Aspects of the spectrum of strings in $\mathbf{AdS}_5\times\mathbf{S}^5$

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a brief review
and 0906.4294 with A. Tseytlin

Solving a 4d gauge theory is hard

$\diamond$ Expectation is that for N=4 SYM the AdS/CFT correspondence provides us with means to do so

- reduces the problem to solving a 2d qft: GS string in $AdS_5 \times S^5$

What would we expect to get:

- spectrum of dimensions
- correlation functions of gauge-invariant operators
- expectation values of Wilson loops
- scattering amplitudes

In string language:

- spectrum of states and the corresponding vertex operators
- correlation functions of vertex operators
- partition functions in presence of minimal surfaces

And then... on to QCD

The plan

- GS string in $AdS_5 \times S^5$
- A review of long operators
- Organization of flat space string states in $AdS_5 \times S^5$ multiplets
- Structure of corrections to E of quantum strings; approaches
- Semiclassical approach: small circular strings and small folded strings vs. states in Konishi multiplet
- Summary

String theory in AdS$_5\times$S^5

Bosonic string: coset model on $\frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}$ symmetric space

Superstring: generalized supercoset model on $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ with WZ term and κ-symmetry

Metsaev, Tseytlin

- Constructed ito left-invariant 1-forms $J = g^{-1}dg = H + Q_1 + P + Q_2$
- Action: $T = \frac{\sqrt{\lambda}}{2\pi}$ $P^{ab}_{IJ} = \eta^{ab}\delta_{IJ} + \epsilon^{ab}s_{IJ}$

$$\begin{aligned} S &= T\int d^2\sigma \left[\eta^{ab}P_aP_b + \epsilon^{ab}Q_{1a}Q_{2b}\right] \\ &= T\int d^2\sigma \left[\eta^{ab}\partial_aX^A\partial_bX^BG_{AB} + P^{ab}_{IJ}\bar{\theta}^I(D(\omega, F_{(5)})\theta)^J + \mathcal{O}(\theta^4)\right] \end{aligned}$$

- 2d Lorentz-invariant
- $PSU(2,2|4)$ invariant
- conformally invariant to 2-loops; expected to all orders
- fix κ-symmetry: need bosonic background (pointlike or extended)
  $\rightarrow$ light-cone or gauges adapted to solution

The spectral problem

gauge theory operators and string states are organized in representations of $PSU(2,2|4)$ – $(E; S_1, S_2; J_1, J_2, J_3) \equiv \mathcal{A}^{E}_{[p_1,q,p_2]_{(s_L,s_R)}}$

Principles of comparison: identify $SO(4) \times SO(6)$ quantum numbers of operators and string states

> additional insight needed, since there are many states with the same global quantum numbers

> there exist "quantum numbers" which do not have immediate relation between gauge and string theory

string energies = dimensions of local Tr (...) operators

$$E(\sqrt{\lambda}, \mathrm{C}, m, ...) = \Delta(\lambda, \mathrm{C}, m, ...) \qquad \mathrm{C} = (S_1, S_2; J_1, J_2, J_3)$$

m stands for other quantum numbers

Good understanding of long operators

Asymptotic Bethe Ansatz (ABA) — Beisert, Eden, Staudacher

- Firmly established; tested to 4/2 loops at weak/strong coupling

String theory tests – in semiclassical regime; states with large quantum numbers, dual to operators with large classical dimension

Various limits: $S = \sqrt{\lambda}\mathcal{S}$, $J = \sqrt{\lambda}\mathcal{J}$

(i) "Fast strings" – locally BPS — Berenstein, Maldacena, Nastase; Frolov, Tseytlin; Beisert, Minahan, Staudacher, Zarembo

GT: $J \gg 1$ & fixed S/J ST: $\mathcal{J} \gg 1$ & fixed $\mathcal{S}/\mathcal{J}$

$$E = S + J + \frac{\lambda}{J}\Big[h_1(\tfrac{S}{J}, m) + \tfrac{1}{J}h_2(\tfrac{S}{J}, m) + ...\Big] + ...$$

(ii) "Slow long strings" – long non-BPS — Gubser, Klebanov, Polyakov

GT: $\ln S \gg J \gg 1$ ST: $\ln \mathcal{S} \gg \mathcal{J} \gg 1$ & $\mathcal{J}$-fixed

$$E = S + f(\lambda)\ln S \quad ; \quad f(\lambda \gg 1) = a_0\sqrt{\lambda} + \dots \quad ; \quad f(\lambda \ll 1) = c_1\lambda + \dots$$

(iii) "Fast long strings" — Belitsky, Gorsky, Korchemsky; Frolov, Tirziu, Tseytlin; RR, Tseytlin

GT: $\ln S \gg J \gg 1$ & $j = J/\ln S$-fixed ST: $\ln \mathcal{S} \gg \mathcal{J} \gg 1$ & $\ell = \mathcal{J}/\ln \mathcal{S}$-fixed

Important example of long operators: $\mathrm{Tr}[\Phi D_+^S \Phi]$

- Dual to spinning strings
- $S \to \infty$ solution becomes homogeneous; folds reach boundary

$E = S$ from massless endpoints

$E - S \simeq \frac{\sqrt{\lambda}}{\pi} \ln S$ from stretching of string

- Semiclassical limit

$\lambda \gg 1$ & $\mathcal{S} = S/\sqrt{\lambda}$-fixed & $\mathcal{S} \gg 1$

$$E = S + f(\lambda) \ln S + \ldots \qquad f(\lambda \gg 1) = \frac{\sqrt{\lambda}}{\pi}\Big[1 + \frac{a_1}{\sqrt{\lambda}} + \frac{a_2}{(\sqrt{\lambda})^2} + \ldots\Big]$$

a_n – from Feynman graphs of GS superstring in $\mathrm{AdS}_5 \times \mathrm{S}^5$

