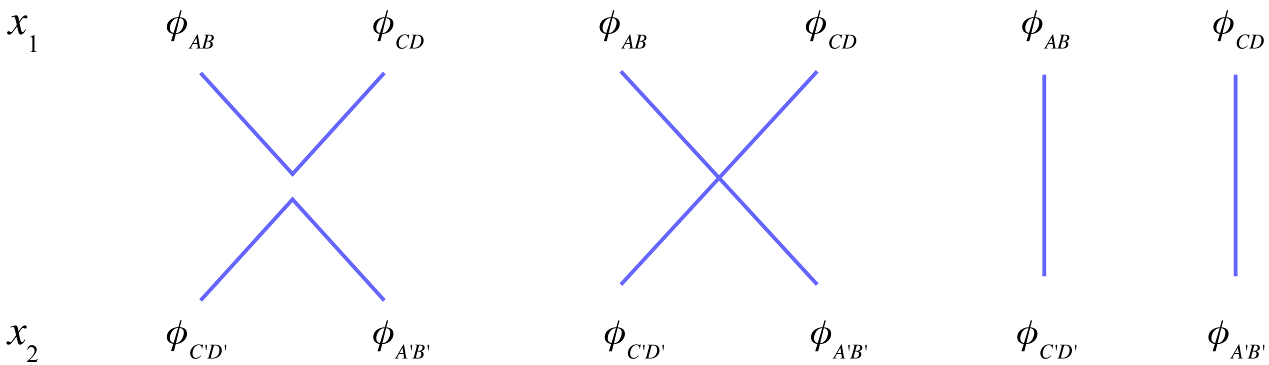
  - This sector is closed at one loop
  - calculation of anomalous dimension mapped to that of the eigenvalues of an integrable Hamiltonian (Minahan & Zarembo)

- Perturbative calculation

  - at one loop in the planar limit only nearest neighbours interact. Effectively the calculation is equivalent to that of the two-point correlator $\langle(\phi_{AB}\phi_{CD})(x_1)(\phi_{A'B'}\phi_{C'D'})(x_2)\rangle$

- General structure of $\langle (\phi_{AB}\phi_{CD})(x_1)(\phi_{A'B'}\phi_{C'D'})(x_2)\rangle$

$$= A\,\epsilon_{ABCD}\epsilon_{A'B'C'D'} + B\,\epsilon_{ABA'B'}\epsilon_{C'D'CD} + C\,\epsilon_{ABC'D'}\epsilon_{A'B'CD}$$



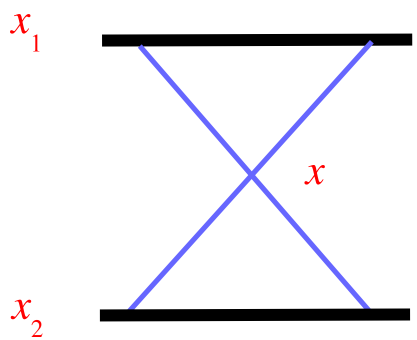
Trace — Permutation — Identity

- from Minahan & Zarembo: $A_{\rm UV} = 1/2$, $B_{\rm UV} = -1$, $C_{\rm UV} = 1$

(more accurately: $A = A_{\rm UV} \times [\lambda/(8\pi^2)] \times (1/(4\pi^2 x_{12}^2))^2 \times (1/\epsilon) + \text{finite}$)

$$\mathbb{H} = \sum_{i=1}^{L} (\mathbb{I} - \mathbb{P} + \frac{1}{2}\mathbb{T})_{ii+1}$$ Hamiltonian of an integrable spin chain

- In x space, prototypical UV divergence from

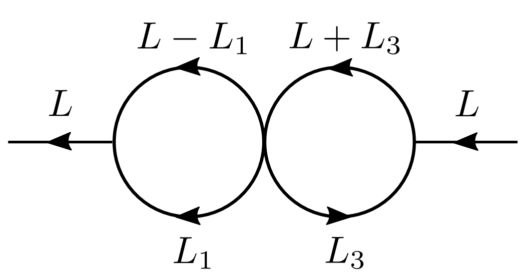


$$= \quad I(x_{12}) := \int^{1/\Lambda} d^4x \, \frac{1}{[(x-x_1)^2]^2[(x-x_2)^2]^2} \sim \frac{2\pi^2}{(x_{12}^2)^2} \log(x_{12}^2\Lambda^2)$$

