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Scaling functions for AdS/CFT

Lisa Freyhult

Based on
arXiv:0901.2749 [hep-th] and arXiv:0905.3536 [hep-th] with Stefan Zieme, Martin Kruczenski and Alin Tirziu

Integrability in gauge and string theory, Golm 2009-07-03

Overview

- Twist operators

$$\mathcal{O} = \mathrm{Tr}\left(D^{s_1} Z D^{s_2} Z \ldots D^{s_n} Z\right)$$

  L =length=twist
  $S = s_1 + s_2 + \cdots + s_n$ =spin
- Study the spectrum of the anomalous dimensions of these operators in the large S limit.
- The anomalous dimensions lies in the band [Belitsky, Gorsky, Korchemsky]

$$f(\lambda) \log S \leq \gamma(\lambda) \leq \frac{L}{2} f(\lambda) \log S$$

1. The ground state
   - Clues for amplitudes
   - Provides functions interpolating between strong and weak coupling ideal for studying AdS/CFT
2. The highest excited state
   - Spiky strings
   - Interpolating functions

Motivation

Two approaches to solving $\mathcal{N} = 4$ SYM

- Spectrum of anomalous dimensions of gauge invariant operators
- Scattering amplitudes

Motivation

BDS conjecture for all-loop n-point on-shell MHV gluon amplitudes in $4-2\epsilon$ dimensions

[Bern, Dixon, Smirnov]

$$\log \frac{\mathcal{A}_n}{\mathcal{A}_n^{tree}} = Div_n + \frac{f(\lambda)}{4} F_n^{(1)}(0) + nk(\lambda) + C(\lambda)$$

$$Div_n = -\sum_{i=1}^{n} \left(\frac{1}{8\epsilon^2} f^{(-2)} \left(\frac{\lambda \mu^{2\epsilon}}{(-s_{i,i+1})^{\epsilon}} \right) + \frac{1}{4\epsilon} g^{(-1)} \left(\frac{\lambda \mu^{2\epsilon}}{(-s_{i,i+1})^{\epsilon}} \right) \right)$$

$$f(\lambda) = \left(\lambda \frac{d}{d\lambda} \right)^2 f^{(-2)}(\lambda) \qquad g(\lambda) = \lambda \frac{d}{d\lambda} g^{(-1)}(\lambda)$$

BDS conjecture does not hold for $n \geq 6$ beyond one-loop. [Alday, Maldacena] [Drummond, Henn, Korchemsky, Sokatchev] [Bartels, Lipatov, Sabio Vera] [Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich] ...

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Twist operators and scaling functions

- $f(\lambda)$ is given by the anomalous dimension of local operators in the $sl(2)$ sector of $N = 4$ SYM

[Sterman, Tejeda-Yeomans] [Belitsky, Gorsky, Korchemsky] [Eden, Staudacher] [Beisert, Eden, Staudacher]

$$\mathcal{O} = \text{Tr}\left(D^{s_1} Z D^{s_2} Z \ldots D^{s_n} Z\right)$$

Scaling limit

$S \to \infty$ logarithmic Sudakov scaling [Belitsky, Gorsky, Korchemsky]

$$\gamma_L(g, S) = f(g) \log S + \ldots$$

$$g^2 = \frac{\lambda}{16\pi^2}$$

Twist operators and scaling functions

- $g(\lambda)$ is the collinear anomalous dimension, operator interpretation?

$$g(\lambda) = G_{eik}(\lambda) + B(\lambda)$$

[Dixon, Magnea, Sterman]

Scaling limit

Obtain $B(\lambda)$ by considering subleading corrections in the large S expansion of the anomalous dimension of twist (two) operators.

$$\gamma_L(g, S) = f(g)\left(\log S + \gamma_E - (L-2)\log 2\right) + B_L(g) + \ldots$$

- $B_L(g)$ is what we call the virtual scaling function
- at strong coupling there is a prediction from string theory

[Beccaria, Forini, Tirziu, Tseytlin]

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Twist operators and scaling functions

- Obtain these quantities from the asymptotic Bethe ansatz

[Minahan, Zarembo] [Beisert, Kristjansen, Staudacher] [Beisert, Dippel, Staudacher] [Arutyunov, Frolov, Staudacher] [Beisert] [Staudacher] [Beisert, Staudacher] [Janik] [Beisert, Eden, Staudacher] [Martins, Melo] ...

All loop equations for $sl(2)$

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}^{S} \frac{u_k - u_j - i}{u_k - u_j + i} \left(\frac{1 - \frac{g^2}{x_k^+ x_j^-}}{1 - \frac{g^2}{x_k^- x_j^+}}\right)^2 e^{2i\theta(u_k, u_j)}$$

Anomalous dimension

$$\gamma(g) = 2g^2 \sum_{k=1}^{S} \left(\frac{i}{x_k^+} - \frac{i}{x_k^-}\right)$$

$$x(u) = \frac{u}{2}\left(1 + \sqrt{1 - \frac{4g^2}{u^2}}\right) \qquad x_\pm = x(u \pm i/2).$$

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Twist operators and scaling functions

Wrapping problem, equations valid to $\mathcal{O}(g^{2L+2})$. [Staudacher] [Beisert, Staudacher]

Bethe equations are asymptotic, suggested extension TBA, Y-system. [Arutyunov, Frolov] [Gromov, Kazakov, Vieira] [Gromov, Kazakov, Kozak, Vieira] [Bombardelli, Fioravanti, Tateo]

Large spin expansion of twist operators

$$\gamma(g, L) = f(g)\left(\log S + \gamma_E - (L-2)\log 2\right) + B_L(g) + \ldots$$

- f(g) is universal, it does not depend on L! [Kotikov, Lipatov, Onishenko, Velizhanin]
- conjecture: not affected by wrapping [Eden, Staudacher]
- First wrapping correction was computed for $L = 2$ and arbitrary spin S, the result is consistent with the conjecture. [Bajnok, Janik, Lukowski] [Velizhanin]
- Strong coupling expansion agrees with the prediction from string theory [Basso, Korchemsky] [Gubser, Klebanov, Polyakov] [Frolov, Tseytlin] [Roiban, Tirziu, Tseytlin] [Roiban, Tseytlin]

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- ⋆ Compute $B_L(g)$ at strong coupling (by resumming all orders in the weak coupling expansion) and compare with the prediction from string theory.
  (Bonus: Another check of AdS/CFT and the dressing phase.)