$a_0 = 1$ Gubser, Klebanov, Polyakov

$a_1 = -3 \ln 2$ Frolov, Tseytlin; Frolov, Tirziu, Tseytlin

$a_2 = -K$ RR, Tseytlin; RR, Tirziu, Tseytlin

- For S=4 it is among the simplest string states

small spin analysis Tirziu, Tseytlin

general “short” operators $\longleftrightarrow$ general (quantum) string states

◇ Integrability-based results, based on improving ABA

- “Luscher corrections” — Ambjorn, Kristjansen, Janik; Bajnok, Janik, Lukowski, Bajnok, Hegedus, Janik, Lukowski
- ABA $\longrightarrow$ TBA — Arutyunov, Frolov
- TBA in Y variables — Gromov, Kazakov, Vieira; Bombardelli, Fioravanti, Tateo

◇ Aim here:

- structure of the corrections to E for general states
- evaluate leading $1/\sqrt{\lambda}$ correction to the dimension of “lightest” massive string state dual to Konishi operator

◇ In general: need better handle on quantum $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring

Organization of string states

- $T \propto \sqrt{\lambda} \to \infty$: AdS$_5\times$S^5 is nearly flat
  $\longrightarrow$ expect that some features of the flat space spectrum survive
- flat space spectrum: $-p^2 = m^2 = 4(2\pi T)(n-1)$ with $n = \frac{1}{2}(N+\tilde{N})$
- states: best constructed in light-cone gauge

- $n=1$: $|0\rangle \equiv (\mathbf{8}_v \oplus \mathbf{8}_c)^{\otimes 2} \mapsto$ IIB supergravity multiplet; 2^8 states
- $n=2$: $(a^i_{-1} \oplus S^a_{-1})^{\otimes 2}|0\rangle \equiv ((\mathbf{8}_v \oplus \mathbf{8}_c) \otimes (\mathbf{8}_v \oplus \mathbf{8}_c))^2 \mapsto 2^{16}$ states
  - organize in SO(9) representations
    $$(\mathbf{44}_b + \mathbf{84}_b + \mathbf{128}_f)^{\otimes 2} \equiv ([2,0,0,0]+[0,0,1,0]+[1,0,0,1])^{\otimes 2}$$
  - all states have same mass and form a single susy multiplet
- AdS$_5\times$S^{5}background: lifts degeneracy

String spectrum in AdS$_5 \times$S^5:

- $PSU(2,2|4)$ symmetry $\mapsto$ (long) multiplets $\mathcal{A}^E_{[k,p,q](s_L,s_R)}$

  $>$ highest weight states have $SO(6) \times SO(4)$ labels: $[k,p,q]_{(s_L,s_R)}$

Remarkably, flat-space string spectrum can be reorganized in multiplets of $SO(2,4) \times SO(6) \subset PSU(2,2|4)$ Bianchi, Morales, Samtleben

$$SO(9) \mapsto SO(4) \times SO(5) \longrightarrow SO(4) \times SO(6) \subset SO(2,4) \times SO(6)$$

$n = 2$ excited string level: same $SO(4) \times SO(6)$ content as the Konishi multiplet:

$$\widehat{T}_1 = (1 + Q + Q \wedge Q + \ldots)[0,0,0]_{(0,0)}$$

$>$ tensor with scalar spherical harmonics $[0,J,0]_{(0,0)} \mapsto$ full KK tower