  - for later applications, go to momentum space:

$$I(x_{12}) := \int \frac{d^D L}{(2\pi)^D} e^{iLx_{12}} \int \frac{d^D L_1}{(2\pi)^D} \frac{1}{L_1^2(L_1-L)^2} \int \frac{d^D L_3}{(2\pi)^D} \frac{1}{L_3^2(L_3+L)^2}$$

FT of Double bubble



$$= \frac{1}{\epsilon} \frac{1}{8\pi^2} \frac{1}{(4\pi^2 x^2)^2} + \text{finite}$$

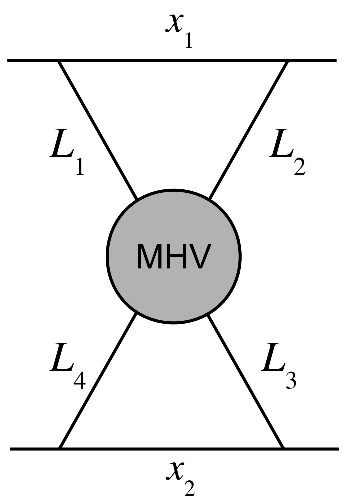
Calculation from MHV diagrams

(Brandhuber, Penante, GT, Young)

- Back to 2004!
  - MHV diagrams: new perturbative expansion of Yang-Mills theory (Cachazo, Svrcek, Witten 2004)
    - vertices are off-shell continuation of the MHV amplitudes, to each internal leg with momentum L one associates the spinor

      $\lambda_\alpha \to L_{\alpha\dot{\alpha}}\,\xi^{\dot{\alpha}}$ $\xi^{\dot{\alpha}}$ = reference spinor
    - connect vertices using scalar propagators
    - derivation from lightcone quantisation of YM + change of variables in the path integral with Jacobian = 1 (Mansfield; Gorsky, Rosly)
  - also works for loops, and without supersymmetry (cut-constructible parts) (Brandhuber, Spence, GT 2004)

- Only one MHV diagram!
  - choose three R-symmetry assignments contributing to one structure at a time



	$ABCD$	$A'B'C'D'$
Tr	1234	2413
$\mathbb{P}$	1213	3424
$\mathbb{1}$	1213	2434

$$A_{\rm MHV}\left(1_{\phi_{AB}}, 2_{\phi_{CD}}, 3_{\phi_{A'B'}}, 4_{\phi_{C'D'}}\right) = \begin{cases} \mathbb{T}: & \dfrac{\langle 13\rangle\langle 24\rangle}{\langle 12\rangle\langle 34\rangle} \\ \mathbb{P}: & -1 \\ \mathbb{I}: & \dfrac{\langle 13\rangle\langle 24\rangle}{\langle 23\rangle\langle 14\rangle} \end{cases}$$

$$A\,\epsilon_{ABCD}\epsilon_{A'B'C'D'} + B\,\epsilon_{ABA'B'}\epsilon_{C'D'CD} + C\,\epsilon_{ABC'D'}\epsilon_{A'B'CD}$$

$\mathbb{T}$ $\mathbb{P}$ $\mathbb{I}$

- New integral to compute:

$$I(x_{12}) := \int \frac{d^D L}{(2\pi)^D} e^{iLx_{12}} \int \frac{d^D L_1}{(2\pi)^D} \frac{1}{L_1^2 (L_1 - L)^2} \int \frac{d^D L_3}{(2\pi)^D} \frac{1}{L_3^2 (L_3 + L)^2} \; A_{\mathrm{MHV}}(1_{\phi_{AB}}, 2_{\phi_{CD}}, 3_{\phi_{A'B'}}, 4_{\phi_{C'D'}})$$

  - Double bubble x one MHV vertex

- Work out the integrands:

  - for P: nothing to be done! (-1 x double bubble, or -1)
  - for I: $\frac{\langle 13\rangle\langle 24\rangle}{\langle 23\rangle\langle 14\rangle} = 1 + \frac{\langle 12\rangle\langle 34\rangle}{\langle 23\rangle\langle 14\rangle}$
  - next using CSW prescription and momentum conservation:

$$1 + \frac{[\xi|L_1 L_2|\xi][\xi|L_3 L_4|\xi]}{[\xi|L_2 L_3|\xi][\xi|L_1 L_4|\xi]} = 1 + \frac{[\xi|L_1 L|\xi][\xi|L_3 L|\xi]}{[\xi|(L - L_1)L_3|\xi][\xi|L_1(L + L_3)|\xi]}$$

UV divergent

manifestly finite
DROP!