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Integral equations

- Take the large S limit in the all loop Bethe equations
- the result is an integral equation [Eden, Staudacher] [Beisert, Eden, Staudacher] [L.F, Rej, Staudacher] [Bombardelli, Fioravanti, Rossi]

Integral equation with S^0 corrections

$$\hat{\sigma}(t) = \frac{t}{e^t - 1}\Big[K(2gt,0)\left(\log S + \gamma_E - (L-2)\log 2\right) - \frac{L}{8g^2 t}\left(J_0(2gt) - 1\right) + \frac{1}{2}\int_0^\infty dt' \left(\frac{2}{e^{t'}-1} - \frac{L-2}{e^{t'/2}+1}\right)\left(K(2gt,2gt') - K(2gt,0)\right) - 4g^2\int_0^\infty dt' K(2gt,2gt')\hat{\sigma}(t')\Big]$$

The anomalous dimension

$$\gamma(g) = 16g^2\hat{\sigma}(0)$$

Integral equations

The kernel is given by

$$K(t,t') = K_0(t,t') + K_1(t,t') + K_d(t,t')$$

where

$$K_0(t,t') = \frac{2}{tt'} \sum_{n=1}^{\infty} (2n-1) J_{2n-1}(t) J_{2n-1}(t')$$

$$K_1(t,t') = \frac{2}{tt'} \sum_{n=1}^{\infty} 2n\, J_{2n}(t) J_{2n}(t')$$

$$K_d(t,t') = 8g^2 \int_0^{\infty} dt''\, K_1(t, 2gt'') \frac{t''}{e^{t''}-1} K_0(2gt'', t')$$

Integral equation, weak coupling expansion

- The BES equation gives

$$f(g) = 8g^2 - \frac{8\pi^2}{3}g^4 + \frac{88\pi^4}{45}g^6 - \left(\frac{584\pi^6}{315} + 64\zeta(3)^2\right)g^8 + \ldots$$

- Agrees with 4- and 5-loop results [Bern, Dixon, Smirnov] [Cachazo, Spradlin, Volovich] [Bern, Czakon, Dixon, Kosower, Smirnov]
- The virtual scaling function

$$B_L(g) = -8(7-2L)\zeta(3)g^4 + 8\left(\frac{4-L}{3}\pi^2\zeta(3) + (62-21L)\zeta(5)\right)g^6 + \ldots$$

- agrees with the maximal transcendentality part of corresponding QCD result, obtained from the virtual part of the splitting function [Moch, Vermaseren, Vogt]

Integral equation, strong coupling expansion

Strong coupling in the BES equation. Much time and effort to find the expansion [Kotikov, Lipatov] [Benna, Benvenuti, Klebanov, Scardicchio] [Alday, Arutyunov, Benna, Eden, Klebanov] [Kostov, Serban, Volin] [Basso, Korchemsky, Kotanski] [Basso, Korchemsky] ...

Decomposing the density in even/odd parts

$$\frac{e^t-1}{t}\hat{\sigma}(t)=\frac{\gamma_+(2gt)}{2gt}+\frac{\gamma_-(2gt)}{2gt}$$

and expanding in Neumann series

$$\gamma_+(t)=2\sum_{n=1}^{\infty}(2n)J_{2n}(t)\gamma_{2n}$$

$$\gamma_-(t)=2\sum_{n=1}^{\infty}(2n-1)J_{2n-1}(t)\gamma_{2n-1}$$

leads to the set of equations [Eden] [Basso, Korchemsky, Kotanski]

$$\int_0^{\infty}\frac{dt}{t}\left(\frac{\gamma_+(t)}{1-e^{-t/2g}}-\frac{\gamma_-(t)}{e^{t/2g}-1}\right)J_{2n}(t)=0$$

$$\int_0^{\infty}\frac{dt}{t}\left(\frac{\gamma_-(t)}{1-e^{-t/2g}}+\frac{\gamma_+(t)}{e^{t/2g}-1}\right)J_{2n-1}(t)=0$$

Integral equation, strong coupling expansion

change of variables

$$2\gamma_{\pm}(t) = \left(1 - \operatorname{sech}\left(\frac{t}{2g}\right)\right)\Gamma_{\pm}(t) \pm \tanh\left(\frac{t}{2g}\right)\Gamma_{\mp}(t)$$

$$\int_0^\infty \frac{dt}{t}\left(\Gamma_+(t) + \Gamma_-(t)\right) J_{2n}(t) = 0$$

$$\int_0^\infty \frac{dt}{t}\left(\Gamma_-(t) - \Gamma_+(t)\right) J_{2n-1}(t) = 0$$

all coupling dependence is in the functions $\Gamma_{\pm}(t)$

$$\Gamma_+(t) = \sum_{k=0}^{\infty}(-1)^{k+1} J_{2k}(t)\Gamma_{2k} \quad \Gamma_-(t) = \sum_{k=0}^{\infty}(-1)^{k+1} J_{2k-1}(t)\Gamma_{2k-1}$$

