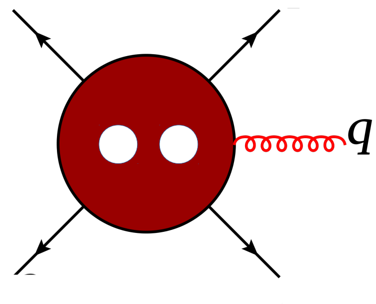
Soft factorization of gauge theory amplitudes beyond one loop

$$\lim_{q \to 0}$$



Hua Xing Zhu

SLAC

With Lance Dixon, work in preparation

Amplitudes 2015, ETH Zürich
10-07-2015

Motivation

- Truly remarkable progress in N=4 planar scattering amplitudes
  - All-loop planar integrand
  - Integrability for planar amplitudes; Modern multi-loop bootstrap program
- Most of these marvelous developments seems to rely on dual conformal invariance and integrability. Both are absent beyond planar limit
- No Canonical definition for non-planar integrand
- The question is, does the simplicity and beauty seen in the amplitudes calculation for planar sector seize to be true for non-planar sector?

Motivation

- There are hints that the answer could be a “NO”
  - Integrand of planar *and* non-planar four-particle MHV amplitudes in N=4 sYM can be represented as *dlog* form weighted by color factors and Parke-Taylor amplitudes

$$A(1^-,2^-,3^+,4^+) \sim \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \times \sum_j C_1^{(j)} d\log\alpha_1^j\, d\log\alpha_2^j \ldots d\log\alpha_8^j$$
$$+ \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 24\rangle\langle 43\rangle\langle 31\rangle} \times \sum_j C_2^{(j)} d\log\beta_1^j\, d\log\beta_2^j \ldots d\log\beta_8^j$$

Arkani-Hamed et.al, 1410.0354; Bern et.al, 1412.8584

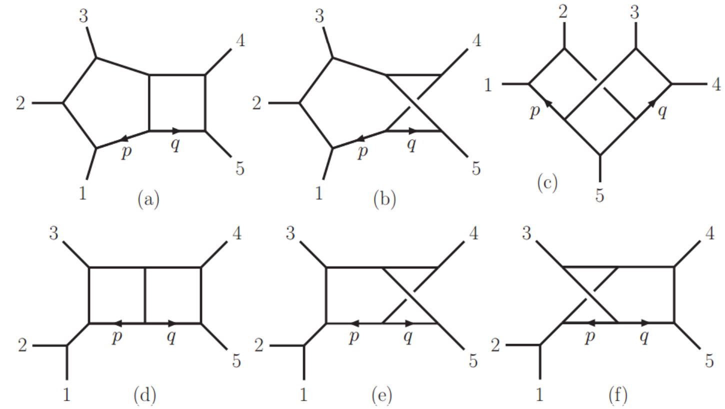
  - Non-planar MHV leading singularities can always be written as positive sum of planar ones with different ordering of legs.

Nikani-Hamed et.al, 1412.8475

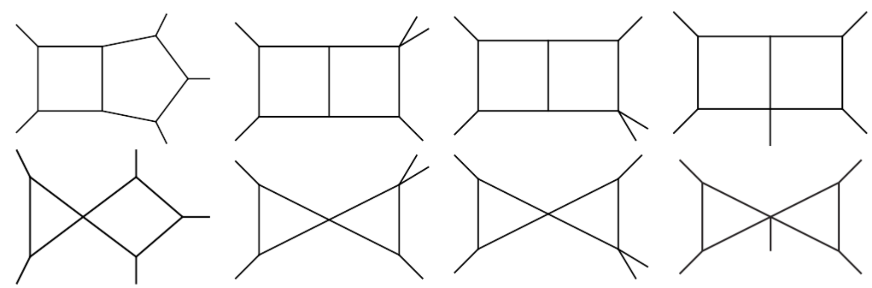
Clearly more non-planar data at two loops and beyond would be very helpful!

Towards five-particle amplitudes

- Non-planar five-particle integrand for N=4 sYM and N=8 supergravity known for some time
- Five-gluon all-positive helicities amplitudes known numerically
- Non-planar five-gluon all-positive helicity integrand (see Badger's talk)
- Master integrals for five-point planar integrals (see Henn's talk)
- Non-planar 3-loop 5-pt integrand (see Carrasco's talk)
- ……



Carrasco, Johansson, 1106.4711



Badger, Frellesvig, Y. Zhang, 1310.1051

Can we have a glimpse of (non-planar) two-loop n-point (n>4) amplitudes by studying their soft factorization limit?