Summarising: $B_{\mathrm{UV}} = -1$, $C_{\mathrm{UV}} = 1$

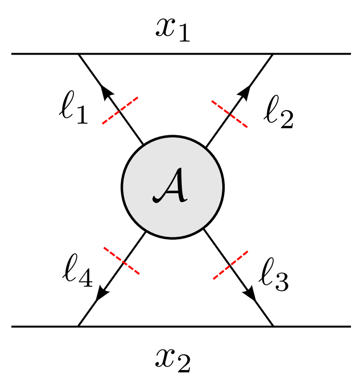
- Summary: $A_{\rm UV} = 1/2$, $B_{\rm UV} = -1$, $C_{\rm UV} = 1$

  (after similar manipulations on Tr)

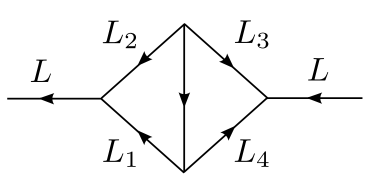
- Comments:
  - ξ-dependence: drops out at the end of the calculation
    - similar to lightcone gauge…
    - explicitly: by Lorentz invariance, final result can depend only on the combination $[\xi|L^2|\xi] = 0$
    - note that the result cannot depend on $L \cdot \xi$ as we have NOT introduced a spinor ξ_α !
  - self-energies: vanish with MHV diagrams (Brandhuber, Spence, GT 04)
    - in lightcone gauge self-energies are finite and do not contribute to anomalous dimension (Belitsky, Derkachov, Korchemsky, Manashov)
  - nice derivation from MHV rules in twistor space (Koster, Mitev, Staudacher)

Once again, with generalised unitarity

(Brandhuber, Penante, GT, Young)



- Compute correlator from cuts
  - Earlier applications (Engelund & Roiban)
  - Ingredients:
    - on-shell amplitudes (no off-shell continuations) & cut propagators
  - Quickly reconstruct the nontrivial cases:
    - for I: $\frac{\langle 13\rangle\langle 24\rangle}{\langle 23\rangle\langle 14\rangle} = 1 + \frac{\langle 12\rangle\langle 34\rangle}{\langle 23\rangle\langle 14\rangle} = 1 + \frac{\langle 12\rangle\langle 34\rangle}{\langle 23\rangle\langle 14\rangle}\frac{[34]}{[34]}$
    - second term: $-\frac{L^2}{2(\ell_1\cdot\ell_4)} \longrightarrow$
    - Kite integral, UV finite in four dimensions
    - Keep just the 1, drop the rest, or $C_{\rm UV} = 1$.
    - Similarly for the Tr structure



- Several generalisation/extensions possible
  - SU(2|3) sector, higher loops….

Amplitude Yangian & Dilatation Operator Yangian

(Brandhuber, Heslop, GT, Young, to appear)

- Yangian symmetry is thought to be a fundamental property of N=4 super Yang-Mills
- Two slightly different manifestations on
  - amplitudes
  - dilatation operator
- Goal: derive the action of the Yangian on the dilatation operator from the Yangian of amplitudes

Amplitude Yangian

- Fact 1: Tree-level amplitudes in N=4 SYM are Yangian invariant (covariant) (Drummond, Henn, Plefka)
  - level-zero generators $J^A \longrightarrow$ ordinary superconformal group
  - level-one generators $Q^A = \sum_{i<j} Q^A_{ij}$
    - $Q^A_{ij} := f^A_{CB} J^B_i J^C_j$ are non-local densities acting on particles i and j
  - level-one generators $\longrightarrow$ dual superconformal group
    - level-one generator $p^{(1)}$ associated to momentum p is (related to) dual conformal K
    - level-one generator $q^{(1)}$ associated to q-supersymmetry is (related to) dual superconformal generator S
  - amplitudes are covariant under dual superconformal transformations (Drummond, Henn, Korchemsky, Sokatchev)