The solution

$$\Gamma_k = -\frac{1}{2}\Gamma_k^{(0)} + \sum_{p=1}^{\infty}\frac{1}{g^p}\left(c_p^- \Gamma_k^{(2p-1)} + c_p^+ \Gamma_k^{(2p)}\right)$$

$$\Gamma_{2m}^{(p)} = \frac{\Gamma(m+p-1/2)}{\Gamma(m+1)\Gamma(1/2)} \qquad \Gamma_{2m-1}^{(p)} = \frac{(-1)^p\Gamma(m-1/2)}{\Gamma(m+1-p)\Gamma(1/2)}$$

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Integral equation, strong coupling expansion

The solution

$c_p^{\pm}$ determined by requiring the correct analyticity properties of $\gamma_{\pm}(t)$, quantization conditions for $c_p^{\pm}$

Scaling function at strong coupling

$$f(g) = 4g - \frac{3\log 2}{\pi} - \frac{1}{g}\frac{K}{4\pi^2} - \frac{1}{g^2}\left(\frac{3K\log 2}{16\pi^3} + \frac{27\zeta(3)}{512\pi^2}\right) + \dots$$

- Matches the string result (first three orders in large g expansion) [Gubser Klebanov, Polyakov] [Frolov, Tseytlin] [Roiban, Tirziu, Tseytlin] [Roiban, Tseytlin]
- Strong coupling expansion is asymptotic
- Nonperturbative contributions has to be included for resummation and continuation to weak coupling to be possible [Basso, Korchemsky] [Alday, Maldacena]

$$\delta\gamma(g) = -\frac{\sigma}{2\sqrt{2}} m^2_{O(6)} + \mathcal{O}(m^4_{O(6)}) \quad m_{O(6)} \sim g^{1/4} e^{-\pi g}$$

Integral equation, strong coupling expansion

- Strong coupling for the virtual scaling function?
- Try to make use of the known solution to the BES equation.
- Integral equations are similar, in particular the kernels are the same.

Drop the part proportional to $K(2gt, 0)$, that is the BES equation rescaled.

Decompose $\hat{\sigma}(t)$

$$\frac{e^t - 1}{t}\hat{\sigma}(t) = \frac{\tilde{\gamma}_+(2gt)}{2gt} + \frac{\tilde{\gamma}_-(2gt)}{2gt}$$

$$\int_0^\infty \frac{dt}{t}\left(\frac{\tilde{\gamma}_+(t)}{1-e^{-t/2g}} - \frac{\tilde{\gamma}_-(t)}{e^{t/2g}-1}\right) J_{2n}(t) = \frac{L}{8ng} + h_{2n}$$

$$\int_0^\infty \frac{dt}{t}\left(\frac{\tilde{\gamma}_-(t)}{1-e^{-t/2g}} + \frac{\tilde{\gamma}_+(t)}{e^{t/2g}-1}\right) J_{2n-1}(t) = h_{2n-1}$$

$$h_n = \frac{1}{4}\int_0^\infty dt \left(\frac{2}{e^t - 1} - \frac{L-2}{e^{t/2}+1}\right)\left(\frac{J_n(2gt)}{gt} - \delta_{n,1}\right)$$

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Integral equation, strong coupling expansion

Introduce a dummy index j.

$$\int_0^\infty \frac{dt}{t}\left(\frac{\tilde{\gamma}_+(t)}{1-e^{-t/2g}} - \frac{\tilde{\gamma}_-(t)}{e^{t/2g}-1}\right) J_{2n}(t) = j\,\frac{L}{8ng} + j\,h_{2n}$$

$$\int_0^\infty \frac{dt}{t}\left(\frac{\tilde{\gamma}_-(t)}{1-e^{-t/2g}} + \frac{\tilde{\gamma}_+(t)}{e^{t/2g}-1}\right) J_{2n-1}(t) = \frac{1-j}{2}\delta_{n,1} + j\,h_{2n-1}$$

- $j = 0$ gives the BES equation
- $j = 1$ gives the equations for the virtual scaling function
- we want to "parametrize" the difference to BES since we hope to reuse that solution

Integral equation, strong coupling expansion

Trick:

- multiply the equations by $2n\gamma_{2n}(g,j')$ and $(2n-1)\gamma_{2n-1}(g,j')$ respectively and sum over all $n \geq 1$
- subtract the resulting equations
- integral kernel invariant under $j \leftrightarrow j'$
- set $j' = 0$ and $j = 1$

$$\gamma_1(g,1) = \frac{1}{4}\int_0^\infty dt \left(\frac{2}{e^t - 1} - \frac{L-2}{e^{t/2}+1}\right) \times \left(\frac{\gamma_-(2gt,0) - \gamma_+(2gt,0)}{gt} - 2\gamma_1(g,0)\right) - \frac{L}{2g}\sum_{n=1}^\infty \gamma_{2n}(g,0)$$

The virtual scaling function is given by

$$B_L(g) = 16g^2\gamma_1(g,1)$$

Integral equation, strong coupling expansion

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The virtual scaling function is given by

$$B_L(g) = 16g^2\gamma_1(g,1)$$

The virtual scaling function is completely determined in terms of the solution to the BES equation!

