The Sinh-Gordon model

– a warm-up for non-compact σ-models?

J. Teschner

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Ultimate goal: Understand string theory on AdS-spaces

Basic building blocks: nonlin. σ-models on super-grps.

$PSL(1,1|2) \rightsquigarrow AdS_3 \times S_3$

$PSL(2,2|4) \rightsquigarrow AdS_5 \times S_5$

Focus on $PSL(1,1|2) \rightsquigarrow$ Eucl. $PSL(2|2)$

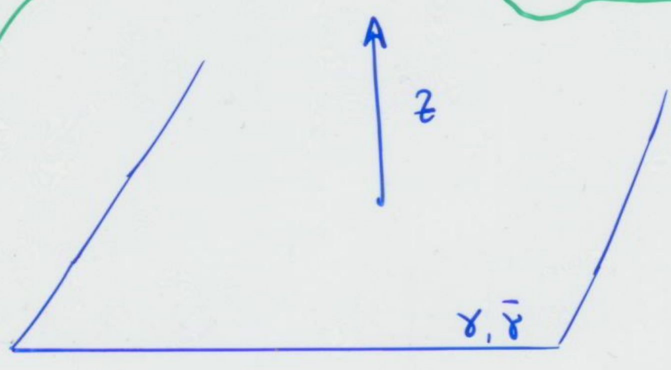
$$S[\hat{g}] = \frac{1}{\lambda^2} S_0[\hat{g}] + k\,\Gamma[\hat{g}]$$

σ-model term   WZ term

At $\lambda^2 = \frac{4\pi}{k}$ (vanishing RR-background)

$$S = S_*[\hat{g}] = S^{WZW}_{H_3}[g'] + S^{WZW}_{SU(2)}[g^\circ]$$

+ fermionic stuff



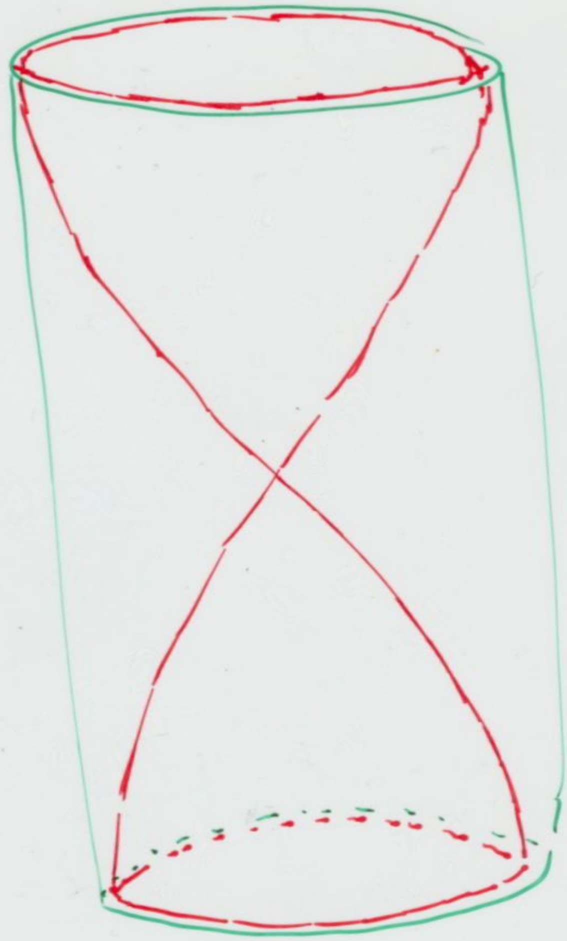
$z = e^{\varphi}$

$$ds^2 = \frac{1}{z^2}\left(dz^2 + d\gamma\, d\bar{\gamma}\right)$$

$$H = \frac{1}{2}\left(\Pi_\varphi^2 + (\varphi')^2\right) + \bar{\gamma}'\Pi_{\bar{\gamma}} - \gamma'\Pi_\gamma + e^{2\varphi}\Pi_\gamma \Pi_{\bar{\gamma}}$$

Term $e^{2\varphi}\,\overline{\Pi}_\gamma \overline{\Pi}_{\bar\gamma}$ ("wall") shields
$z \longrightarrow \infty$ (regularity in interior of AdS_3),
disappears near boundary