Dilatation operator Yangian

- Fact 2: The complete one-loop dilatation operator is Yangian invariant up to boundary terms (Dolan, Nappi, Witten)

$$[Q^A\,,\,H] \sim J_1^A - J_L^A$$

  - Equivalent to showing $[Q_{12}^A\,,\,H_{12}] \sim J_1^A - J_2^A$
    - $H = \sum_{i=1}^{L} H_{ii+1}\,,$ where H_{12} acts on sites 1 and 2
  - proof is based on acting on irreps of PSU(2,2|4)
  - LHS ~ one loop, RHS ~ tree level. Proof relies on identity

$$\big[h(j) - h(j-1)\big]/j = 1 \qquad h\,(j) = j^{\text{th}} \text{ harmonic number}$$

- Next: derive Fact 2 from Fact 1

- **Main tool:** an intriguing formula relating the dilatation operator to four-point superamplitudes found by Zwiebel (+ unpublished work of Beisert)
- Building blocks of this formula:
  - tree-level four-point superamplitude
    - recall: at one-loop, only two fields interact, 2 →2 structure
  - tree-level minimal form factors $\langle 0|\,(\Phi_1 \cdots \Phi_L)(0)\,|\Phi_1 \cdots \Phi_L\rangle$
    - represent the states on which the dilatation operator acts
  - effectively computes two-particle cuts of one-loop minimal form factors of (non-protected) operators (Wilhelm)
- **Idea:** use known action of Yangian generators on amplitudes to derive action on the dilatation operator

- States & single-trace operators
  - A state corresponds to a single-trace operator $\mathrm{Tr}\big(\Phi_1 \cdots \Phi_L\big)(x)$
  - The letters Φ_i : $F^{\alpha\beta},\ \psi^{\alpha ABC},\ \phi^{[AB]},\ \bar{\psi}^{\dot{\alpha}A},\ \bar{F}^{\dot{\alpha}\dot{\beta}}$
    (and symmetrised covariant derivatives D acting on them)
  - oscillator representation of the states (Gunaydin & Marcus)

$$[a_\alpha, a^{\dagger\beta}] = \delta_\alpha^\beta\,, \quad [b_{\dot{\alpha}}, b^{\dagger\dot{\beta}}] = \delta_{\dot{\alpha}}^{\dot{\beta}}\,, \quad \{d_A, d^{\dagger B}\} = \delta_A^B\,, \quad \alpha,\beta = 1,2, \quad \dot{\alpha},\dot{\beta} = 1,2, \quad A = 1,\ldots,4$$

    - a, b bosonic oscillators, d fermionic oscillators
  - map to the states:

$$\bar{F} \leftrightarrow b^\dagger b^\dagger\,, \quad \bar{\psi} \leftrightarrow b^\dagger d^\dagger\,, \quad \phi \leftrightarrow d^\dagger d^\dagger\,, \quad \psi \leftrightarrow a^\dagger d^\dagger d^\dagger d^\dagger\,, \quad F \leftrightarrow a^\dagger a^\dagger d^\dagger d^\dagger d^\dagger d^\dagger\,, \quad D \leftrightarrow a^\dagger b^\dagger$$

- Spinor-helicity translation:

$$a^{\dagger\alpha} \leftrightarrow \lambda^\alpha\,, \quad b^{\dagger\dot{\alpha}} \leftrightarrow \tilde{\lambda}^{\dot{\alpha}}, \quad d^{\dagger A} \leftrightarrow \eta^A$$
$$a_\alpha \leftrightarrow \partial_\alpha\,, \quad b_{\dot{\alpha}} \leftrightarrow \partial_{\dot{\alpha}}, \quad d_A \leftrightarrow \partial_A$$

- States in spinor-helicity language:
  - combine $\Lambda^a := (\lambda^\alpha, \tilde{\lambda}^{\dot{\alpha}}, \eta^A)$
  - a state is a polynomial $P(\Lambda_1, \ldots, \Lambda_L)$ in the Λ's satisfying the physical state condition (vanishing central charge)
- Examples:
  - half-BPS $\cdots \phi^{12}\phi^{12} \cdots \leftrightarrow (\eta_1^1 \eta_1^2)(\eta_2^1 \eta_2^2)$ ←R-symmetry ← position
  - Konishi $\cdots \epsilon_{ABCD} \phi^{AB} \phi^{CD} \cdots \leftrightarrow \epsilon_{ABCD} (\eta_1^A \eta_1^B)(\eta_2^C \eta_2^D)$
- $P(\Lambda_1, \ldots, \Lambda_L)$ = tree-level minimal form factor of the corresponding operator (Wilhelm)
  - E.g. half-BPS

$$\cdots (\eta_1^1 \eta_1^2)(\eta_2^1 \eta_2^2) \cdots = \langle 0 | \big(\cdots \phi^{12} \phi^{12} \cdots \big)(0) \big| \cdots \phi^{12} \phi^{12} \cdots \rangle$$