Trick: rewrite the solution to this new equation in terms of the solution to the BES equation

$$B_L(g) = 4g^2 \int_0^\infty dt \left[\frac{2}{e^t - 1} - \frac{L-2}{e^{t/2}+1}\right] \times \left[\frac{\gamma_-(2gt) - \gamma_+(2gt)}{gt} - 2\gamma_1(g)\right] - 4gL \int_0^\infty \frac{dt}{t}\gamma_+(2gt).$$

Using the solution of Basso, Korchemsky and Kotanski

$$B_L(g) = \frac{1}{16g^2}(L-2)\left(\epsilon_1(g) + f(g)\log 2\right) + (-\gamma_E - \log g)f(g) - 4g(1-\log 2) - \left(1 - \frac{6\log 2}{\pi} + \frac{3(\log 2)^2}{\pi}\right) + \mathcal{O}(1/g)$$

- $\epsilon_1(g) = -1 + \mathcal{O}(e^{-\pi g})$, a quantity analyzed in the context of the generalized scaling function and the O(6) model [Basso, Korchemsky] [Fioravanti, Grinza, Rossi] [L.F., Rej, Staudacher]
- $f(g)$ is the universal scaling function

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$$\begin{aligned} B_L(g) = & \frac{1}{16g^2}(L-2)\left(\epsilon_1(g) + f(g)\log 2\right) \\ & + (-\gamma_E - \log g) f(g) - 4g(1-\log 2) \\ & - \left(1 - \frac{6\log 2}{\pi} + \frac{3(\log 2)^2}{\pi}\right) + \mathcal{O}(1/g) \end{aligned}$$

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- $f(g)$ is the universal scaling function

Comparison with string theory

This can be compared with the result from string theory ($L = 2$), we predict

$$E - S = L + \gamma_L\left(\frac{\sqrt{\lambda}}{4\pi}, S\right)\Big|_{L=2} = \left(\frac{\sqrt{\lambda}}{\pi} - \frac{3\log 2}{\pi}\right)\log\frac{4\pi S}{\sqrt{\lambda}}$$

$$+ \frac{\sqrt{\lambda}}{\pi}(\log 2 - 1) + 1 + \frac{6\log 2}{\pi} - \frac{3(\log 2)^2}{\pi}$$

in agreement with the result of Beccaria, Forini, Tirziu and Tseytlin, the string result contains an arbitrary constant c which we determine to be

$$c = 6\log 2 + \pi$$

In agreement with the result obtained by the algebraic curve approach. [Gromov]
Also agrees with the numerical results. [Fioravanti, Grinza, Rossi]

More

- More orders in the strong coupling expansion

$$B_2(g+c_1) = (\log \tfrac{2}{g} - \gamma_{\mathrm{E}}) f(g+c_1) - 4g - 1 + \frac{1}{g}\frac{\mathrm{K}}{2\pi^2} - \frac{1}{g^2}\frac{9\zeta(3)}{2^8\pi^3} + \frac{1}{g^3}\Big(\frac{9\,\beta(4)}{2^7\pi^4} - \frac{\mathrm{K}^2}{2^7\pi^4}\Big) - \frac{1}{g^4}\Big(\frac{6831\,\zeta(5)}{2^{18}\pi^5} - \frac{423\,\mathrm{K}\,\zeta(3)}{2^{13}\pi^5}\Big) + \mathcal{O}(1/g^5),$$

where $c_1 = \frac{3\log 2}{4\pi}$.

- More orders in the large spin expansion

$$\gamma_{L=2}(g,S) = f(g)\Big(\log S + \gamma_{\mathrm{E}} + \frac{f(g)}{2}\frac{\log S + \gamma_{\mathrm{E}}}{S} + \frac{1+B_2(g)}{2\,S}\Big) + B_2(g) + \dots,$$

  - in agreement with string theory [Beccaria, Forini, Tirziu, Tseytlin].
  - Next order $\mathcal{O}((logS/S)^2)$ will be affected by wrapping [Bajnok, Janik, Lukovski] [Beccaria, Forini]

Summary so far

- Provided an integral equation for the virtual scaling function

$$\gamma_L(g,S) = f(g)\,(\log S + \gamma_E - (L-2)\log 2) + B_L(g)\ldots$$

- demonstrated its expansion at weak and strong coupling
- Found complete agreement with the prediction from string theory, no wrapping problem at this order.
- provides some further clues for amplitudes

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The upper part of the spectrum

- Treat the highest excited state similarly to the lowest.
- Find the duals of the spiky strings explicitly from the Bethe ansatz.
- Construct an integral equation valid for all values of the coupling constant.
- Leading anomalous dimension

$$\gamma(\lambda) = \frac{L}{2} f(\lambda) \log S$$

[Belitsky, Gorsky, Korchemsky], [Belitsky, Korchemsky, Pasechnik], [Kazakov, Zarembo], [Dorey], [Dorey, Losi]

- subleading corrections $\mathcal{O}(S^0)$, string theory prediction for the first term at strong coupling [Beccaria, Forini,Tirziu, Tseytlin]
- For a general state the subleading corrections are parametrized by the integers $l_1, l_2, \ldots, l_{n-1}$ satisfying $l_1 + l_2 + \cdots + l_{n-1} = S$.
  Leading contribution is $\gamma(\lambda) = \frac{n}{2} f(\lambda) \log S$.

One loop from the Bethe ansatz

$sl(2)$ Bethe equations

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{j\neq k}^{S} \frac{u_k - u_j - i}{u_k - u_j + i}$$

momentum condition

$$\prod_{j=1}^{S} \frac{u_k + i/2}{u_k - i/2} = 1$$

- $$\frac{L}{u_k} = 2\pi n_k - 2\sum_{j\neq k} \frac{1}{u_k - u_j}$$
- Continuum approximation with density $\rho_0(u) = \sum_k \delta(u - u_k)$

Leading order equations

$$⨍ du \frac{\rho_0(u)}{u' - u} = \pi n(u'), \quad ⨍ du \frac{\rho_0(u)}{u} = 0, \quad \int \rho_0(u) = S$$