$\rightsquigarrow$ long strings : (cont. spectrum)



Away from the WZ-point $\lambda^2 = \frac{4\pi}{k}$

$$H = H^{WZW} + x\, H^6 ,$$

$$H^6 = e^{-2\varphi}\left((x_1')^2 + (x_2')^2 \right) + $$

$$\gamma = x_1 + i\, x_2$$

second "wall", shields boundary

Expectation: Long strings disappear
disc. spectrum

Simplify life, focus on radial motion:

$$\boxed{\begin{array}{c} H_3\text{-WZNW} \\ \ldots + \Pi_\gamma \Pi_{\bar\gamma}\, e^{2\varphi} \end{array}} \xrightarrow[\Pi_\gamma = \text{const.}]{} \boxed{\begin{array}{c} \text{Liouville} \\ \partial\varphi\bar\partial\varphi + \mu e^{2\varphi} \end{array}}$$

Remark: There ex. reconstruction

$$H_3\text{-WZNW} \longleftarrow \text{Liouville}$$

Put second "wall":

$$\mathcal{L}_{\text{Liou}} + \mu e^{-2\varphi} = \partial\varphi\bar\partial\varphi + 2\mu\cosh 2\varphi$$
$$\equiv \mathcal{L}_{\text{Sinh-Gordon}}$$

Remark: Liouville-th. controls UV-behavior of Sinh-Gordon!

Max. chiral symmetry	Integrable deformation
Liouville $\xleftarrow{\text{UV}}$	$\xrightarrow{\text{UV}}$ Sinh-Gordon
$\Leftarrow$ H_3-WZNW	
$\downarrow$	
$PSL_*(2\vert 2)_k$ WZNW $\xrightarrow{\text{marginal (??) def.}}$	$PSL(2\vert 2)_{\lambda,k}$-WZNW

Aim: Construct integrable lattice regularization of the Sinh-Gordon model:

$$\mathcal{T} = (\mathcal{H}, \mathsf{H}, \mathcal{Q}),$$

- Hilbert space $\mathcal{H}$
- Hamiltonian H,
- set $\mathcal{Q} = \{\mathsf{T}_0, \mathsf{T}_1, \dots\}$ of conserved charges such that

$$\begin{aligned}
&\text{(A)} \quad \mathsf{T}^\dagger = \mathsf{T} \quad \forall\, \mathsf{T} \in \mathcal{Q},\\
&\text{(B)} \quad [\mathsf{T}, \mathsf{T}'] = 0 \quad \forall\, \mathsf{T},\, \mathsf{T}' \in \mathcal{Q},\\
&\text{(C)} \quad [\mathsf{T}, \mathsf{H}] = 0 \quad \forall\, \mathsf{T} \in \mathcal{Q},\\
&\text{(D)} \quad \text{if } [\mathsf{T}, \mathsf{O}] = 0 \text{ for all } \mathsf{T} \in \mathcal{Q}, \text{ then } \mathsf{O} = \mathsf{O}(\mathcal{Q}).
\end{aligned}$$

Definition of $\mathcal{H}$:

Discretize Sinh-Gordon variables as

$$\Pi_n \to \Pi(x)\,\Delta\,, \quad \Phi_n \to \Phi(x)\,, \quad x = n\Delta\,.$$

Quantize:

$$[\,\Pi_n\,,\,\Phi_n\,] = \frac{1}{i}\delta_{n,m}\,.$$

$\Rightarrow$ Hilbert space $\mathcal{H} \equiv (L^2(\mathbb{R}))^{\otimes \mathrm{N}}$.

