YANGIANS AND INTEGRABILITY IN ADS/CFT

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PLAN OF THE TALK

- **S-matrix and Hopf algebra**
- **Non-abelian symmetries and the Yangian**
- **Quest for universal R-matrix and the quantum Double**
- **Classical r-matrix**
- **A secret symmetry**
- **Yangian representations and the Bound State S-matrix**
- **Applications and Open problems**

... a Journey through Symmetries [1]*...*

[1]Sincere apologies to many beautiful papers which have not been included

S-MATRIX and HOPF ALGEBRA

Elementary excitations (magnons) scatter through S-matrix

[Staudacher ’04; Beisert ’05]

$$R \; : \quad V_1 \otimes V_2 \longrightarrow V_1 \otimes V_2$$

$$S \; = \; PR \qquad P = (graded)\, perm$$

V_i **carries a representation of (two-copies of) centrally-extended** $\mathfrak{psu}(2|2) := A$

S-matrix encodes info on dynamics, therefore its symmetries are important

Action of symmetry generators on 2-particle states (‘in’) given by ‘coproduct’

$$\Delta \; : \quad A \longrightarrow A \otimes A$$

such that $[\Delta(a), \Delta(b)] = \Delta([a,b])$ **(homo) and**

$$(P\Delta)\, R \; = \; R\, \Delta$$

$P\Delta$ **is called the ‘opposite’ coproduct** Δ^{op} **(‘out’)**

To begin with: LIE SUPERALGEBRA SYMMETRY

$$[\mathbb{L}_a{}^b, \mathbb{J}_c] = \delta_c^b \mathbb{J}_a - \tfrac{1}{2}\delta_a^b \mathbb{J}_c \qquad [\mathbb{R}_\alpha{}^\beta, \mathbb{J}_\gamma] = \delta_\gamma^\beta \mathbb{J}_\alpha - \tfrac{1}{2}\delta_\alpha^\beta \mathbb{J}_\gamma$$
$$[\mathbb{L}_a{}^b, \mathbb{J}^c] = -\delta_a^c \mathbb{J}^b + \tfrac{1}{2}\delta_a^b \mathbb{J}^c \qquad [\mathbb{R}_\alpha{}^\beta, \mathbb{J}^\gamma] = -\delta_\alpha^\gamma \mathbb{J}^\beta + \tfrac{1}{2}\delta_\alpha^\beta \mathbb{J}^\gamma$$
$$\{\mathbb{Q}_\alpha{}^a, \mathbb{Q}_\beta{}^b\} = \epsilon_{\alpha\beta}\epsilon^{ab}\mathbb{C} \qquad \{\mathbb{G}_a{}^\alpha, \mathbb{G}_b{}^\beta\} = \epsilon^{\alpha\beta}\epsilon_{ab}\mathbb{C}^\dagger$$
$$\{\mathbb{Q}_\alpha^a, \mathbb{G}_b^\beta\} = \delta_b^a \mathbb{R}_\alpha{}^\beta + \delta_\alpha^\beta \mathbb{L}_b{}^a + \tfrac{1}{2}\delta_b^a \delta_\alpha^\beta \mathbb{H}$$

Dynamical Spin-Chain Picture

$$\mathbb{H}\,|p\rangle \;=\; \epsilon(p)\,|p\rangle$$

$$\mathbb{C}\,|p\rangle \;=\; c(p)\,|p\,Z^-\rangle \qquad\qquad \mathbb{C}^\dagger\,|p\rangle \;=\; \bar{c}(p)\,|p\,Z^+\rangle$$

$Z^{+(-)}$**: one site of the chain is added (removed)**

On 2-particle states the action is non-local:

$$\mathbb{C}\otimes 1\,|p_1\rangle \otimes |p_2\rangle =$$

$$\mathbb{C}\otimes 1 \sum_{n_1<<n_2} e^{i\,p_1\,n_1 + i\,p_2\,n_2}\,|\cdots Z\,Z\,\phi_1\,\underbrace{Z\cdots Z}_{n_2-n_1-1}\,\phi_2\,Z\cdots\rangle$$

$$(rescale\,n_2) \quad = \; c(p_1)\,e^{ip_2}\,|p_1\rangle\otimes|p_2\rangle$$

$$S\,\Delta(\mathbb{C}) \;= S\,[\mathbb{C}\otimes 1 + 1\otimes\mathbb{C}] \;= S\,[e^{ip_2}\mathbb{C}_{local}\otimes 1 + 1\otimes\mathbb{C}_{local}]$$

$$\Delta(\mathbb{C}_{local}) = \mathbb{C}_{local}\otimes e^{ip} + 1\otimes\mathbb{C}_{local}$$