Clues from the string dual

What are the mode numbers corresponding to the spiky string? Get clues from the spiky string in flat space, $ds^2 = -dt^2 + dx^2 + dy^2$. String world sheet parametrized by

$$\sigma_+ = \tau + \sigma$$
$$\sigma_- = \tau - \sigma$$

Solutions are obtained by superpositions of right and left moving waves

$$x = A\cos((n-1)\sigma_+) + A(n-1)\cos(\sigma_-)$$
$$y = A\sin((n-1)\sigma_+) + A(n-1)\sin(\sigma_-)$$
$$t = 2A(n-1)\tau = A(n-1)(\sigma_+ + \sigma_-)$$

[Kruczenski]

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Scaling functions for AdS/CFT

Lisa Freyhult

Clues from the string dual

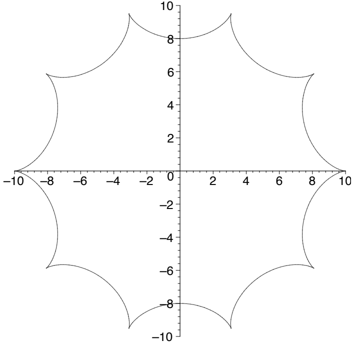

Figure: Rotating string with 10 spikes in flat space. We plot (x, y) parametrically as a function of $\sigma = 0 \rightarrow 2\pi$ for $n = 10$, $A = 1$ and $\tau = 0$.

- $n_R = A^2(n-1)^2$ right moving excitations with wave number $k_R = 1$
- $n_L = A^2(n-1)$ left moving excitations with wave number $k_L = n - 1$.

One loop from the Bethe ansatz

- Distribution of Bethe roots with mode number

$$n(u) = \begin{cases} -1, & d < u < 0 \\ n-1, & 0 < u < a \end{cases}$$

- real roots
- one cut [Belitsky, Gorsky, Korchemsky], [Belitsky, Korchemsky, Pasechnik]
- Endpoints, a and d, will be determined as part of the solution.

$$\int_d^a du \frac{\rho_0(u)}{u'-u} = \pi n(u'), \quad \int_d^a du \frac{\rho_0(u)}{u} = 0, \quad \int_d^a \rho_0(u) = S$$

is solved by

$$\rho_0(u) = \frac{1}{\pi} \int_d^a \left| \frac{F(u)}{F(u')} \right| \frac{n(u')}{u'-u} du'$$

with $F(u) = \sqrt{u-a}\sqrt{u-d}$ and

$$a = \frac{S}{n} \tan \frac{\pi}{2n} \quad d = -\frac{S}{n} \cot \frac{\pi}{2n}.$$

One loop from the Bethe ansatz

$$\rho_0(u) = \frac{n}{\pi} \log \left| \frac{\sqrt{a(u-d)} + \sqrt{-d(a-u)}}{\sqrt{a(u-d)} - \sqrt{-d(a-u)}} \right|$$

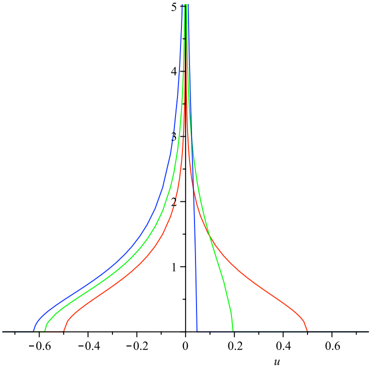

Figure: Root density $\rho_0(\bar{u}) = \rho_0(\frac{u}{S})$ for $n = 2$, $n = 3$ and $n = 6$. The larger the n the more skewed the distribution becomes.

One loop from the Bethe ansatz

$$\rho_0(u) = \frac{n}{\pi} \log \left| \frac{\sqrt{a(u-d)} + \sqrt{-d(a-u)}}{\sqrt{a(u-d)} - \sqrt{-d(a-u)}} \right|$$

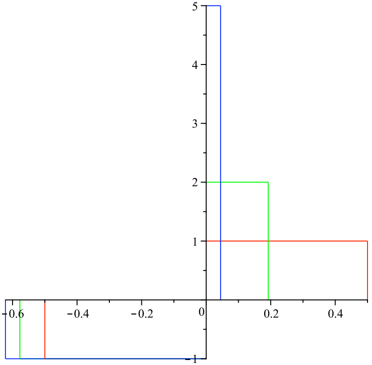

Figure: Wave numbers $n_w(\bar{u}) = n_w(\frac{u}{S})$ for n=2,3,6. By analogy with flat space, we take the modes with $\bar{u} < 0$ to have $n_w = -1$ and those with $\bar{u} > 0$ to have $n_w = n - 1$.

One loop from the Bethe ansatz

One-loop energy

$$\frac{E-S}{2g^2} = 2n \log S + \mathcal{O}(S^0)$$

For the highest excited state $n = L$

One loop from the Bethe ansatz

One-loop energy

$$\frac{E-S}{2g^2} = 2n \log S + \mathcal{O}(S^0)$$

For the highest excited state $n = L$

All loops from the Bethe ansatz

Integral equation for the density

$$\frac{n}{u^2+1/4} + in\frac{d}{du}\log\frac{1+g^2/(x^-(u))^2}{1+g^2/(x^+(u))^2} = 2\pi\rho(u)$$
$$-2\int_a^d \frac{\rho(u')}{(u-u')^2+1}$$
$$+\int_a^d du'\rho(u')\frac{d}{du}\left(2i\log\frac{1-g^2/x^+(u)x^-(u')}{1-g^2/x^+(u')x^-(u)} - 2\theta(u,u')\right)$$

- Back to one-loop

$$\frac{n}{u^2+1/4} = 2\pi\rho(u) - 2\int_a^d du'\frac{\rho_0(u')}{(u-u')^2+1}$$

- rescale roots $u = S\bar{u}$

$$\frac{2n}{S}\delta(\bar{u}) = 2\pi\rho_0(\bar{u}) - 2\int_{a/S}^{d/S} d\bar{u}'\rho_0(\bar{u}')\left(\pi\delta(\bar{u}-\bar{u}') + \frac{1}{S}\frac{\mathcal{P}}{(\bar{u}-\bar{u}')^2}\right)$$