- Zwiebel's formula: (second term slightly rewritten)

$$H_{12}|1,2\rangle = \int d\Lambda_3 d\Lambda_4\, A(1,2,3,4)\left[P(-4,-3) - \left(\frac{\langle 12\rangle}{\langle 34\rangle}\right)^2 P(1,2)\right]$$

"un-integrated form"

  - phase-space measure $d\Lambda_i := d^2\lambda_i d^2\tilde{\lambda}_i d^4\eta_i$ (mod little group)
  - superamplitude $A(1,2,3,4) = \frac{\delta^{(4)}\left(\sum_i \lambda_i\tilde{\lambda}_i\right)\delta^{(8)}\left(\sum_i \lambda_i\eta_i\right)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$
  - $P(1{,}2)$ represents the operator/state $|\cdots 1,2,\cdots\rangle$

- Connection to dilatation operator

  - integrating out the momentum delta function one gets: (Zwiebel)

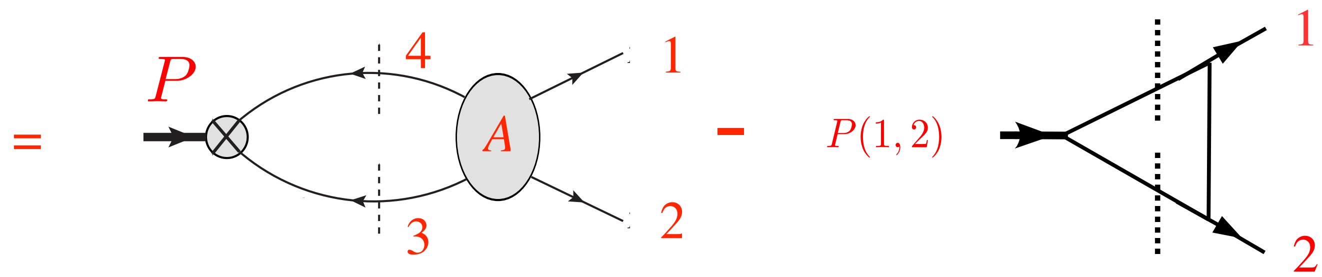
$$H_{12}|1,2\rangle = -\frac{1}{\pi}\int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} d\theta \left[e^{2i\phi}P(1',2') - P(1,2)\right]$$

"integrated form"

  - $\lambda_1' = \lambda_1 \cos\theta - e^{i\phi}\lambda_2 \sin\theta\,, \quad \lambda_2' = \lambda_1 \sin\theta + e^{i\phi}\lambda_2\cos\theta$ (similarly for $\tilde{\lambda}', \eta'$)
  - Neatly reproduces Beisert's harmonic action form of the complete dilatation operator at one loop

• Connection to form factors (Wilhelm)

$$H_{12}|1,2\rangle \;=\; \int d\Lambda_3 d\Lambda_4\, A(1,2,3,4)\Big[P(-4,-3)\;-\;\left(\frac{\langle 12\rangle}{\langle 34\rangle}\right)^2 P(1,2)\Big] \;=$$



- first term is the two-particle cut of the one-loop form factor of the operator represented by P
  - contains IR divergences (triangle) and also UV divergences (bubbles)
- second term subtracts the IR divergence of the same cut
- Leftover = discontinuity of a (UV-divergent) bubble, whose coefficient is ~ the dilatation operator
  - note: the discontinuity of a bubble if finite

- Summarising:
  - Unintegrated form:

$$H_{12}|1,2\rangle \;=\; \int d\Lambda_3 d\Lambda_4\, A(1,2,3,4)\Big[P(-4,-3) \;-\; \left(\frac{\langle 12\rangle}{\langle 34\rangle}\right)^2 P(1,2)\Big]$$

  - integrated form:

$$H_{12}|1,2\rangle = -\frac{1}{\pi}\int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} d\theta\, \left[e^{2i\phi}P(1',2') - P(1,2)\right]$$

- It is not at all obvious to see how the relation

$$[Q^A_{12}\,,\,H_{12}] \sim J^A_1 - J^A_2$$

is realised when acting on the integrated form

Amplitudes to the rescue!