$$H_1 = \sum_{J=0}^{\infty} [0,J,0]_{(0,0)} \times \widehat{T}_1$$

- assume nonintersection principle Polyakov

Konishi multiplet $\longleftrightarrow$ $n = 2$ excited level

E	$[q_1,p,q_2]_{(s_L,s_R)} = [J_2-J_3, J_1-J_2, J_2+J_3]_{((S_1+S_2)/2,(S_1-S_2)/2)}$
Δ	$[0,0,0]_{(0,0)}$
$\Delta+\frac{1}{2}$	$[0,0,1]_{(0,\frac{1}{2})} + [1,0,0]_{(\frac{1}{2},0)}$
$\Delta+1$	$[0,0,0]_{(\frac{1}{2},\frac{1}{2})} + [0,0,2]_{(0,0)} + [0,1,0]_{(0,1)+(1,0)} + [1,0,1]_{(\frac{1}{2},\frac{1}{2})} + [2,0,0]_{(0,0)}$
$\Delta+\frac{3}{2}$	$[0,0,1]_{(\frac{1}{2},0)+(\frac{1}{2},1)+(\frac{3}{2},0)} + [0,1,1]_{(0,\frac{1}{2})+(1,\frac{1}{2})} + [1,0,0]_{(0,\frac{1}{2})+(0,\frac{3}{2})+(1,\frac{1}{2})} + [1,0,2]_{(\frac{1}{2},0)}$ $+[1,1,0]_{(\frac{1}{2},0)+(\frac{1}{2},1)} + [2,0,1]_{(0,\frac{1}{2})}$
$\Delta+2$	$[0,0,0]_{(0,0)+(0,2)+(1,1)+(2,0)} + [0,0,2]_{(\frac{1}{2},\frac{1}{2})+(\frac{3}{2},\frac{1}{2})} + [0,1,0]_{2(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})} + [2,0,2]_{(0,0)} + [2,1,0$ $+[0,1,2]_{(1,0)} + [0,2,0]_{2(0,0)+(1,1)} + [1,0,1]_{(0,0)+2(0,1)+2(1,0)+(1,1)} + [1,1,1]_{2(\frac{1}{2},\frac{1}{2})} + [2,0,0]_{(\frac{1}{2},\frac{1}{2})+(}$
$\Delta+\frac{5}{2}$	$[0,0,1]_{(0,\frac{1}{2})+(0,\frac{3}{2})+2(1,\frac{1}{2})+(1,\frac{3}{2})+(2,\frac{1}{2})} + [0,0,3]_{(\frac{3}{2},0)} + [0,1,1]_{3(\frac{1}{2},0)+2(\frac{1}{2},1)+(\frac{3}{2},0)+(\frac{3}{2},1)} + [0,2,1]_{(0,\frac{1}{2})+(}$ $+[1,0,0]_{(\frac{1}{2},0)+2(\frac{1}{2},1)+(\frac{1}{2},2)+(\frac{3}{2},0)+(\frac{3}{2},1)} + [1,0,2]_{(0,\frac{1}{2})+2(1,\frac{1}{2})} + [1,1,0]_{3(0,\frac{1}{2})+(0,\frac{3}{2})+2(1,\frac{1}{2})+(1,\frac{3}{2})}$ $+[1,1,2]_{(\frac{1}{2},0)} + [1,2,0]_{(\frac{1}{2},0)+(\frac{1}{2},1)} + [2,0,1]_{(\frac{1}{2},0)+2(\frac{1}{2},1)} + [2,1,1]_{(0,\frac{1}{2})} + [3,0,0]_{(0,\frac{3}{2})}$
$\Delta+3$	$[0,0,0]_{(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})+(\frac{3}{2},\frac{3}{2})} + [0,0,2]_{2(0,0)+(1,0)+2(1,1)+(2,0)} + [0,1,0]_{3(0,1)+3(1,0)+2(1,1)+(1,2)+(2,1}$ $+[0,1,2]_{2(\frac{1}{2},\frac{1}{2})+(\frac{3}{2},\frac{1}{2})} + [0,2,0]_{3(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})} + [0,2,2]_{(0,0)} + [0,3,0]_{(0,1)+(1,0)}$ $+[1,0,3]_{(1,0)} + [1,1,1]_{2(0,0)+2(0,1)+2(1,0)+2(1,1)} + [1,2,1]_{(\frac{1}{2},\frac{1}{2})} + [2,0,0]_{2(0,0)+(0,1)+(0,2)+2(1,1)}$ $+[2,0,2]_{(\frac{1}{2},\frac{1}{2})} + [2,1,0]_{2(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})} + [2,2,0]_{(0,0)} + [3,0,1]_{(0,1)} + [1,0,1]_{4(\frac{1}{2},\frac{1}{2})+2(\frac{1}{2},\frac{3}{2})+2(\frac{3}{2},\frac{1}{2})+(\frac{3}{2},\frac{3}{2})}$
$\Delta+\frac{7}{2}$	$[0,0,1]_{2(\frac{1}{2},0)+3(\frac{1}{2},1)+(\frac{3}{2},0)+2(\frac{3}{2},1)+(\frac{3}{2},2)} + [0,0,3]_{(0,\frac{1}{2})+(1,\frac{1}{2})} + [0,1,1]_{3(0,\frac{1}{2})+(0,\frac{3}{2})+4(1,\frac{1}{2})+2(1,\frac{3}{2})+(2,\frac{1}{2})}$ $+[0,1,3]_{(\frac{1}{2},0)} + [0,2,1]_{2(\frac{1}{2},0)+2(\frac{1}{2},1)+(\frac{3}{2},0)} + [0,3,1]_{(0,\frac{1}{2})} + [1,0,0]_{2(0,\frac{1}{2})+(0,\frac{3}{2})+3(1,\frac{1}{2})+2(1,\frac{3}{2})+(2,\frac{3}{2})}$ $+[1,0,2]_{2(\frac{1}{2},0)+2(\frac{1}{2},1)+(\frac{3}{2},0)+(\frac{3}{2},1)} + [1,1,0]_{3(\frac{1}{2},0)+4(\frac{1}{2},1)+(\frac{1}{2},2)+(\frac{3}{2},0)+2(\frac{3}{2},1)} + [1,1,2]_{(0,\frac{1}{2})+(1,\frac{1}{2})}$ $+[1,2,0]_{2(0,\frac{1}{2})+(0,\frac{3}{2})+2(1,\frac{1}{2})} + [1,3,0]_{(\frac{1}{2},0)} + [2,0,1]_{2(0,\frac{1}{2})+(0,\frac{3}{2})+2(1,\frac{1}{2})+(1,\frac{3}{2})} + [2,1,1]_{(\frac{1}{2},0)+(\frac{1}{2},1)}$ $+[3,0,0]_{(\frac{1}{2},0)+(\frac{1}{2},1)} + [3,1,0]_{(0,\frac{1}{2})}$
$\Delta+4$	$[0,0,0]_{3(0,0)+3(1,1)+(2,2)} + [0,0,2]_{3(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})+(\frac{3}{2},\frac{3}{2})} + [0,1,0]_{4(\frac{1}{2},\frac{1}{2})+2(\frac{1}{2},\frac{3}{2})+2(\frac{3}{2},\frac{1}{2})+2(\frac{3}{2},\frac{3}{2})}$ $+[0,1,2]_{(0,0)+2(0,1)+2(1,0)+(1,1)} + [0,2,0]_{3(0,0)+(0,1)+(0,2)+(1,0)+3(1,1)+(2,0)} + [0,2,2]_{(\frac{1}{2},\frac{1}{2})}$ $+[0,3,0]_{2(\frac{1}{2},\frac{1}{2})} + [0,4,0]_{(0,0)} + [1,0,1]_{(0,0)+3(0,1)+3(1,0)+4(1,1)+(1,2)+(2,1)} + [1,0,3]_{(\frac{1}{2},\frac{1}{2})} + [0,0,4]$ $+[1,1,1]_{4(\frac{1}{2},\frac{1}{2})+2(\frac{1}{2},\frac{3}{2})+2(\frac{3}{2},\frac{1}{2})} + [1,2,1]_{(0,0)+(0,1)+(1,0)} + [2,0,0]_{3(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})+(\frac{3}{2},\frac{3}{2})}$ $+[2,0,2]_{(0,0)+(1,1)} + [2,1,0]_{(0,0)+2(0,1)+2(1,0)+(1,1)} + [2,2,0]_{(\frac{1}{2},\frac{1}{2})} + [3,0,1]_{(\frac{1}{2},\frac{1}{2})} + [4,0,0]_{(0,0)}$
$\Delta+\frac{9}{2}$	$[0,0,1]_{2(0,\frac{1}{2})+(0,\frac{3}{2})+3(1,\frac{1}{2})+2(1,\frac{3}{2})+(2,\frac{3}{2})} + [0,0,3]_{(\frac{1}{2},0)+(\frac{1}{2},1)} + [0,1,1]_{3(\frac{1}{2},0)+4(\frac{1}{2},1)+(\frac{1}{2},2)+(\frac{3}{2},0)+2(\frac{3}{2},1)}$ $+[0,1,3]_{(0,\frac{1}{2})} + [0,2,1]_{2(0,\frac{1}{2})+(0,\frac{3}{2})+2(1,\frac{1}{2})} + [0,3,1]_{(\frac{1}{2},0)} + [1,0,0]_{2(\frac{1}{2},0)+3(\frac{1}{2},1)+(\frac{3}{2},0)+2(\frac{3}{2},1)+(\frac{3}{2},2)}$ $+[1,0,2]_{2(0,\frac{1}{2})+(0,\frac{3}{2})+2(1,\frac{1}{2})+(1,\frac{3}{2})} + [1,1,0]_{3(0,\frac{1}{2})+(0,\frac{3}{2})+4(1,\frac{1}{2})+2(1,\frac{3}{2})+(2,\frac{1}{2})} + [1,1,2]_{(\frac{1}{2},0)+(\frac{1}{2},1)}$ $+[1,2,0]_{2(\frac{1}{2},0)+2(\frac{1}{2},1)+(\frac{3}{2},0)} + [1,3,0]_{(0,\frac{1}{2})} + [2,0,1]_{2(\frac{1}{2},0)+2(\frac{1}{2},1)+(\frac{3}{2},0)+(\frac{3}{2},1)} + [2,1,1]_{(0,\frac{1}{2})+(1,\frac{1}{2})}$ $+[3,0,0]_{(0,\frac{1}{2})+(1,\frac{1}{2})} + [3,1,0]_{(\frac{1}{2},0)}$