All loops from the Bethe ansatz

and the one-loop equation is recovered

$$0 = \pi n \delta(\bar{u}) - \int_{a/S}^{d/S} d\bar{u}' \frac{\rho_0(\bar{u}')}{(\bar{u}-\bar{u}')^2}$$

Obtain the *leading order* all loop density using the trick of splitting the density

$$\rho(u) = \rho_0(u) + 2g^2 \tilde{\sigma}(u)$$

[Eden, Staudacher]

$$0 = 2\pi\tilde{\sigma}(u) - 2\int_{-\infty}^{\infty} \frac{\tilde{\sigma}(u')}{(u-u')^2+1} + \frac{1}{2g^2}\int_d^a du' \rho_0(u') \frac{d}{du}\left(2i \log \frac{1-g^2/x^+(u)x^-(u')}{1-g^2/x^+(u')x^-(u)}\right) + \int_{-\infty}^{\infty} du' \tilde{\sigma}(u') \frac{d}{du}\left(2i \log \frac{1-g^2/x^+(u)x^-(u')}{1-g^2/x^+(u')x^-(u)}\right).$$

For simplicity we have excluded the dressing phase, we will restore its contribution in the end.

All loops from the Bethe ansatz

Use the one-loop density to compute

$$\frac{1}{2g^2}\int_d^a du'\rho_0(u')\frac{d}{du}\left(2i\log\frac{1-g^2/x^+(u)x^-(u')}{1-g^2/x^+(u')x^-(u)}\right)$$
$$=n\log S\frac{d}{du}\left(\frac{1}{x^+(u)}-\frac{1}{x^-(u)}\right)+\dots$$

Fourier transformation and redefinition $\hat{\sigma}(t)=-2ne^{t/2}\hat{\hat{\sigma}}(t)\log S$

Integral equation for the highest excited state

$$\hat{\sigma}(t)=\frac{t}{e^t-1}\left(K(2gt,0)-4g^2\int_0^\infty dt'K(2gt,2gt')\hat{\sigma}(t')\right)$$

Energy for the highest excited state

$$E-S=\frac{n}{2}16g^2\hat{\sigma}(0)\log S+\mathcal{O}(S^0)$$

All loops from the Bethe ansatz

The kernels

$$K(t,t') = K_0(t,t') + K_1(t,t')$$

since we excluded the dressing phase. Now including it modifies the kernels as

$$K(t,t') = K_0(t,t') + K_1(t,t') + K_d(t,t')$$

and the integral equation we obtained is the BES equation rescaled

Energy for the highest excited state

$$E - S = \frac{n}{2} f(g) \log S + \mathcal{O}(S^0)$$

where $f(g)$ is the scaling function.

Subleading corrections to the large S limit

Use a similar trick of splitting the density. At one-loop

$$\frac{n}{u^2+1/4} = 2\pi\rho_0(u) - 2\int_d^a du'\,\frac{\rho_0(u')}{(u-u')^2+1}$$

which is solved approximately by the density

$$\tilde{\rho}_0(u) = \frac{n}{\pi}\log\left|\frac{\sqrt{a(u-d)}+\sqrt{-d(a-u)}}{\sqrt{a(u-d)}-\sqrt{-d(a-u)}}\right|$$

$$= \frac{n}{\pi}\log S - \frac{n}{2\pi}\log(u^2) + \frac{n}{\pi}\log\left(\frac{2}{n}\sin\frac{\pi}{n}\right) + \mathcal{O}(1/S).$$

Split: $\rho_0(u) = \tilde{\rho}_0(u) + r(u)$

$$0 = 2\pi r(u) + 2n\log S - n\log(u^2) + 2n\log\left(\frac{2}{n}\sin\frac{\pi}{n}\right) - \frac{n}{u^2+1/4}$$
$$-2\int_{-\infty}^{\infty} dv\left(\frac{n}{\pi}\log S - \frac{n}{2\pi}\log(u^2) + \frac{n}{\pi}\log\left(\frac{2}{n}\sin\frac{\pi}{n}\right)\right)\frac{1}{(u-v)^2+1}$$
$$-2\int_{-\infty}^{\infty} dv\,\frac{r(v)}{(u-v)^2+1}.$$

Subleading corrections to the large S limit

Solution by Fourier transform

$$\hat{r}(t) = n\left(\frac{e^{-|t|/2}}{1-e^{-|t|}} - \frac{1}{|t|}\right)$$

One-loop energy

$$\frac{E-S-n}{2g^2} = \int_d^a du \frac{\tilde{\rho}_0(u)}{u^2+1/4} + \int_{-\infty}^{\infty} \frac{r(u)}{u^2+1/4}$$
$$= 2n\left(\log S + \gamma_E + \log\left(\frac{2}{n}\sin\frac{\pi}{n}\right)\right) + \dots$$

In agreement with [Kruczenski]

Subleading corrections to the large S limit, all loops

- Continue splitting, $\rho(u) = \rho_0(u) + 2g^2\tilde{\sigma}(u)$

$$\frac{n}{2g^2}\left(\frac{d}{du}\log\frac{x(i/2+u)}{x(i/2-u)} - \frac{1}{u^2+1/4}\right) = 2\pi\tilde{\sigma}(u)$$

$$-2\int_{-\infty}^{\infty}\frac{\tilde{\sigma}(u')}{(u-u')^2+1} + I(u)$$

$$+\int_{-\infty}^{\infty} du'\tilde{\sigma}(u')\frac{d}{du}\left(2i\log\frac{1-g^2/x^+(u)x^-(u')}{1-g^2/x^+(u')x^-(u)}\right)$$

where

$$I(u) = \frac{1}{2g^2}\int_d^a du'\rho_0(u')\frac{d}{du}\left(2i\log\frac{1-g^2/x^+(u)x^-(u')}{1-g^2/x^+(u')x^-(u)}\right).$$

- For simplicity we have also here excluded the dressing phase. It is easily restored in the end.