QISM, Step 1: Construction of $\mathcal{Q}$. Let

$$\mathsf{M}(u) \equiv \begin{pmatrix} \mathsf{A}(u) & \mathsf{B}(u) \\ \mathsf{C}(u) & \mathsf{D}(u) \end{pmatrix} \equiv L_N(u) \cdot \ldots \cdot L_2(u) \cdot L_1(u)\,,$$

where

$$L(u) \equiv \left(\begin{array}{cc} L_{11}(u) & L_{12}(u) \\ L_{21}(u) & L_{22}(u) \end{array} \right),$$

$$\begin{aligned}
L_{11}(u) &= e^{+\frac{\beta}{8}(\Pi_n+2s)} \left(1 + e^{-\beta(\Phi_n+s)}\right) e^{+\frac{\beta}{8}(\Pi_n+2s)} \\
L_{12}(u) &= \sinh\bigl(\pi b u + \frac{\beta}{2}\Phi_n\bigr) \\
L_{21}(u) &= \sinh\bigl(\pi b u - \frac{\beta}{2}\Phi_n\bigr) \\
L_{22}(u) &= e^{-\frac{\beta}{8}(\Pi_n-2s)} \left(1 + e^{+\beta(\Phi_n-s)}\right) e^{-\frac{\beta}{8}(\Pi_n-2s)}
\end{aligned}$$

M satisfies commutation relations of the form

$$R_{12}(u)\,\mathsf{M}_{13}(u+v)\,\mathsf{M}_{23}(v) = \mathsf{M}_{23}(v)\,\mathsf{M}_{13}(u+v)\,R_{12}(u)$$

Consider the one–parameter family of operators:

$$\mathsf{T}(u) \,=\, \mathrm{tr}\,\bigl(\mathsf{M}(u)\bigr) = \mathsf{A}(u) + \mathsf{D}(u)\,.$$

$\Rightarrow$ The operators T_m which appear in the expansion

$$\mathsf{T}(u) \,=\, e^{\pi b N u} \sum_{m=0}^{N} (-e^{-2\pi b u})^m\, \mathsf{T}_m,$$

are positive self–adjoint and commuting, $[\mathsf{T}_m, \mathsf{T}_n] = 0$.

QISM, Step 2: Construction of H

Definition 1. *Let*

$$\boxed{\begin{gathered}\mathsf{H} = \frac{i}{\pi b}\mathsf{U}^{-1}\left[\frac{\partial}{\partial u}\mathrm{tr}_a\left(\mathsf{R}_{aN}(u)\cdot\ldots\cdot\mathsf{R}_{a2}(u)\cdot\mathsf{R}_{a1}(u)\right)\right]_{u=0}\\ \mathsf{R}_{12}(u) = \mathsf{P}\,\frac{w_b(\mathsf{s}_{12}+u)}{w_b(\mathsf{s}_{12}-u)}\end{gathered}}$$

where

- $\cancel{\mathsf{s}_{12}}$ $4\cosh^2 \pi b\, s_{12}$ *is the Casimir of* $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ *in* $\mathcal{P}_s \otimes \mathcal{P}_s$,
- $w_b(x)$ *is the* *) *quantum dilogarithm,*
- U *is the cyclic shift operator,*
- P *is the permutation operator.*

*) non-compact (self-dual $b \to b^{-1}$)

Theorem 1. [BT]

(a) The Hamiltonian is local:

$$\mathsf{H} = \sum_{n=1}^{N} H_{n,n+1}\,.$$

(b) H *commutes with the conserved charges* T_k*:*

$$[\,\mathsf{H}\,,\,\mathsf{T}_k\,] = 0, \quad \text{for } k = 0,\ldots,N.$$

Classical continuum limit

Classical limit:

$$
\begin{aligned}
&\tfrac{\mu}{4}e^{\gamma H^{\rm cl}_{n,n+1}} = \\
&\quad = \tfrac{1}{2}\cosh\tfrac{\beta}{4}\left(\Pi_n+\Pi_{n+1}\right)+\tfrac{1+\mu^2}{2}\cosh\tfrac{\beta}{2}\left(\Phi_n-\Phi_{n+1}\right)\\
&\qquad +\tfrac{\mu}{2}\cosh\beta\big(\Phi_n-\tfrac{1}{4}(\Pi_n+\Pi_{n+1})\big)\\
&\qquad +\tfrac{\mu}{2}\cosh\beta\big(\Phi_{n+1}-\tfrac{1}{4}(\Pi_n+\Pi_{n+1})\big)\\
&\qquad +\mu\,\cosh\tfrac{\beta}{2}(\Phi_n+\Phi_{n+1})\\
&\qquad +\tfrac{\mu^2}{4}\cosh\beta\big(\Phi_n+\Phi_{n+1}-\tfrac{1}{4}(\Pi_n+\Pi_{n+1})\big)\,,
\end{aligned}
$$