Approaches to computation of corrections to E of quantum strings:

(i) vertex operator approach:

Construct classical vertex operators; use $\mathrm{AdS}_5\times\mathrm{S}^5$ string sigma model perturbation theory to find leading terms in their anomalous dimensions; diagonalize Polyakov; Tseytlin

(ii) "light-cone" quantization approach:

Use light-cone gauge $\mathrm{AdS}_5\times\mathrm{S}^5$ string action and compute corrections to energy of Fock states Metsaev, Thorn, Tseytlin Berenstein, Maldacena, Nastase; Arutyunov, Frolov, Plefka, Zamaklar

(iii) semiclassical:

Identify short string state as small-charge limit of semiclassical string state; expected structure of strong-coupling corrections to short operators Frolov, Tseytlin; Tirziu, Tseytlin

(iv) space-time effective action approach:

Use near-flat-space expansion and NSR vertex operators to reconstruct $1/\sqrt{\lambda}$ corrections to massive string state equation of motion; spontaneous symmetry breaking in effective action Burrington, Liu

Structure of the expansion

RR, Tseytlin

- “Short” strings + weak curvature $\rightarrow$ mass insensitive to curvature to leading order

$$E(\sqrt{\lambda}, \mathrm{C}) \;=\; \underbrace{2\sqrt{n-1}}_{\text{flat space}}\lambda^{1/4} + \underbrace{\sum_{k=0}^{\infty} \frac{b_k}{\lambda^{k/4}}}_{\text{Corrections from e.g. diagonalization of 2d anom. dim. matrix}}$$

Structure of the expansion

RR, Tseytlin

- “Short” strings + weak curvature → mass insensitive to curvature to leading order

$$
\begin{aligned}
E(\sqrt{\lambda}, \mathrm{C}) &= 2\sqrt{n-1}\lambda^{1/4} + \sum_{k=0}^{\infty} \frac{b_k}{\lambda^{k/4}} = E^{(\mathrm{an})} + E^{(\mathrm{nan})} \\
E^{(\mathrm{an})} &= \lambda^{1/4}\left[2\sqrt{n-1} + \sum_{k=1}^{\infty} \frac{b_{2k}}{(\sqrt{\lambda})^k}\right] \\
&= \lambda^{1/4}\left[2\sqrt{n-1} + \frac{b_1}{\sqrt{\lambda}} + \frac{b_3}{(\sqrt{\lambda})^2} + \dots\right] \\
E^{(\mathrm{nan})} &= b_0 + \sum_{k=1}^{\infty} \frac{b_{2k+1}}{(\sqrt{\lambda})^k} = b_0 + \frac{b_2}{\sqrt{\lambda}} + \dots
\end{aligned}
$$

- $E^{(\mathrm{an})}$: corrections to 2d anomalous dimension/string masses

- $E^{(\mathrm{nan})}$: potential origin in diagonalization of 2d anomalous dimension matrix; semiclassically – 2d light modes

- flat space

$$2 = \gamma_{2d} = 2n - \frac{1}{\pi T}(E^2 - p^2)$$

- AdS$_5\times$S^5 $(E^2 - p^2) \rightarrow$ some quadratic combination of charges

$$2 = \gamma_{2d} = 2n - \frac{1}{2\sqrt{\lambda}}\Big[\, E(E+a_1) + a_2 E + a_3 J(J+a_4) + a_5 \Big] + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}^2}\right)$$

- consistent with differential operator interpretation of γ_{2d}

$$\left[2 - 2n + \frac{\alpha'}{2}\nabla^2 + \alpha'(c_1 \mathsf{R} + c_2 \mathsf{F}_5 \mathsf{F}_5) + \mathcal{O}(\alpha'^2)\right]\Psi = 0$$

- Supersymmetry constraints: $\hat{\mathsf{C}} = (E; s_L, s_R; p_1, q, p_2) = (E, \mathsf{C})$

$$E(\sqrt{\lambda}, \mathsf{C}) = \left[b_0 + \frac{b_2}{\sqrt{\lambda}} + \ldots \right] + \lambda^{1/4} \left[2\sqrt{n-1} + \frac{b_1}{\sqrt{\lambda}} + \ldots \right]$$

$$|\ell, \mathsf{C}\rangle = Q^\ell |0, \mathsf{C}\rangle \text{ and } [D, Q] = \tfrac{1}{2} Q \;; \quad D|\ell, \mathsf{C}\rangle = E(\ell)|\ell, \mathsf{C}\rangle$$