Soft factorization in QED

- Factorization of tree amplitudes when one of the external massless gauge boson becomes soft was studied in the early days of quantum field theory
- For U(1) gauge boson, this is the well-known Weinberg's soft photon theorem:

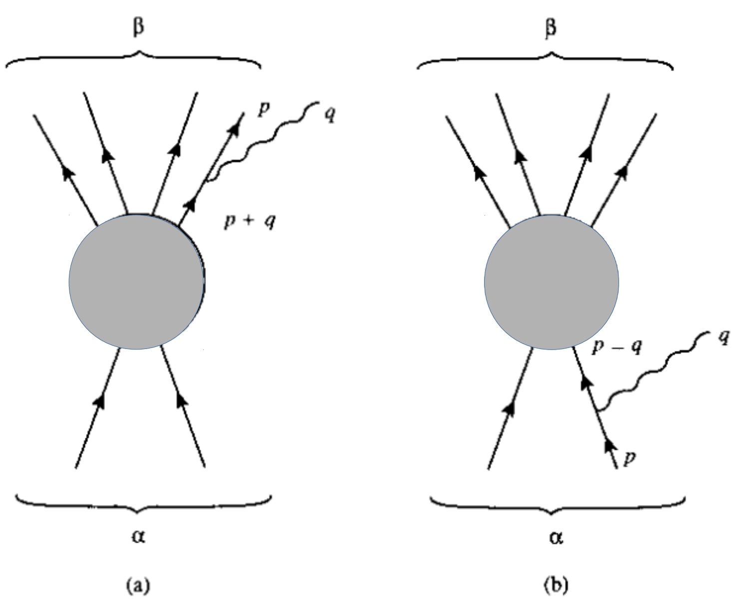


Figure 13.1. Dominant graphs for the emission of soft photons in an arbitrary process $\alpha \to \beta$. Straight lines are particles in the states α and β (including possible hard photons); wavy lines are soft photons.

Tree-level amplitudes of n hard particles plus one soft photon factorize into amplitudes of n hard particles multiplied with eikonal factors

$$\varepsilon_\mu(q) M^\mu_{n+1}(q) \overset{E_q\to 0}{=} M_n \varepsilon_\mu(q) \sum_n \frac{e_n p_n^\mu}{p_n \cdot q + i\epsilon}$$

The Quantum Theory of Fields I, Weinberg

Soft factorization in QCD

- The soft factorization formula can be easily generalized to QCD by replacing electric charge with color charge T

$$M_{n+1}^{a,\mu}(q) = \sum_{i=1}^{n} \frac{p_i^\mu}{p_i \cdot q} (T_i^a) M_n$$

$$e_i \Rightarrow T_i^a \quad \begin{cases} (T_i^a)_{\alpha\beta} = t_{\alpha\beta}^a & \text{quark} \\ (T_i^a)_{\alpha\beta} = -t_{\beta\alpha}^a & \text{antiquark} \\ (T_i^a)_{bc} = i f_{bac} & \text{gluon} \end{cases}$$

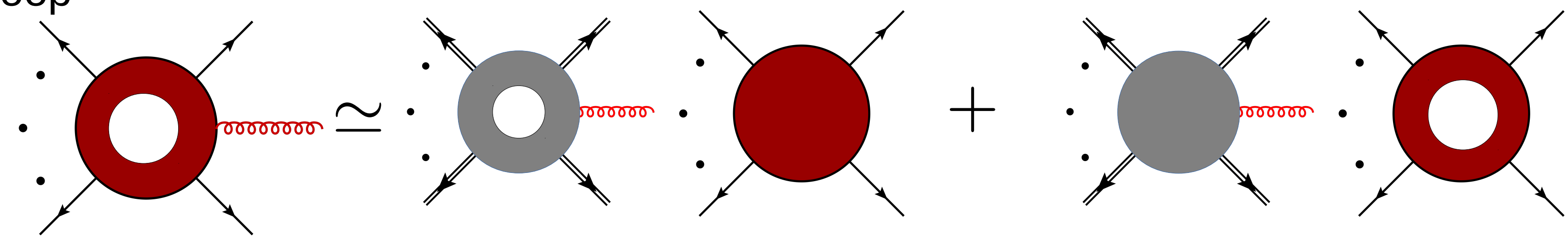
Rescaling invariance

- The physical picture is clear: the emitted soft gluon has very long wave length, therefore can only resolve the directions but not energy of the hard particles in the amplitudes.
- The eikonal factors can be regarded as amplitudes for emitting a soft gluon from n Wilson lines, which are specified by direction $\beta^\mu = p^\mu / p^0$ and color charge T^a of the hard particle