[Gomez-Hernandez ’06; Plefka-Spill-AT ’06]

Similar coproduct arises for the other (super)charges, controlled by a quantum number $[[Q]]$ s.t.

$$\Delta(\mathbb{Q}) = \mathbb{Q} \otimes e^{i[[Q]]p} + 1 \otimes \mathbb{Q}$$

In the presence of *central* elements C, there is consistency requirement:

$$P\Delta(\mathbb{C})\, R \;=\; R\,\Delta(\mathbb{C}) \;=\; \Delta(\mathbb{C})\, R$$

therefore

$$P\Delta(\mathbb{C}) \;=\; \Delta(\mathbb{C})$$

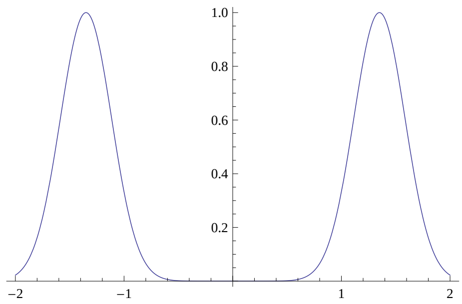
(coproduct is said to be *co*-commutative)

In our case, this is guaranteed by physical request

$$e^{ip} = \kappa\, C \;+\; 1$$

for a constant κ [straightforw. proof]

- **One can check all the axioms of Hopf algebras are satisfied**



STRING WORLDSHEET PICTURE

Coproduct reproduced from Bernard-LeClair procedure

[Klose-McLoughlin-Roiban-Zarembo '06]

Alternative classical argument. Light-cone worldsheet supercharges have non-locality

$$Q = \int_{-\infty}^{\infty} d\sigma\, J(\sigma)\, e^{i \int_{-\infty}^{\sigma} d\sigma'\, \partial x^-(\sigma')}$$

[Arutyunov-Frolov-Plefka-Zamaklar '06]

Imagine two well-separated soliton excitations ("scattering state"). Define semiclassical action of charge

$$\begin{aligned} Q_{|profile} &= \int_{-\infty}^{\infty} d\sigma\, J(\sigma)_{|profile}\, e^{i \int_{-\infty}^{\sigma} d\sigma'\, \partial x^-(\sigma')_{|profile}} \\ &= \int_{-\infty}^{0} d\sigma\, J(\sigma) e^{i \int_{-\infty}^{\sigma} d\sigma'\, \partial x^-(\sigma')} \\ &\quad + \int_{0}^{\infty} d\sigma\, J(\sigma)\, e^{i \int_{-\infty}^{0} d\sigma'\, \partial x^-(\sigma')}\, e^{i \int_{0}^{\sigma} d\sigma'\, \partial x^-(\sigma')} \\ &\sim\; Q_1 + e^{ip_1} Q_2 \quad \longrightarrow \quad \Delta(Q) = Q \otimes 1 + e^{ip} \otimes Q \end{aligned}$$

CROSSING SYMMETRY

Hopf-algebra antipode $\Sigma : A \longrightarrow A$ is defined as

$$m\,(\Sigma \otimes 1)\Delta(Q) \;=\; \text{“}\mathbf{0}\text{”}$$

where

$$m\,(a \otimes b) \;=\; ab$$

Derive from it antiparticle representation $\tilde{Q}$:

$$\Sigma(Q) \;=\; C^{-1}\,\tilde{Q}^{st}\,C$$

with C charge-conjugation matrix.

Possible to write down crossing symmetry of S-matrix

[Janik '06]

$$(\Sigma \otimes 1)\,R \;=\; (1 \otimes \Sigma^{-1})\,R \;=\; R^{-1}$$

directly from the dynamical generators: $\Sigma(Q) = -e^{-ip}Q$

- **Reformulation as a Faddeev-Zamolodchikov algebra**

[Arutyunov-Frolov-Zamaklar '06]

$$A_1\,A_2 \;=\; S\,A_2\,A_1$$

YANGIANS

∃ Lie superalgebra Q^A. Suppose ∃ additional charges $\hat{Q}^A$

$$[Q^A, Q^B] = i f^{AB}_C \, Q^C \qquad [Q^A, \hat{Q}^B] = i f^{AB}_C \, \hat{Q}^C$$