$\longrightarrow E(\ell) = \frac{\ell}{2} + E(0)$ holds both at weak and at strong coupling

$\mapsto b_0$ is $\lambda = 0$ dimension up to ℓ-independent shift: $b_0 = \mathsf{b}_0 + \Delta_0$

- Vertex operator language: marginality condition

$$E^2 - 2(\mathsf{b}_0 + \Delta_0)E + (\mathsf{b}_0 + \Delta_0)^2 - 4b_1 = 4(n-1)\sqrt{\lambda} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)$$

- generally: $2 = 4(n-1) - \dfrac{1}{2\sqrt{\lambda}} \left(\displaystyle\sum_{A,B=1}^{6} u_{AB} \hat{\mathsf{C}}^A \hat{\mathsf{C}}^B + \sum_A v_{\ell A} \hat{\mathsf{C}}^A + h_\ell \right) + \mathcal{O}\left(\dfrac{1}{\sqrt{\lambda}^2}\right)$

$\diamond$ most coefficients vanish or expressed ito (universal) h_0 and h_2

$$\mathsf{b}_0 = -2 + \tfrac{1}{2}(h_2 - h_0 - 1) \quad b_1 = \tfrac{1}{16}(h_2 - h_0 - 1)^2 - \tfrac{1}{4} h_0$$

Semiclassical approach

- Standard expansion: $\lambda \gg 1$ & $\mathcal{J} = \frac{J}{\sqrt{\lambda}}$ =fixed

$$E = E(\frac{J}{\sqrt{\lambda}}, \sqrt{\lambda}) = \sqrt{\lambda}\mathcal{E}_0(\mathcal{J}) + \mathcal{E}_1(\mathcal{J}) + \frac{1}{\sqrt{\lambda}}\mathcal{E}_2(\mathcal{J}) + ...$$

- If $\mathcal{J} \ll 1$, each coefficient is expandable in $\mathcal{J}$

$$\mathcal{E}_k = \sqrt{\mathcal{J}}\ (a_{0k} + a_{1k}\mathcal{J} + a_{2k}\mathcal{J}^2 + ...) + \mathcal{E}_k^{(\text{nan})}$$

$$\mathcal{E}_k^{(\text{nan})} = c_{0k} + c_{1k}\mathcal{J} + ...$$

- If all terms were known, one could resum, use $\mathcal{J} = \frac{J}{\sqrt{\lambda}}$, fix J and re-expand at large λ

$$E = \sqrt{\sqrt{\lambda}J}\ \left[a_{00} + \frac{a_{10}J + a_{01}}{\sqrt{\lambda}} + \frac{a_{20}J^2 + a_{11}J + a_{02}}{(\sqrt{\lambda})^2} + ...\right] + E^{(\text{nan})}$$

$$E^{(\text{nan})} = c_{01} + \frac{c_{11}J + c_{02}}{\sqrt{\lambda}} + ...$$

◇ the first k terms in semiclassical expansion determine the $(\sqrt{\lambda})^{-k}$ term in short string expansion

String in flat space at 1st excited level:

- A classical solution

$$\mathrm{x}_x \equiv x_1 + ix_2 = \ a\ e^{i(\tau+\sigma)} \qquad \mathrm{x}_y \equiv x_3 + ix_4 = \ a\ e^{i(\tau-\sigma)}$$

$$E = \sqrt{4(2\pi T)\, J} \qquad J = a^2(2\pi T)$$

- Promote to quantum state: only a_{-1} is excited $\mapsto$ vertex operator

$$e^{-iEt}\left[(\partial \mathrm{x}_x \bar{\partial} \mathrm{x}_x)^{\frac{J_1}{2}} (\partial \mathrm{x}_y \bar{\partial} \mathrm{x}_y)^{\frac{J_2}{2}} + \ldots\right], \qquad \frac{E^2}{2\pi T} = 2(J_1 + J_2 - 2)$$

- Level matching: $J_1 = J_2 = J \longrightarrow E = \sqrt{4(2\pi T)(J-1)}$

◇ $J_1 = J_2 = J = 2$ corresponds to the 1st excited level

◇ embed in $AdS_5 \times S^5$:	both planes in S^5	$[2,0,2]_{(0,0)} \equiv (0,0;2,2,0)$
	both planes in AdS_5	$[0,0,0]_{(2,0)} \equiv (2,2;0,0,0)$
	one plane each	$[0,2,0]_{(1,1)} \equiv (2,0;2,0,0)$

$J_1 = J_2$ in S^5

$R^4 \mapsto S^5$ defn'd $X_1^2 + ... + X_6^2 = 1$: string on *small* sphere inside S^5:

$$X_1 + iX_2 = a\ e^{i(\tau+\sigma)}, \qquad X_3 + iX_4 = a\ e^{i(\tau-\sigma)}$$
$$X_5 + iX_6 = \sqrt{1-2a^2}, \qquad t = \kappa\tau$$
$$\mathcal{J} = \mathcal{J}_1 = \mathcal{J}_2 = a^2, \qquad \mathcal{E}^2 = \kappa^2 = 4\mathcal{J} < 2$$

Frolov, Tseytlin

[different branch of (unstable) circular string with $\mathcal{J} > 1/2$ with $E_0 = \sqrt{\lambda + J^2}$]

Remarkably, exact E_0 is just as in flat space

$$E_0 = \sqrt{\lambda}\mathcal{E} = \sqrt{4\sqrt{\lambda}J}\ , \qquad J = \sqrt{\lambda}\mathcal{J}$$

◇ interpret as quantum state: shift $J \mapsto J - 1$ (cf. flat space lim)

- perform this shift in quantum corrections as well

1-loop quantum string correction to the energy:

- sum of bosonic and fermionic frequencies or path integral

> Bosons (2 massless + massive):

$$
\begin{aligned}
AdS_5: \quad & 4\times \quad \omega_n^2 = n^2 + 4\mathcal{J} \\
S^5: \quad & 1\times \quad \omega_{n\pm}^2 = n^2 + 4(1-\mathcal{J}) \pm 2\sqrt{4(1-\mathcal{J})n^2 + 4\mathcal{J}^2} \\
& 2\times \quad \omega_{n\pm}^2 = n^2
\end{aligned}
$$

> Fermions:

$$
4\times \quad \omega_{n\pm}^2 = n^2 + 1 + \mathcal{J} \pm \sqrt{4(1-\mathcal{J})n^2 + 4\mathcal{J}}
$$

◇ Correction to energy:

$$
E_1 = \frac{1}{2\kappa} \sum_{n=-\infty}^{\infty} \left[4\omega_n + 2n + (\omega_{n+} + \omega_{n-}) - 4(\omega^f_{n+} + \omega^f_{n-}) \right]
$$

- good UV convergence; expand in small $\mathcal{J}$ and do sums potential nonanalyticity in $\mathcal{J}$ for $n = 0, \pm 1, \pm 2$, e.g.

$$
\Omega_0 = -4 + 8\sqrt{\mathcal{J}} - 2\mathcal{J} - 2\mathcal{J}^2 - \mathcal{J}^3 + \ldots
$$

However, nonanalyticity cancels out and

$$E_1^{(\text{an})} = \frac{1}{\sqrt{\mathcal{J}}}\Big[\mathcal{J} - \frac{1}{8}(1+8\zeta(3))\mathcal{J}^2 + \dots\Big]$$
$$E_1^{(\text{nan})} = 0$$

$$E = E_0 + E_1 = 2\sqrt{\sqrt{\lambda}J}\Big[1 + \frac{1}{2\sqrt{\lambda}} + \dots\Big]$$

- no $\frac{1}{\sqrt{\lambda}}$ present; expect remains this way when 2-loops are included
- interpolate to small $J = J_1 = J_2$; correct flat space limit: $J \mapsto J-1$
- for $J = J_1 = J_2 = 2$ it is level-2 massive string state in $[2,0,2]_{(0,0)}$

  Konishi descendants in this rep for $\Delta_0 = 2 + \frac{\ell}{2} = 4, 6, 8$

> in flat space it appears in $Q_L^2 Q_R^2|0\rangle$ $\rightarrow$ suggests $\ell = 4,\ \Delta_0 = 4$

$\rightarrow$ $b_1 = 1$ and $\mathrm{b}_0 = -4$; expect $b_2 = 0$, b_3 contains $\zeta(3)$

$\rightarrow$ dual operator is $\mathrm{Tr}\left[[\Phi_1, \Phi_2]^2\right]$

$S_1 = S_2$ in AdS^5

$R^4 \mapsto$AdS^5 defn'd as $-Y_5^2 - Y_0^2 + Y_1^2 + ... + Y_4^2 = -1$

$$Y_0 + iY_5 = \sqrt{1 + 2r^2}\ e^{i\mathbf{k}t} \quad Y_1 + iY_2 = r\ e^{i(w\tau+\sigma)} \quad Y_3 + iY_4 = r\ e^{i(w\tau-\sigma)}$$

$$r = \sinh \rho_0 = \frac{1}{4}\kappa^2 \qquad w^2 = 1 + \kappa^2 = 1 + 4\mathcal{S} - 8\mathcal{S}^2 + \dots$$

$$S_1 = S_2 = S = \sqrt{\lambda}\ \mathcal{S} \qquad \mathcal{S} = \frac{1}{4}\kappa^2\sqrt{\kappa^2 + 1}$$

$$E_0 = \sqrt{\lambda}\ \mathcal{E}_0 \qquad \mathcal{E}_0 = \kappa + \frac{2\kappa\mathcal{S}}{\sqrt{\kappa^2 + 1}}$$

curvature corrections

$$E_0 = 2\sqrt{\sqrt{\lambda}S}\ \left[1 + \frac{S}{\sqrt{\lambda}} - \frac{3S^2}{2\lambda} + \mathcal{O}\left(\frac{S^3}{\lambda^{3/2}}\right)\right]$$

◇ quantum state in flat space limit (fixed S, $\lambda \to \infty$): shift $S \to S - 1$

◇ Similarly to $J_1 = J_2$ state, has $\ell = 4$ in flat space lim.

$\rightarrow$ expect $b_1 = 1$ and b$_0 = -4$

1-loop quantum string correction to the energy:

- sum of bosonic and fermionic frequencies or path integral

> Bosons (2 massless + massive):

$$
\begin{aligned}
AdS_5: \quad & 2\times \quad \omega_n^2 = n^2 \\
AdS_5: \quad & 4\times \quad \omega_n^6 + c_1\omega_n^4 + c_2\omega_n^2 + c_3 = 0 \ , \ c_1 = -8 - 10\kappa^2 - 3n^2 \\
& \qquad\quad c_2 = 16 + 40\kappa^2 + 24\kappa^4 + 8\kappa^2 n^2 + 3n^4 \\
& \qquad\quad c_3 = -n^2(n^2-4)(n^2-4-2\kappa^2) \\
S^5: \quad & 5\times \quad \omega_{n\pm}^2 = n^2
\end{aligned}
$$

> Fermions:

$$
4\times \quad \omega_{n\pm}^2 = n^2 + 1 + \frac{5}{4}\kappa^2 \pm \sqrt{4n^2 + \kappa^2 + 3n^2\kappa^2 + \kappa^4}
$$

◇ Correction to energy:

$$
E_1 = \frac{1}{2\kappa} \sum_{n=-\infty}^{\infty} \left[5n + \omega_n^{(1)} + \omega_n^{(2)} + \omega_n^{(3)} - 4(\omega_{n+}^f + \omega_{n-}^f) \right]
$$