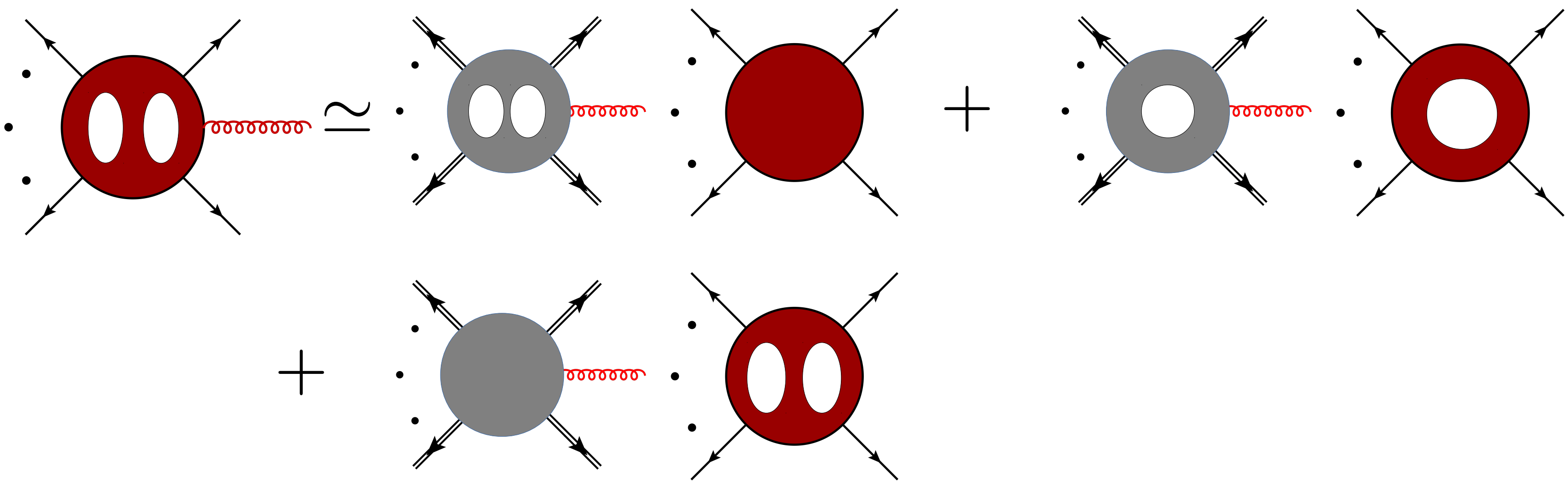
Multi-loop generalization of soft fact.

- The tree-level soft factorization formula has a very natural multi-loop generalization

1-loop



2-loop



Multi-loop generalization of soft fact.