(plus Serre) with coproducts

$$\Delta(Q^A) = Q^A \otimes 1 + 1 \otimes Q^A$$
$$\Delta(\hat{Q}^A) = \hat{Q}^A \otimes 1 + 1 \otimes \hat{Q}^A + \frac{i}{2} f^A_{BC} Q^B \otimes Q^C$$

[Drinfeld '86]

(Infinite) Spin-Chain [Dolan-Nappi-Witten '03, Agarwal-Rajeev '04, Zwiebel '06, Beisert-Zwiebel '07]

Classical String [Mandal-Suryanarayana-Wadia '02, Bena-Polchinski--Roiban '03, Hatsuda-Yoshida '04, Das-Maharana-Melikyan-Sato '04]

S-matrix Yangian [Beisert '07]

We know modification

$$\Delta(Q^A) = Q^A \otimes 1 + e^{i\,[[A]]\,p} \otimes Q^A$$

Additionally, ∃ centrally-extended $\mathfrak{psu}(2|2)$ Yangian

$$\Delta(\hat{Q}^A) = \hat{Q}^A \otimes 1 + e^{i\,[[A]]\,p} \otimes \hat{Q}^A + \frac{i}{2} f^A_{BC} \, Q^B \, e^{i\,[[C]]\,p} \otimes Q^C$$

REMARKS

- **Evaluation representation:**

$$\hat{Q}^A = u\,Q^A = ig\,(x^+ + \frac{1}{x^+} - \frac{i}{2g})\,Q^A$$

(satisfies comm. rel.s)

- f^A_{BC} **needs** f^{AB}_C **and inverse Killing form** G^{-1}_{AB}

For $\mathfrak{psu}(2|2)$**, this does not exist, yet table of coproducts can be fully determined (cf. extension by automorph.s**

[Spill 'dipl.thesis, Beisert '06])

- **Yangian coproduct is non-local**

- **Traditionally, Yangian symmetry in evaluation representation implies** *difference form*

$$S = S(u_1 - u_2)$$

S-matrix is known *NOT* **to possess this symmetry (**u **depends on** $x^{\pm}$**), but let us keep it in mind...**

- **For higher bound-states, either YBE or Yangian symmetry have to be used to completely fix S-matrix**

[Arutyunov-Frolov '08, de Leeuw '08]

DO WE HAVE A CONTINUUM PICTURE?

Take a 2D classical field theory, with local currents

$$J_\mu = J^A_\mu \, T_A \qquad \partial^\mu J^A_\mu \, = \, 0 \qquad Q^A \, = \, \int_{-\infty}^{\infty} dx \, J^A_0$$

satisfying flatness (Lax pair)

$$\partial_0 J_1 - \partial_1 J_0 + [J_0, J_1] \, = \, 0$$

The following non-local current is conserved

$$\hat{J}^A_\mu = \epsilon_{\mu\nu} \, J^{\nu,A} + \frac{i}{2} \, f^A_{BC} \, J^B_\mu \int_{-\infty}^{x} dx' \, J^C_0(x')$$

$$\frac{d}{dt} \, \hat{Q}^A = \frac{d}{dt} \int_{-\infty}^{\infty} dx \, \hat{J}^A_0(x) = 0$$

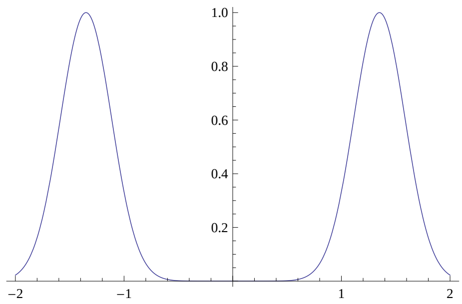
Prototype: Principal Chiral Model

$$L = Tr[\partial_\mu \, g^{-1} \, \partial^\mu \, g] \qquad g \in Lie$$

(left,right) global symmetry $g \longrightarrow e^{i\lambda} g, \, g \, e^{i\lambda}$

Noether current is flat

$$J^{L,R} = (\partial_\mu g) g^{-1}, \, g^{-1} \, (\partial_\mu g) \, \in lie$$



Let us repeat semiclassical argument:

$$\hat{Q}^A = \int_{-\infty}^{\infty} dx\, J_1^A(x) + \frac{i}{2} f^A_{BC} \int_{-\infty}^{\infty} dx\, J_0^B(x) \int_{-\infty}^{x} dx'\, J_0^C(x')$$