Continuum limit:

$$
\left\{\begin{array}{l} N\to\infty\\ \Delta\to 0\\ s\to\infty\end{array}\right\}\quad\text{with}\quad\left\{\begin{array}{l} R=N\Delta/2\pi\\ m=\frac{4}{\Delta}e^{-\pi b s}\end{array}\right\}\quad\text{fixed.}
$$

$$
\Pi_n\to\Pi(x)\,\Delta\,,\qquad\Phi_n\to\Phi(x)\,,\qquad x=n\Delta\,.
$$

$$
\begin{aligned}
&\sum_n\frac{1}{\Delta}H^{\mathsf{G},\rm cl}_{n,n+1}\to\\
&\quad\to\int_0^{2\pi R}dx\,\left(\tfrac{1}{2}\Pi^2+\tfrac{1}{2}(\partial_x\Phi)^2+\tfrac{m^2}{\beta^2}\cosh\beta\Phi\right)+\text{const}
\end{aligned}
$$

Good news:

We have succeeded in constructing a integrable lattice regularization of the Sinh-Gordon model.

Count of DOF $\Rightarrow$

Expect that spectral problem for H is equivalent to:

> **Auxiliary Spectral Problem:** Find spectrum of $\mathsf{T}(u)$
> $\Leftrightarrow$ joint spectral decomposition for $\mathcal{Q} = \{\mathsf{T}_0, \ldots, \mathsf{T}_{\mathrm{N}}\}$.

Bad news:

$$\boxed{\text{Algebraic Bethe ansatz fails!}}$$

Failure of algebraic Bethe ansatz

1^st^ attempt:

$$\mathcal{L}^n_{21}(u)\,\Psi_0 \overset{!}{=} 0 \qquad \mathcal{L}^n_{21} = \mathrm{sh}\left(\pi b u - \tfrac{\beta}{2}\Phi_n\right)$$

$$\Rightarrow \quad e^{\pm\frac{\beta}{2}\phi_n}\,\Psi_0 = 0$$

Relax, this has happened for Sine-G.

Consider $\mathfrak{L}^n_{21}(u) \equiv \mathcal{L}^n_{21}(u)\,\mathcal{L}^{n+1}_{21}(u)$

$$-i\,\mathfrak{L}^n_{21} = e^{\pi b u}\big(\underbrace{k\otimes e + f\otimes k}_{\to 0}\big) - e^{-\pi b u}\big(\underbrace{k^{-1}\otimes e + f\otimes k^{-1}}_{\to 0}\big)$$

On Ψ_0:

$$e_n = e^{\pi b x_n}\,\mathrm{ch}\,\pi b(p_n - s)\,e^{\pi b x_n} \qquad k_n = e^{-\pi b p_n}$$

$$f_n = e^{-\pi b x_n}\,\mathrm{ch}\,\pi b(p_n + s)\,e^{-\pi b x_n}$$

$$2\pi b p_n = -\beta\phi_n \qquad 4\pi b x_n = \beta\left(\tfrac{1}{2}\pi_n - \phi_n\right)$$

Note that $k\otimes e + f\otimes k$

$$= (\Omega\otimes 1)\cdot(k^{-1}\otimes e + e\otimes k)\cdot(\Omega\otimes 1)$$

$$= (\Omega\otimes 1)\cdot\Delta(e)\cdot(\Omega\otimes 1)$$

Furthermore: $\Delta(e) = C^{\dagger}_{21}\cdot e^{2\pi b x}\cdot C_{21}$

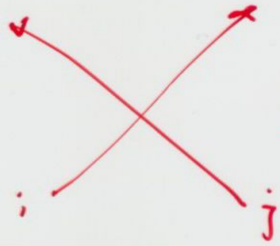
Clebsch-Gordan maps (PT)