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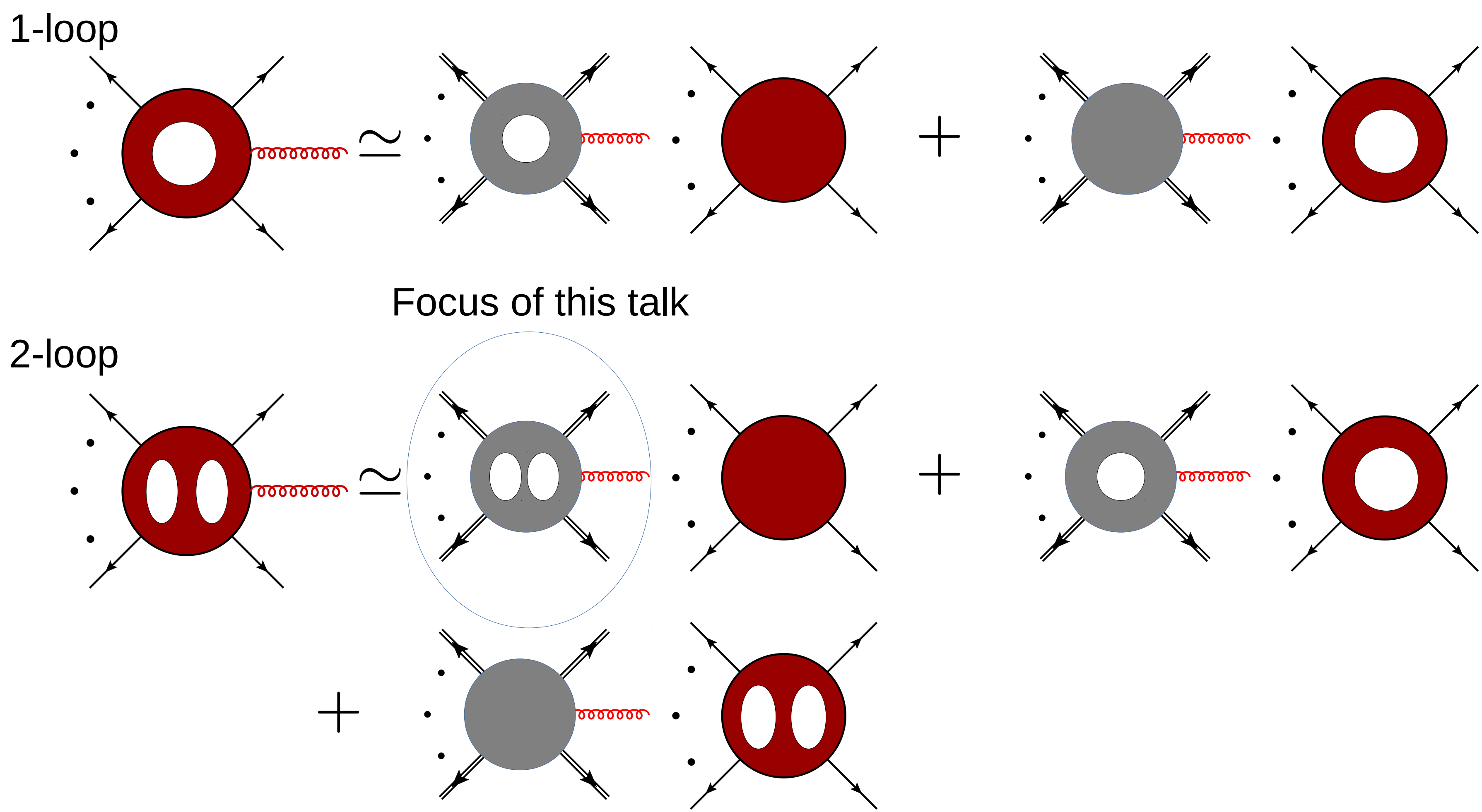
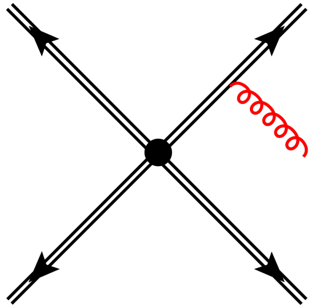
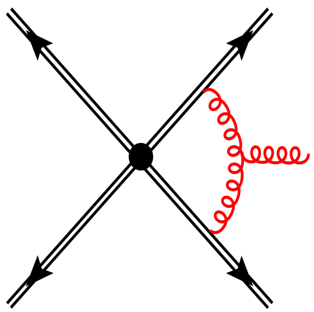


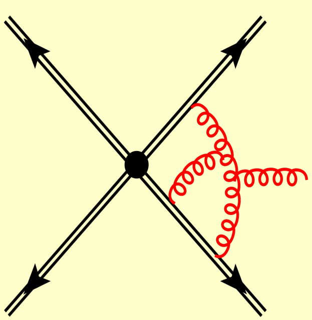
Feynman diagrams for soft factor

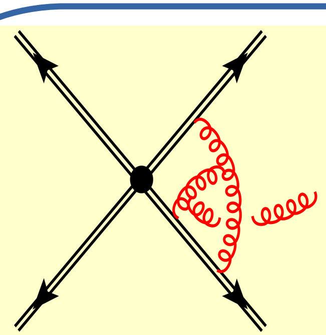


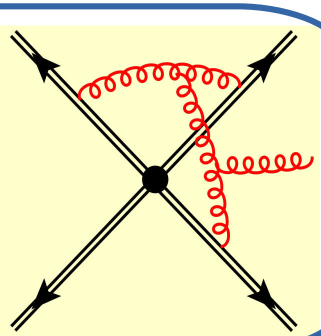
1 diagram
Bassetto, Ciafaloni, Marchesini (1983)
Dokshitzer, Khoze, Mueller, Troyan (1991)
……

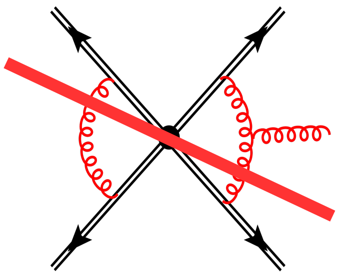


1 non-vanishing diagram
Bern, Chalmers (1995); Bern, Del Duca, Schmidt (1998);
Bern, Del Duca, Kilgore, Schmidt (1999)
Catani, Grazzini (2000)









Badger, Glover (2004)
Y. Li, H.X.Z (2013)
Duhr, Gehrmann (2013)

This talk!