Subleading corrections to the large S limit, all loops

- The all loop energy is given by

$$\frac{E-S-n}{2g^2} = \int_d^a \rho(u)\left(\frac{i}{x^+(u)} - \frac{i}{x^-(u)}\right)$$
$$= \int_d^a \rho_0(u)\left(\frac{i}{x^+(u)} - \frac{i}{x^-(u)}\right)$$
$$+ 2g^2 \int_0^\infty du\,(\tilde{\sigma}(u)+\tilde{\sigma}(-u))\left(\frac{i}{x^+(u)} - \frac{i}{x^-(u)}\right)$$

- As previously we use the one-loop density to construct an integral equation for the higher loop density.
- Fourier transformation and redefinition

$$\hat{\sigma}(t) = -\frac{1}{8}e^{-t/2}\left(\hat{\hat{\sigma}}(t)+\hat{\hat{\sigma}}(-t)\right)$$

Subleading corrections to the large S limit, all loops

All loop integral equation including $\mathcal{O}(S^0)$

$$\begin{aligned}\hat{\sigma}(t) = \frac{t}{e^t - 1}\Bigg[& K(2gt, 0)\frac{n}{2}\left(\log S + \gamma_E + \log\left(\frac{2}{n}\sin\frac{\pi}{n}\right)\right) \\ & - \frac{n}{2}\frac{2}{8g^2 t}(J_0(2gt) - 1) \\ & + \frac{n}{2}\int_0^\infty dt' \frac{1}{e^{t'} - 1}\left(K(2gt, 2gt') - K(2gt, 0)\right) \\ & - 4g^2 \int_0^\infty dt' K(2gt, 2gt')\hat{\sigma}(t')\Bigg].\end{aligned}$$

All loop energy

$$E - S - n = 16g^2\hat{\sigma}(0)$$

Subleading corrections to the large S limit, all loops

- $K(t,t') = K_0(t,t') + K_1(t,t')$, including the dressing phase modifies the kernel to $K(t,t') = K_0(t,t') + K_1(t,t') + K_d(t,t')$
- Compare to the integral equation for the ground state of the operators with twist 2
- The all loop result can be directly read off

$$E - S = n + f(g)\frac{n}{2}\left(\log S + \gamma_E + \log\left(\frac{2}{n}\sin\frac{\pi}{n}\right)\right) + \frac{n}{2}B_2(g) + \dots$$

Subleading corrections to the large S limit, all loops

- Strong coupling is now easily obtained using

$$f(g) = 4g - \frac{3\log 2}{\pi} + \mathcal{O}(1/g)$$

$$B_2(g) = (-\gamma_E - \log g)\, f(g) - 4g(1 - \log 2) - \left(1 - \frac{6\log 2}{\pi} + \frac{3(\log 2)^2}{\pi}\right) + \mathcal{O}(1/g)$$

Strong coupling expansion

$$E - S = 2ng\left(\log\frac{S}{g} + \log\left(\frac{4}{n}\sin\frac{\pi}{n}\right) - 1\right) + \frac{n}{2}\left(1 + \frac{6\log 2}{\pi} - \frac{3\log 2}{\pi}\log\left(\frac{4}{n}\sin\frac{\pi}{n}\right) - \frac{3\log 2}{\pi}\log\frac{S}{g}\right) + \mathcal{O}(1/g)$$

Subleading corrections to the large S limit, all loops

- First part computed from string theory and agrees [Beccaria, Forini, Tirziu, Tseytlin]
- For $n = 2$ we recover the result of the folded string including one-loop corrections [Beccaria, Forini, Tirziu, Tseytlin]
- Second part not directly computed from the sigma model but from general considerations we expect agreement. [Gromov, Vieira]
- Higher orders?
- Wrapping plays no role to this order in expansion and it is enough to use the asymptotic Bethe equations.

Two cut solution

- Consider a solution where the Bethe roots condense on two cuts and the mode numbers are

$$n(u) = \begin{cases} -1, & d < u < c \\ n-1, & b < u < a \end{cases}$$

- Similarly to the one cut solution we find the root distribution

$$\rho_{0L}(u) = -\frac{2n}{\pi}\sqrt{\frac{(c-u)(b-u)}{(u-d)(a-u)}}\frac{1}{\sqrt{(a-c)(b-d)}}\left((a-d)\Pi[\frac{(b-a)(u-d)}{(b-d)(u-a)}, r] + (u-a)K[r]\right) + \frac{J}{2\pi^2 u}(B_1(u) - B_2(u))$$

for the left interval $u \in [d, c]$, and

$$\rho_{0R}(u) = \frac{2n}{\pi}\sqrt{\frac{(u-c)(u-b)}{(u-d)(a-u)}}\frac{1}{\sqrt{(a-c)(b-d)}}\left((d-a)\Pi[\frac{(d-c)(u-a)}{(a-c)(u-d)}, r] + (u-d)K[r]\right) + \frac{J}{2\pi^2 u}(-B_1(u) + B_2(u))$$

for the right one $u \in [b, a]$.