- UV convergent; expand at small S; isolate low-lying modes

$$E^{(\mathrm{an})} = -\sqrt{\mathcal{S}} + \mathcal{O}(\mathcal{S}^{3/2})$$
$$E^{(\mathrm{nan})} \stackrel{?}{=} \underset{\substack{\uparrow \\ \text{from } n=0,\pm1,\pm2 \text{ modes}}}{-2} + \mathcal{O}(\mathcal{S})$$

◇ leading term in $E^{(\mathrm{nan})}$ is an artifact of representing E_1 in terms of characteristic frequencies. In a path integral/determinant approach it is absent. Adopt this approach.

$$
\begin{aligned}
E^{(\mathrm{an})} &= -\sqrt{\mathcal{S}} + \mathcal{O}(\mathcal{S}^{3/2}) \\
E^{(\mathrm{nan})} &\stackrel{?}{=} \; 0 + \mathcal{O}(\mathcal{S})
\end{aligned}
$$

$$
E = E_0 + E_1 = 2\sqrt{\sqrt{\lambda} S}\,\left[1 + \frac{S - \frac{1}{2}}{\sqrt{\lambda}} + \mathcal{O}(\frac{S^2}{\lambda})\right] + \mathcal{O}(\frac{S}{\sqrt{\lambda}})
$$

- interpolate to small $S = S_1 = S_2$; correct flat space lim: $S \mapsto S - 1$
- for $S = S_1 = S_2 = 2$ it is level-2 massive string state in $[0,0,0]_{(2,0)}$

  analogous solution exists in $[0,0,0]_{(0,2)}$

  Konishi descendants in this rep for $\Delta_0 = 2 + \frac{\ell}{2} = 4, 8$

$>$ in flat space it appears in $Q_L^2 Q_R^2 |0\rangle$ $\rightarrow$ suggests $\ell = 4,\ \Delta_0 = 4$

$\rightarrow$ $b_1 = 1$ and $\mathrm{b}_0 = -4$; expect $b_2 = 0$ after 2-loops

$\rightarrow$ dual operator is $\mathrm{Tr}\left[[D_{1+i2}, D_{3+i4}]^2\right]$

$S = J$ in AdS$_5\times$S^5

- put together the previous two solutions:

$$Y_0 + iY_5 = \sqrt{1+r^2}\ e^{i\kappa t}\ , \qquad Y_1 + iY_2 = r\ e^{i(w\tau+\sigma)}\ ,$$
$$X_1 + iX_2 = a\ e^{i(\tau-\sigma)}\ , \qquad X_3 + iX_4 = \sqrt{1-a^2}\ .$$

- EOM and Virasoro constraints

$$\left.\begin{aligned} w^2 &= \kappa^2 + 1 \\ (1+r^2)\kappa^2 &= r^2(w^2+1) + 2a^2 \\ \mathcal{S} = r^2 w &= a^2 = \mathcal{J} \end{aligned}\right\} \Rightarrow \kappa = \sqrt{\sqrt{\frac{1}{4}+2\mathcal{S}} - \frac{1}{2} + 2\mathcal{S}}$$

- Classical energy:

$$E_0 = \sqrt{\lambda}\mathcal{E}_0 = 2\sqrt{\sqrt{\lambda}S}\ \left[1 + \frac{S}{2\sqrt{\lambda}} - \frac{5S^2}{8\lambda} + \mathcal{O}(\frac{S^3}{\lambda^{3/2}})\right]$$

◇ quantum state in flat space limit (fixed S, $\lambda \to \infty$): shift $S \to S-1$

◇ Similarly to $J_1 = J_2$ state, has $\ell = 4$ in flat space lim.

$\rightarrow$ expect $b_1 = 1$ and b$_0 = -4$

1-loop quantum string correction to the energy:

- Adopt path integral approach for all modes

$$E_1 = \frac{1}{2\kappa}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\sum_{n=-\infty}^{\infty}\ln\frac{P_B(\omega,n,\mathcal{S})}{P_F(\omega,n,\mathcal{S})}$$

$$P_B = (\omega-n)^5(\omega+n)^6\Big[(\omega-n)^2-4(1-\mathcal{S})\Big]\Big[\omega^2-n^2+\tfrac{1}{2}(1-4\mathcal{S}-\sqrt{1+8\mathcal{S}})\Big]^2$$
$$\times\Big[(\omega-n)[(\omega+n)^2-4]+(3-8\mathcal{S}-3\sqrt{1+8\mathcal{S}})\omega-(1-\sqrt{1+8\mathcal{S}})n\Big]$$

$$P_F = [\omega^2-(n+1)^2]^3\ [\omega^2-(n-1)^2]^3$$
$$\times\Big[[\omega^2-(n+1)^2][\omega^2-(n-1)^2]+(1-3\omega^2+4\omega n+n^2)\mathcal{S}\Big]+\mathcal{O}(\mathcal{S}^2)$$

- $n=0,\pm1,\pm2$ may contribute to $E_1^{(\mathrm{nan})}$; analyze separately

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\ln\frac{P_B(\omega,-1,\mathcal{S})}{P_F(\omega,-1,\mathcal{S})}\frac{P_B(\omega,0,\mathcal{S})}{P_F(\omega,0,\mathcal{S})}\frac{P_B(\omega,+1,\mathcal{S})}{P_F(\omega,+1,\mathcal{S})}=\mathcal{O}(\mathcal{S})$$

$\rightarrow$ No leading order non-analyticity! $E_1^{(\mathrm{nan})}=\mathcal{O}(\frac{S}{\sqrt{\lambda}})$

- Extract leading $\mathcal{O}(\mathcal{S}^{1/2})$ analytic term in E_1

$\rightarrow$ expand in $\mathcal{S}$ and integrate: it vanishes $\rightarrow$ $E_1 = \mathcal{O}\left(\frac{S}{\sqrt{\lambda}}\right)$