Vanishes trivially

The need for soft factor for higher order cross section calculation

- For example NLO cross section with m final state particles is

$$\sigma_{NLO} = \int_m d\sigma_V + \int_{m+1} d\sigma_R$$

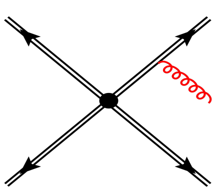
- Introduce counter terms which have the same soft and collinear singularities as the (m+1)-point amplitudes

$$\sigma_{NLO} = \int_m d\sigma_V + \int_{m+1} d\sigma_A + \int_{m+1} (d\sigma_R - d\sigma_A)$$

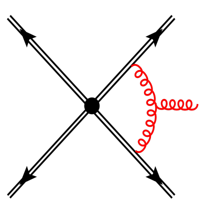
IR pole cancel analytically

Numerically integrable

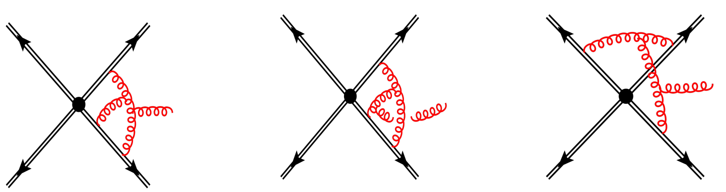
- The soft factor is one piece of the subtracted matrix elements $d\sigma_A$ which has the same soft singularities as the full matrix elements.



For NLO calculation



For NNLO calculation



For N^3LO calculation

The diagrammatic calculation

- Diagrams generated by QGRAF
- Reducing tensor integrals into scalar integrals (Anastasiou, Glover, Oleari, 1999)
- Integrals reduced to Master Integrals (11 MIs) via Integration-By-Parts identities (Tkachov, Chetyrkin, 1981)
- Solve the master integrals using the method of differential equation for Feynman integrals (Kotikov; Remiddi; Gehrmann-Remiddi)
- Going to a canonical basis of D.E. to simplify the calculation (Henn, 1304.1806)
- Calculating the boundary constants by Mellin-Barnes.

Constraint from rescaling symmetry

- The rescaling symmetry imposes strong constraint on the soft factor, but not completely fix it.
- For dipole contribution, where the soft particles (real or virtual) only attach to two Wilson line, there is one constant to be determined

$$\left(\frac{\mu^2 s_{ij}}{s_{iq}s_{qj}}\right)^{2L\epsilon} C_{ij}^{(L)}(\epsilon)$$

- For tripole contribution, where the soft particles (real or virtual) attach to three Wilson lines with momentum p1, p2, p3, the result depends on rescaling invariant cross ratios and Kallen function of the ratios

$$u = \frac{s_{12}s_{3q}}{s_{13}s_{2q}} \qquad v = \frac{s_{23}s_{1q}}{s_{13}s_{2q}} \qquad \Delta = \sqrt{1+u^2+v^2-2u-2v-2uv}$$

- It's convenient to parametrize u and v by a single complex variable z

$$u = (1-z)(1-\bar{z}) \qquad v = z\bar{z} \qquad \Delta = \pm(z-\bar{z})$$

$$z \to 0 : p_q \parallel p_1, \qquad z \to 1 : p_q \parallel p_3, \qquad z \to \infty : p_q \parallel p_2$$

- In the Euclidean region, the soft factor is a single-valued function in z

A pleasant simplification

- The singularities occurring in the differential equation for master integrals are

$$\{z, \bar{z}, (1-z), (1-\bar{z}), z-\bar{z}\}$$

- Individual master integrals will be single-valued multiple polylogarithms G(...;z). First entry of the symbols constraint by physical branching cut. Must be u or v.
- The double and single pole of the master integrals contain polylogarithms, which are canceled out in the gauge invariant contribution
- The finite terms of the master integrals contain polylogarithms with symbol entries $z-\bar{z}$, which are also canceled out in the gauge invariant contribution
- The final results through to finite terms are single-valued harmonic polylogarithms (SVHPLs) F. Brown 2004

SVHPLs

- Harmonic polylogarithms (HPLs):