Two cut solution

with

$$r = \frac{(a-b)(c-d)}{(a-c)(b-d)}$$

$$B_{1,2}(u) = \sqrt{|F(u)|}A_{4,5}, \quad F(u) = \sqrt{u-a}\sqrt{u-b}\sqrt{u-c}\sqrt{u-d}$$

The momentum condition and the normalization of the density give the relations

$$n = 1 - \frac{A_2}{A_3}, \qquad L = \frac{2\pi A_1}{A_5 - A_4}(1 - \frac{A_2}{A_3})$$

$$S = A_6 + \frac{A_1(A_2 - A_3)^2}{A_3(A_4 - A_5)} - A_7\frac{A_2}{A_3}$$

$$A_4 - \frac{A_2}{A_3}A_5 + \frac{A_1(A_8 - A_9)(A_3 - A_2)}{A_3(A_5 - A_4)} = 0$$

Two cut solution

$$A_1 = \int_b^a \frac{dx}{|F(x)|} = \frac{2}{\sqrt{(a-c)(b-d)}} K[r]$$

$$A_2 = \int_d^c \frac{xdx}{|F(x)|} = \frac{2}{\sqrt{(a-c)(b-d)}} [(d-a)\Pi[\frac{d-c}{a-c}, r] + aK[r]]$$

$$A_3 = \int_b^a \frac{xdx}{|F(x)|} = \frac{2}{\sqrt{(a-c)(b-d)}} [(a-d)\Pi[\frac{b-a}{b-d}, r] + dK[r]]$$

$$A_4 = \int_d^c \frac{dx}{x|F(x)|} = \frac{2}{ad\sqrt{(a-c)(b-d)}} [(a-d)\Pi[\frac{a(d-c)}{d(a-c)}, r] + dK[r]]$$

$$A_5 = \int_b^a \frac{dx}{x|F(x)|} = \frac{2}{ad\sqrt{(a-c)(b-d)}} [(d-a)\Pi[\frac{d(b-a)}{a(b-d)}, r] + aK[r]]$$

$$A_6 = \int_d^c \frac{x^2dx}{|F(x)|} = \frac{1}{\sqrt{(a-c)(b-d)}} [(a-c)(b-d)E[r]$$

$$+ (a(a+c) + d(a-c))K[r] - (a-d)(a+b+c+d)\Pi[\frac{d-c}{a-c}, r]]$$

$$A_7 = \int_b^a \frac{x^2dx}{|F(x)|} = \frac{1}{\sqrt{(a-c)(b-d)}} [(a-c)(b-d)E[r]$$

$$+ (c(a+c) + b(c-a))K[r] - (c-b)(a+b+c+d)\Pi[\frac{a-b}{a-c}, r]]$$

$$A_8 = \int_d^c \frac{dx}{x^2|F(x)|} = \frac{1}{a^2d^2bc\sqrt{(a-c)(b-d)}} [ad(a-c)(b-d)E[r]$$

$$+ bd(-a^2 + cd + a(c+d))K[r] + (a-d)(bcd + acd + abc + abd)\Pi[\frac{a(d-c)}{d(a-c)}, r]]$$

$$A_9 = \int_b^a \frac{dx}{x^2|F(x)|} = \frac{1}{c^2b^2ad\sqrt{(a-c)(b-d)}} [cb(a-c)(b-d)E[r]$$

$$+ bd(-c^2 + ab + c(a+b))K[r] + (c-b)(dab + cab + acd + cbd)\Pi[\frac{c(b-a)}{b(c-a)}, r]]$$

Two cut solution, limits

1. 1-cut solution: $b = \epsilon$, $c = -\eta\epsilon$, $d < 0$, $a > 0$, $\epsilon \to 0$
2. $d \to -\infty$, a, b, c fixed gives

"pp-wave like limit"

$$n >> 1,\ S \sim n^2,\ L \sim n,\ E - S \sim n.$$

$$S = \frac{2}{\pi}\frac{(cK[q] + (a-c)E[q])^2}{a-c}n^2, \qquad L = \frac{2\pi a K[q]}{\Pi[\frac{a-b}{a}, q] + \Pi[\frac{a}{a-c}, q] - K[q]}n$$

$$2\Pi[\frac{a-b}{a}, q] = \frac{ab+bc+ac}{bc}K[q]$$

$$\frac{E-S}{2g^2} = \frac{2n}{\sqrt{a(a-c)}}(1+4b^2)^{\frac{1}{4}}(1+4c^2)^{\frac{1}{4}}\sin[\frac{1}{2}(\tan^{-1}\frac{1}{2b} + \tan^{-1}\frac{1}{2c})](K[q] - \Pi[\frac{a-b}{a}, q])$$

$|d| \gg b, c \gg 1$

$$\frac{E-S}{2g^2} = -\frac{n(b+c)}{\sqrt{a-c}\sqrt{-abc}}(K[q] - \Pi[\frac{a-b}{a}, q])$$

- This limit is considered from string theory and there is hence a prediction for strong coupling [Ishizeki, Kruczenski, Tirziu, Tseytlin]
- It is interesting to note that this type of limit seems to make sense both at weak and strong coupling. All loops in this limit?

Conclusions

- The anomalous dimensions of twist operators in the $sl(2)$ sector occupies a band.
- We studied the lowest state and found the subleading corrections $\mathcal{O}(S^0)$ explicitly at weak and strong coupling.
- This "virtual scaling function" also appear in the context of amplitudes.
- We studied the highest excited state and found the leading,$\mathcal{O}(\log S)$, and subleading contributions, $\mathcal{O}(S^0)$, explicitly at weak and strong coupling.
- This allows us to understand spiky strings directly from the Bethe ansatz as a specific distribution of Bethe roots.

For the future

- ⋆ collinear anomalous dimension
- ⋆ "pp-wave limit" at all loops
- ⋆ the other states in the band
- ⋆ connection to TBA and the Y-system?
- ⋆ so far all interpolating functions proposed are for the $sl(2)$ subsector, is it possible to go beyond that?

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