$\Longrightarrow \quad \Psi \underset{\text{equiv.}}{\overset{\text{unit.}}{\sim}}$ eigenvect. of $e^{2\pi b x}$ with eigenval. 0

$$\boxed{\text{Reason: Non-compactness!}}$$

The factorized scattering-th. is known

1 massive particle



$S(\Theta_i - \Theta_j)$

$$S(\Theta) = \frac{\mathrm{sh}\,\Theta - i \sin \pi p}{\mathrm{sh}\,\Theta + i \sin \pi p} \qquad p = \frac{\beta^2}{1+\beta^2}$$

For "well-separated" particles

$$(*) \quad \Psi(x_1, \dots, x_N) = e^{i \sum_j p_j x_j} \sum_{\pi \in S_N} A_\pi \, \Theta(x_\pi)$$

- $A_\pi = A_{\pi'} \, S(\Theta_i - \Theta_j)$ iff π, π' differ by (ij)
- $\Theta(x_\pi) = \begin{cases} 1 & \text{if } x_{\pi(1)} < \dots < x_{\pi(N)} \\ 0 & \text{otherwise} \end{cases}$

Periodic b.c.:

$$e^{i L m_j \mathrm{sh}\,\Theta_j} \prod_{k \neq j} S(\Theta_j - \Theta_k) = 1 \qquad (*)$$

However: $(*)$ is approximation, valid only if distances $\gg \frac{1}{m}$!

$*$ works if $L \to \infty$ (TBA)

Q–operator and Baxter equation

Theorem 2. *There exists a family of operators* $\mathsf{Q}(u)$ *with the properties*

$$\left[\begin{array}{ll} \text{(a)} & \mathsf{Q}(u) \text{ is normal, } \ \mathsf{Q}(u)\mathsf{Q}^*(v) = \mathsf{Q}^*(v)\mathsf{Q}(u), \\ \text{(b)} & \mathsf{Q}(u)\,\mathsf{Q}(v) = \mathsf{Q}(v)\,\mathsf{Q}(u)\,, \\ \text{(c)} & \mathsf{Q}(u)\,\mathsf{T}(u) = \mathsf{T}(u)\,\mathsf{Q}(u)\,, \\ \text{(d)} & \mathsf{Q}(u)\mathsf{T}(u) = \big(a(u)\big)^{\mathrm{N}}\mathsf{Q}(u-ib) + \big(d(u)\big)^{\mathrm{N}}\mathsf{Q}(u+ib), \end{array}\right]$$

$$Q_u(\mathbf{x},\mathbf{x}') =$$

$$= \big(D_{-s}(u)\big)^{\mathrm{N}} \prod_{r=1}^{\mathrm{N}} D_{\frac{1}{2}(\bar{\sigma}-u)}(x_r - x_r')\, D_{\frac{1}{2}(\bar{\sigma}+u)}(x_{r-1} - x_r') \times D_{-s}(x_r - x_{r-1})\,,$$

Existence of Q–operator + Separation of variables

⇒ Refinement of Auxiliary Spectral Problem:

$$\mathsf{T}(u)\,\Psi_t = t(u)\,\Psi_t\,,$$
$$\mathsf{Q}(u)\,\Psi_t = q_t(u)\,\Psi_t\,.$$

Theorem 3.

$$\left[\begin{array}{ll} \text{(i)} & t(u)q_t(u) = \big(a(u)\big)^{\mathrm{N}} q_t(u-ib) + \big(d(u)\big)^{\mathrm{N}} q_t(u+ib), \\ \text{(ii)} & q_t(u) \ \textit{is meromorphic with poles in } \Upsilon_{-s} \cup \bar{\Upsilon}_s, \\ & \textit{where } \Upsilon_s = \big\{ s + i\big(\tfrac{Q}{2} + nb + mb^{-1}\big),\ n,m \in \mathbb{Z}^{\geq 0} \big\}, \\ \text{(iii)} & q_t(u) \sim \begin{cases} e^{+\pi i \mathrm{N}(s+\frac{i}{2}Q)u} \text{ for } |u| \to \infty,\ |\arg(u)| < \frac{\pi}{2}, \\ e^{-\pi i \mathrm{N}(s+\frac{i}{2}Q)u} \text{ for } |u| \to \infty,\ |\arg(u)| > \frac{\pi}{2}. \end{cases} \end{array}\right]$$