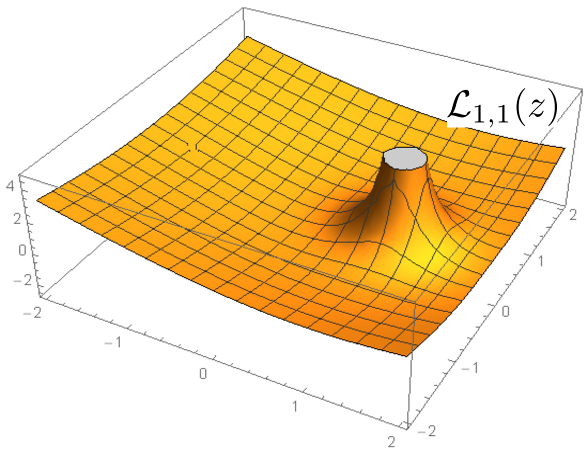
$$H_{0w}(z) = \int_0^z dt \frac{H_w(t)}{t}, \quad H_{1w}(z) = \int_0^z dt \frac{H_w(t)}{1-t}$$

- SVHPLs are built from bilinear combination of HPLs in z and $\bar{z}$ such the branch cuts are canceled

- Explicit construction can be found in

  (Duhr, Dixon, Pennington, 1207.0186)

$$\mathcal{L}_0(z) = H_0 + \bar{H}_0$$
$$\mathcal{L}_1(z) = H_1 + \bar{H}_1 \qquad \bar{H} = H(\ldots;\bar{z})$$
$$\mathcal{L}_{1,1}(z) = H_{1,1} + \bar{H}_{1,1}$$
$$\ldots$$



- SVHPLs occur in multi-Regge limit of six-point remainder function (Duhr, Dixon, Pennington, 1207.0186) and three-loop soft anomalous dimension (Almelid, Duhr, Gardi, 1507.00047)

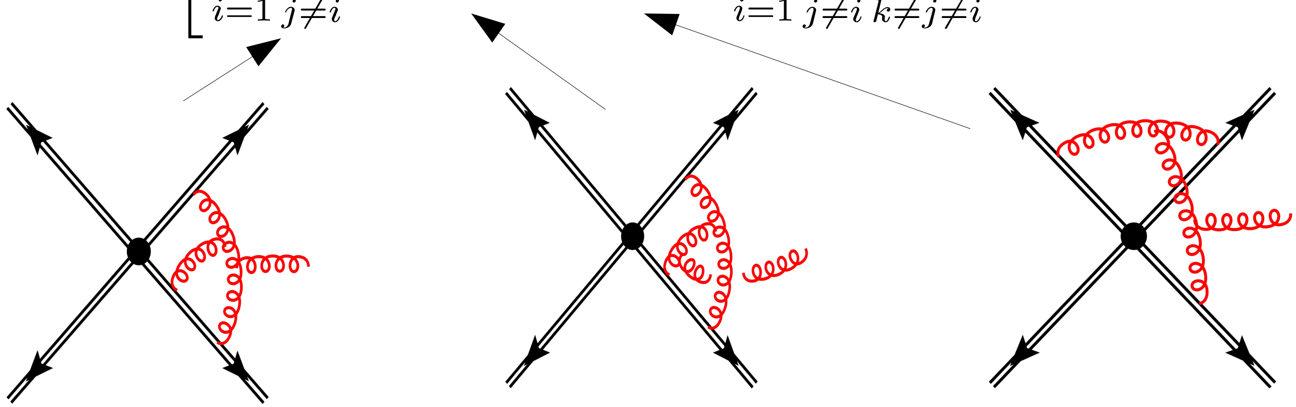
Dipole contribution

- Soft factorization of two-loop amplitude

$$M_{n+1}^{a,\mu(2)}(q) = \varepsilon_\mu(q) J_q^{a,\mu(2)} \cdot M_n^{(0)} + \varepsilon_\mu(q) J_q^{a,\mu(1)} \cdot M_n^{(1)} + \varepsilon_\mu(q) J_q^{a,\mu(0)} \cdot M_n^{(2)}$$

- The two-loop soft factor contains both dipole contribution and tripole contribution

$$J_q^{a,\mu(2)} = \left(\frac{\alpha_s}{4\pi}\right)^2 \left[\sum_{i=1} \sum_{j\neq i} J_{ij}^{a,\mu(2)}(q;p_i,p_j) + \sum_{i=1} \sum_{j\neq i} \sum_{k\neq j\neq i} J_{ijk}^{a,\mu(2)}(q;p_i,p_j,p_k) \right]$$



$$J_{ij}^{a,\mu(2)}(q;p_i,p_j) = i f_{abc} T_i^b T_j^c \left(\frac{p_i^\mu}{p_i \cdot q} - \frac{p_j^\mu}{p_j \cdot q} \right) \times \left(C_A \left(\frac{1}{2\epsilon^4} - \frac{11}{12\epsilon^3} - \frac{67}{36\epsilon^2} + \frac{\zeta_2}{\epsilon^2} + \dots \right) + \dots \right)$$