- interpolate to small $S = J$; correct flat space lim: $S \mapsto S - 1$

  $\frac{1}{2\pi T} E^2 = 2(S + J - 2)$

$$E = 2\sqrt{\sqrt{\lambda}(S-1)}\ \left[1 + \frac{(S-1)}{2\sqrt{\lambda}} + \mathcal{O}(\frac{S^2}{\lambda})\right] + \mathcal{O}(\frac{S}{\sqrt{\lambda}})$$

- for $S = J = 2$ it is level-2 massive string state in $[0,2,0]_{(1,1)}$

  Konishi descendants in this rep for $\Delta_0 = 2 + \frac{\ell}{2} = 4(1), 6(3), 8(1)$

$>$ in flat space it appears in $Q_L^2 Q_R^2 |0\rangle$ $\rightarrow$ suggests $\ell = 4,\ \ \Delta_0 = 4$

$\rightarrow$ $b_1 = 1$ and $b_0 = -4$; expect $b_2 = 0$ after 2-loops

Other solutions: folded strings

• Folded string in AdS: counterpart of

$$t = \kappa\tau \ , \qquad \mathrm{x}_1 \equiv x_1 + i x_2 = \ a \sin\sigma \ e^{i\tau} \qquad E_{\rm flat} = \sqrt{\tfrac{2}{\alpha'} S} \qquad S = \frac{a^2}{2\alpha'}$$

which is the semiclassical version of state on the leading Regge trajectory with $\frac{1}{2\pi T} E^2 = 2(S-2)$ and $V = e^{-itE}((\partial \mathrm{x} \bar\partial \mathrm{x})^{S/2} + \dots)$

$$E = E_0 + E_1 = \sqrt{2\sqrt{\lambda} S}\ \Big[1 + \frac{\frac{3}{8}S - \frac{1}{4}}{\sqrt{\lambda}} + \mathcal{O}(\frac{S^2}{\lambda})\Big] + 2 + \mathcal{O}(\frac{S}{\sqrt{\lambda}})$$

- correct flat space limit: $S \mapsto S-2$
- for $S=4$ it is level-2 massive string state in $[0,0,0]_{(2,2)}$

$$E = 2\lambda^{1/4}\Big[1 + \frac{1}{2\sqrt{\lambda}} + \mathcal{O}(\frac{1}{\lambda})\Big] + 2 + \mathcal{O}(\frac{1}{\sqrt{\lambda}})$$

unique Konishi descendants in this rep for $\Delta_0 = 2 + \frac{\ell}{2} = 6(1)$

◇ consistent with $\mathrm{b}_0 = -4$ and $b_1 = 1$

- folded string in S^5

$$E = E_0 + E_1 = \sqrt{2\sqrt{\lambda}(J-2)}\,\left[1 + \frac{\frac{1}{8}(J-2)+\frac{1}{4}}{\sqrt{\lambda}} + \mathcal{O}(\frac{J^2}{\lambda})\right] + 2 + \mathcal{O}(\frac{J}{\lambda})$$

- $J = 4$ corresponds to 1st excited level in rep $[0,4,0]_{(0,0)}$

  unique Konishi descendants in this rep for $\Delta_0 = 2 + \frac{\ell}{2} = 6(1)$

◇ consistent with $b_0 = -4$ and $b_1 = 1$

- folded string with 2 spins $S = J$: cancellation of quantum corrections to analytic part; flat space limit $S \mapsto S - 1$

$$E = 2\sqrt{\sqrt{\lambda}(S-1)}\,\left[1 + \frac{(S-1)}{2\sqrt{\lambda}} + \mathcal{O}(\frac{S^2}{\lambda})\right] + 2 + \mathcal{O}(\frac{S}{\sqrt{\lambda}})$$

- $S = 2$ corresponds to 1st excited level in rep $[0,2,0]_{(1,1)}$

  Konishi descendants in this rep for $\Delta_0 = 2 + \frac{\ell}{2} = 4(1), 6(3), 8(1)$

◇ symmetry and analogy w/ circular strings: choose $\Delta_0 = 6$

consistent with $b_0 = -4$ and $b_1 = 1$

Summary or, what does it all mean

- argued that, if interpolation of semiclassical results to small charges holds, the strong coupling dimension of Konishi multiplet is

$$E = 2\lambda^{1/4} + (\Delta_0 - 4) + \frac{1}{\lambda^{1/4}} + \mathcal{O}(\lambda^{-3/4})$$

for $\Delta_0 = 6$ it is also a solution of $E(E-4) = 4\sqrt{\lambda} + 0 + \mathcal{O}(\sqrt{\lambda}^{-1})$

- b_0 and b_1 are rational and b_3 is transcendental
- conjectured that b_2 vanishes; shift+ series in $\lambda^{(2k+1)/4}$; $\zeta(3) \in b_3$
- 3 circular strings at $\Delta_0 = 4$; 3 folded strings at $\Delta_0 = 6$, latter on Regge trajectory
- differs from result from TBA in Y variables in $SL(2)$ sector

$$E = 2\lambda^{1/4} + (\Delta_0 - 4) + \frac{2}{\lambda^{1/4}} + \mathcal{O}(\lambda^{-3/4})$$

Gromov, Kazakov, Vieira

How many can be correct?

Both: issues with interpretation

$>$ TBA/Y and continuation of semiclassical results describe different operators at same string level (e.g. slightly different KK modes)

$>$ interpolation to small charges does not yield E but corrections to mass. Then

$$E(E-p)+q=E_{sc}^2$$

Principle determining p and q?

One: interpolation of semiclassical results (or some other manipulations) is not trustworthy

e.g. for circular strings $J\mapsto J-1$; what if $J\mapsto J-1+\mathcal{O}((\sqrt{\lambda})^{-1})$

e.g. exponentially small corrections become important

None?

Summary or, what does it all mean

- argued that, if interpolation of semiclassical results to small charges holds, the strong coupling dimension of Konishi multiplet is

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$$E = 2\lambda^{1/4} + (\Delta_0 - 4) + \frac{2}{\lambda^{1/4}} + \mathcal{O}(\lambda^{-3/4})$$

Gromov, Kazakov, Vieira

A better understanding of GS string in $AdS_5 \times S^5$ appears necessary