- Act with level-one generator $p^{(1)}$ on un-integrated form:

$$[Q_{12}, H_{12}]|1,2\rangle = Q_{12}\int d\Lambda_3 d\Lambda_4\ A(1,2,3,4)\big[P(-4,-3) - r\,P(1,2)\big]$$
$$- \int d\Lambda_3 d\Lambda_4\ A(1,2,3,4)\big[Q_{-4,-3}P(-4,-3) - r\,Q_{12}P(1,2)\big]$$

$$r := \left(\frac{\langle 12\rangle}{\langle 34\rangle}\right)^2$$

  - $Q_{ij} \;=\; \Big(m_j{}^{\gamma}{}_{\alpha}\delta^{\dot\gamma}_{\dot\alpha} + \bar m_j{}^{\dot\gamma}{}_{\dot\alpha}\delta^{\gamma}_{\alpha} - d_j\delta^{\gamma}_{\alpha}\delta^{\dot\gamma}_{\dot\alpha}\Big)p_{i\,\gamma\dot\gamma} \;+\; \bar q_{j\dot\alpha C}\, q^{C}_{i\alpha} - (i \leftrightarrow j)$ from DHP

- Preliminary check: half-BPS operators, e.g. Tr $(\phi^{12}\ \phi^{12})$

  - First line vanishes since $P^{\phi^{12}\phi^{12}}(-4,-3) \;=\; r\,P^{\phi^{12}\phi^{12}}(1,2)$
    (Brandhuber, Spence, GT, Yang)

  - Second line: only constant part of d survives

$$[Q_{12}, H_{12}]|\phi^{12}\phi^{12}\rangle = P^{\phi^{12}\phi^{12}}(1,2)\int d\Lambda_3 d\Lambda_4\ A(1,2,3,4)\cdot r\,\big[p_3 - p_4 - (p_1 - p_2)\big]$$

- Result of explicit integration of RHS:

$$[Q_{12}, H_{12}]|1,2\rangle \;=\; 2(p_1 - p_2)|1,2\rangle$$

- Ingredients of the general proof (for arbitrary states):
  - after IBP, combination of generators acting on amplitude is

$$Q_{12} + Q_{34} = \sum_{i<j} Q_{ij} - (Q_{13} + Q_{14} + Q_{23} + Q_{24})$$

  - $\sum_{i<j} Q_{ij}$ related to dual conformal K, which annihilates amplitude
  - $(Q_{13} + Q_{14} + Q_{23} + Q_{24})\ A = 0$ since

$$(Q_{13} + Q_{14} + Q_{23} + Q_{24})^A = f^A_{CB}(J_1 + J_2)^B J^C - \frac{1}{2} f^A_{CB} f^{BC}_D (J_1 + J_2)^D$$

    - J^C is a symmetry of the amplitude
    - $f^A_{CB} f^{BC}_D$ proportional to the (vanishing) dual Coxeter number of PSU(2,2|4)
    - alternative proof: use that Yangian on amplitudes is compatible with cyclicity!
  - thus $[Q_{12}, H_{12}]|1,2\rangle = P(1,2) \int d\Lambda_3 d\Lambda_4\ A(1,2,3,4) \cdot r\left[p_3 - p_4 - (p_1 - p_2)\right]$
    - remaining term already computed in half-BPS case…

- Result:

$$[Q_{12}, H_{12}]|1,2\rangle \;=\; 2(p_1 - p_2)|1,2\rangle$$

- Comments:

1. very simple extension to show that a similar formula holds if Q is the level-one generator associated to supersymmetry q:

$$[Q_{12}, H_{12}]|1,2\rangle \;=\; 2(q_1 - q_2)|1,2\rangle$$

2. not obvious to see this result on the "integrated form" of Zwiebel's formula (without amplitudes)!

3. RHS looks like a tree-level quantity!

4. can check other commutators