Tripole contribution

- The tripole contribution contain nontrivial dependence on the cross ratio

$$J_{123}^{a,\mu(2)}(q;p_1,p_2,p_3) = f_{qa_2b}f_{a_1a_3b}T_1^{a_1}T_2^{a_2}T_3^{a_3}\frac{p_1^\mu}{p_1\cdot q}D_1(z,\bar{z}) + \text{ permutation of } 1,\,2,\,3$$

$$\begin{aligned}
D_1(z,\bar{z}) =& \frac{1}{3\epsilon^2}(\mathcal{L}_{0,1}+\mathcal{L}_{1,0}) - \frac{2}{3\epsilon}(\mathcal{L}_{0,1,1}+\mathcal{L}_{1,0,1}+\mathcal{L}_{1,1,0}) \\
&+ \frac{4}{3}(\mathcal{L}_{0,1,1,1}+\mathcal{L}_{1,0,1,1}+\mathcal{L}_{1,1,0,1}+\mathcal{L}_{1,1,1,0}) \\
&+ \frac{1}{3}(\mathcal{L}_{0,1,0,1}+\mathcal{L}_{1,0,1,0}) + \frac{\zeta_2}{3}(\mathcal{L}_{0,1}+\mathcal{L}_{1,0}) + \frac{4\zeta_3}{3}\mathcal{L}_1 \\
&+ \left(\frac{z+\bar{z}}{z-\bar{z}}\right)\Big(\mathcal{L}_{1,0,1,0}-\mathcal{L}_{0,1,0,1}+2\zeta_2(\mathcal{L}_{0,1}-\mathcal{L}_{1,0})+4\zeta_3\mathcal{L}_1\Big) \\
&+ \left(\frac{z\bar{z}-z-\bar{z}}{z-\bar{z}}\right)\Big(\frac{2}{3}\Big(\mathcal{L}_{0,0,0,1}-\mathcal{L}_{0,0,1,0}+\mathcal{L}_{0,1,0,0}-\mathcal{L}_{1,0,0,0}+\mathcal{L}_{1,0,1,0}-\mathcal{L}_{0,1,0,1}\Big) \\
&+ 4\zeta_2(\mathcal{L}_{0,1}-\mathcal{L}_{1,0}) + \frac{8}{3}\zeta_3\mathcal{L}_1\Big)
\end{aligned}$$

$$z = \frac{\langle 23\rangle\langle 1q\rangle}{\langle 13\rangle\langle 2q\rangle} \qquad \bar{z} = \frac{[23][1q]}{[13][2q]}$$

- The soft factor is gauge invariant, $q_\mu J_{123}^{a,\mu(2)} = 0$ when summing over the permutation
- It's a function of maximal transcendental weight [L(1)] = 1, [ζ_n] = n, [1/e] = 1.
- Not a pure function.

Soft factor with definite soft-gluon helicity

$$z = \frac{\langle 23\rangle\langle 1q\rangle}{\langle 13\rangle\langle 2q\rangle}$$

$$\bar{z} = \frac{[23][1q]}{[13][2q]}$$

- Suppose the soft gluon has positive helicity

$$S_a^{+,(2)} = \varepsilon^+(q)\cdot J_q^{a(2)}$$

- Again separate into dipole contribution and tripole contribution

$$S_a^{+,(2)} = \left(\frac{\alpha_s}{4\pi}\right)^2 \sum_{i=1}\sum_{j\neq i} S_{a,\mathrm{ij}}^{+,(2)} + \left(\frac{\alpha_s}{4\pi}\right)^2 \sum_{i=1}\sum_{j\neq i}\sum_{k\neq j\neq i} S_{a,\mathrm{ijk}}^{+,(2)}$$

- Dipole contribution

$$S_{a,\mathrm{ij}}^{+,(2)} = \sqrt{2}\frac{\langle ij\rangle}{\langle iq\rangle\langle qj\rangle}\left(\frac{\mu^2 s_{ij}}{s_{iq}s_{qj}}\right)^{2\epsilon} i f_{aa_ia_j} T_i^{a_i} T_j^{a_j}\left(C_A\left(\frac{1}{2\epsilon^4} - \frac{11}{12\epsilon^3} - \frac{67}{36\epsilon^2} + \frac{\zeta_2}{\epsilon^2} + \ldots\right) + \ldots\right)$$

- Tripole contribution

Square bracket from $\bar{z}$ cancel out!