Characteristics (i)-(iii) $\Leftrightarrow$ Bethe ansatz:

$$Q_t(u) := \big(\Gamma_b(\tfrac{Q}{2} - i(u+s))\Gamma_b(\tfrac{Q}{2} - i(u-s))\big)^{-\mathrm{N}} q_t(u),$$

$$\left[\begin{array}{l} \text{(a) } Q_t(u) \ \text{ is entire analytic of order 2 in } \mathbb{C}, \\ \text{(b) } t(u)\, Q_t(u) = \big(A(u)\big)^{\mathrm{N}} Q_t(u-ib) + \big(D(u)\big)^{\mathrm{N}} Q_t(u+ib), \end{array}\right]$$

(a) $\Rightarrow$ $Q_t(u)$ is uniquely characterized by its zeros:

$$Q_t(u) = e^{r(u)} \prod_{k=1}^{\infty}{}' \left(1 - \frac{u}{u_k}\right),$$

(b) $\Rightarrow$ equations of Bethe ansatz type for the zeros:

$$-1 = \frac{(A(u_k))^{\mathrm{N}}}{(D(u_k))^{\mathrm{N}}} \prod_{l=1}^{\infty} \frac{u_l - u_k - ib}{u_l - u_k + ib}, \qquad k \in \mathbb{N}.$$

Main result:

Conditions which characterize the spectrum of the **continuum** sinh–Gordon model:

$$\left[\begin{array}{ll} \text{(i)} & q_t(u) \text{ is entire analytic,} \\ \text{(ii)} & q_t(u) \text{ decays rapidly for } |\mathrm{Re}(u)| \to \infty, \;\; u \in \mathcal{S}, \\ \text{(iii)} & q_t(u) \text{ satisfies a difference equation of the form} \\ & \qquad t(u)\, q_t(u) \;=\; q_t(u - ib) + q_t(u + ib), \\ & \text{where } t(u) \text{ is periodic under } u \to u + ib^{-1}, \\ \text{(iv)} & q_t(u) \text{ satisfies the following relation} \\ & q_t(u+\delta_+)q_t(u-\delta_+) - q_t(u+\delta_-)q_t(u-\delta_-) = 1. \end{array}\right]$$

Continuum limit:

Easy in this representation

Conjecture:

Function $q_0(u)$ which characterizes the ground state satisfies conds. (i)-(iv) above **and**

$$\text{(v)} \qquad q_0(u) \text{ is nonvanishing within } \mathcal{S}\,.$$

Reformulation:

Conditions (i)-(v) $\Leftrightarrow$ $q_0(u)$ is given as

$$\log q_0(u) = -\frac{mR}{2}\frac{\cosh\frac{\pi}{Q}u}{\sin\frac{\pi}{Q}b} + \int_{\mathbb{R}} \frac{dv}{2Q}\,\frac{\log(1+Y(v))}{\cosh\frac{\pi}{Q}(u-v)},$$

which expresses $q_0(u)$ in terms of the solution $Y(u)$ to the nonlinear integral equation

$$\log Y(u) = -mR\cosh\tfrac{\pi}{Q}u + \int_{\mathbb{R}} \frac{dv}{2Q}\,S(u-v)\log(1+Y(v)),$$

where the kernel $S(u-v)$ is explicitly given as follows:

$$S(u) = \frac{2\,\sin\frac{\pi}{Q}b\,\cosh\frac{\pi}{Q}u}{\sinh\frac{\pi}{Q}(u+ib)\,\sinh\frac{\pi}{Q}(u-ib)}\,.$$

Matches results from TBA for the Sinh-Gordon model!

What has been found:

- First decent integrable lattice regularization of the Sinh-Gordon model.

- Algebraic Bethe ansatz fails. *)

- Separation of variables works — logical structure clarified
  (Q-operator, Baxter eqn.)

- Quantization conditions* found.
)* Baxter + Analyticity + Asymptotics

- Continuum limit: Easy.

What remains to be done:

A lot.

Integrable models with
- noncompact target
- exponential interactions $e^{\pm\beta\phi}$

$\equiv$ new class, different from Sine-G....

Prediction: Bethe ansatz will fail for this class
($PSL(1,1|2)$, $PSL(2,2|4)$ ..)