$$S_{a,123}^{+,(2)} = \sqrt{2} f^{aa_2b} f^{a_1a_3b} T_1^{a_1} T_2^{a_2} T_3^{a_3}\left(\frac{\mu^2 s_{13}}{s_{1q}s_{q3}}\right)^{2\epsilon}\left[\frac{\langle 12\rangle}{\langle 1q\rangle\langle q2\rangle}F(z,\bar{z}) + \frac{\langle 23\rangle}{\langle 2q\rangle\langle q3\rangle}F(1-z,1-\bar{z})\right]$$

- F is a pure function of maximal transcendental weight

Consistency check: collinear limit

- Taking the collinear limit of soft factor is equivalent to taking the soft gluon limit of splitting amplitude. Let's take the $p_1||q$ limit
- Dipole contribution

$$\lim_{q||p_1} \sum_i \sum_{j\neq i} \sqrt{2} \frac{\langle ij\rangle}{\langle iq\rangle\langle qj\rangle} \left(\frac{\mu^2 s_{ij}}{s_{iq} s_{qj}}\right)^{2\epsilon} i f_{a a_i a_j} T_i^{a_i} T_j^{a_j} \left(C_A \left(\frac{1}{2\epsilon^4} - \frac{11}{12\epsilon^3} + \ldots \right) + \ldots \right)$$

$$= \left(\frac{\mu^2}{w s_{q1}}\right)^{2\epsilon} C_1 \left(\underbrace{C_A \left(\frac{1}{2\epsilon^4} - \frac{11}{12\epsilon^3} + \ldots \right)} + \ldots \right) \lim_{q||p_1} S_a^{+,(0)}$$

Soft limit of planar collinear splitting amplitude

- Tripole contribution:

$$\lim_{q||p_1 \text{ or } p_2 \text{ or } p_3} \left[\frac{\langle 12\rangle}{\langle 1q\rangle\langle q2\rangle} F(z,\bar{z}) + \frac{\langle 23\rangle}{\langle 2q\rangle\langle q3\rangle} F(1-z, 1-\bar{z}) \right] = \text{no collinear singular terms}$$

- **Soft-collinear limit of gluon emission in planar and non-planar theory are the same.**

Analytic continuation from Euclidean to physical kinematics

- As an example, analytic continuate to the region where p2 and p3 are incoming, p1 and q are outgoing.

$$u = \frac{s_{12}s_{3q}}{s_{13}s_{2q}} \to u\,, \qquad v = \frac{s_{23}s_{1q}}{s_{13}s_{2q}} \to v\exp(-2\pi i)\,, \qquad z = \frac{1}{2}(1-u+v-\sqrt{-4v+(1-u+v)^2})$$

- Compute the discontinuities of SVHPLs using bottom up approach

$$\mathcal{L}_0 = \log v \to \mathcal{L}_0 - 2\pi i \qquad \mathcal{L}_1 = -\log u \to \mathcal{L}_1$$

- For higher weight SVHPLs, compute the discontinuity of their derivatives

$$\mathrm{Disc}\left[\frac{d\mathcal{L}_{0,1}}{dz}\right] = \frac{\mathrm{Disc}[\mathcal{L}_1]}{z}$$

- Integrate to get the discontinuity, up to a constant. Fix the constant by letting $z = \bar{z}$

Single soft limit of five-point MHV amplitudes in N=4 sYM

- An obvious application of the soft factorization formula is calculating the non-planar five-gluon MHV amplitude in N=4 sYM

$$\lim_{p_5\to 0} \mathcal{A}_5^{(2),a_1\ldots a_5}(--+++) = S_a^{+,(2)}(p_5)\cdot \mathcal{A}_4^{(0),a_1\ldots a_4}(--++) + S_a^{+,(1)}(p_5)\cdot \mathcal{A}_4^{(1),a_1\ldots a_4}(--++)$$
$$+ S_a^{+,(0)}(p_5)\cdot \mathcal{A}_4^{(2),a_1\ldots a_4}(--++)$$

Bern, Rozowsky, B. Yan, 97'

- 24 single color-trace coeff., 20 double color-trace coeff.
- Parke-Taylor factor PT(12345) and permutation of 2,3,4.

$$A_5^{(2)}(1^-,2^-,3^+,4^+,5^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\times(\text{pure function of uniform weight})$$
$$+\,(\text{permutations of PT})\ \times(\text{pure function of uniform weight})$$

- IR divergences agree with Catani's formula through to 1/e

Catani (1998) Aybat, Dixon, Sterman (2006) Becher, Neubert (2009) Gardi, Magnea (2009)

Summary

- We computed the soft factor necessary for the factorization of two-loop amplitude in the sinlge-soft-gluon limit
- The soft factor are described by single-valued harmonic polylogarithms in Euclidean region
- We obtained two-loop five-gluon MHV amplitudes in N=4 sYM in the single-soft limit