Evaluating on profile

$$\begin{aligned}\hat{Q}^A_{profile} \;\; = \;\; & \int_{-\infty}^{0} J_1^A + \frac{i}{2} f^A_{BC} \int_{-\infty}^{0} J_0^B \int_{-\infty}^{x} J_0^C \\ & + \int_0^{\infty} J_1^A + \frac{i}{2} f^A_{BC} \int_0^{\infty} J_0^B \int_0^{x} J_0^C \\ & + \frac{i}{2} f^A_{BC} \int_0^{\infty} J_0^B \int_{-\infty}^{0} J_0^C\end{aligned}$$

[Luescher-Pohlmeyer ’78, MacKay ’92]

naturally brings to

$$\Delta(\hat{Q}^A) = \hat{Q}^A \otimes 1 + 1 \otimes \hat{Q}^A + \frac{i}{2} f^A_{BC}\, Q^B \otimes Q^C$$

Quantization of this action **in absence of anomalies** $\rightarrow$
Hopf algebra rep on Hilbert space (see also [Luescher ’78]**).**

[Something similar should happen for string w.sheet...]

SOME EXPECTED CONSEQUENCES

Yangian is an infinite-dimensional non-abelian symmetry algebra

- **(Semiclassical and quantum) S-matrix gets very constrained**

- **Spectrum degeneracies are organized in Yangian irreps:** *spectrum generating algebra* [**cf. angular momentum**]

$$H\,\hat{Q}\,|\psi\rangle \;=\; \hat{Q}\,H\,|\psi\rangle \;=\; \epsilon\,\hat{Q}\,|\psi\rangle$$

- **Transfer matrix may enjoy Kirillov-Reshetikhin benefits**

$\longrightarrow$

- **Whole mathematics of Yangian doubles and rational R-matrices enters the game (quantum groups, in general)**

{*for rev*}[Chari-Pressley '94, Etingof-Schiffman '98, MacKay '04, Molev '07]

$\longrightarrow$

YANGIANS AND BETHE ANSATZ

[Kirillov-Reshetikhin '86, '87]

Given a rational solution of YBE $R_{12}(u)$, Yangian Y can be generated by T^k_{ij}, $k \geq 1$ and $i, j = 1, \cdots N$, s.t.

$$R_{12}(u)\, T_1(u+v)\, T_2(v) = T_2(v)\, T_1(u+v)\, R_{12}(u)$$

$$T(u) = 1 + \sum_{n\geq 1} u^{-n}\, T^k_{ij}\, E^{ij}$$

with $(E^{ij})_{kl} = \delta^i_k\, \delta^j_l$ and Hopf algebra coproduct

$$\Delta(T_{ij}(u)) = T_{ik}(u+v) \otimes T_{kj}(u)$$

$\exists$ abelian subalgebra $\mathcal{T}$ generated by $t_k(u)$, obtained antisymmetrizing k-th tensor product of $T_{ij}(u)$

'Quantum det' $t_N(u)$ generates center of the Yangian

- **Example:** $gl(N)$ $\qquad T_{ij}(u) = \delta_{ij} + \frac{E_{ij}}{u-v}$

Common eigenvectors of $\mathcal{T}$ are in one-to-one with solutions of Bethe equations

$\mathcal{T}$ commutes with $gl(N) \subset Y$, therefore it classifies multiplicities of irreps in tensor products of $gl(N)$

UNIVERSAL R-MATRIX

Given H non *co*-commutative Hopf algebra ($P\Delta \neq \Delta$), suppose $\exists$ abstract solution $R \in$ H $\otimes$ H of

$$(P\Delta)\, R \,=\, R\,\Delta$$

Universal means independent of representations in each factors of $\otimes$

Stand. Yangian is one such H: *"There is so much symmetry, that S-matrix can be written purely in terms of generators of symmetry algebra!"*

Theorem (Drinfeld): if R satisfies Quasi-Triangularity (rep-independent version of bootstrap principle)

$$(\Delta \otimes 1)R = R_{13}\, R_{23}$$
$$(1 \otimes \Delta)R = R_{13}\, R_{12}$$

then it also satisfies YBE and Crossing •

Direct proof of properties of S-matrix

Complete solution to scattering problem reduces to:

find the abstract tensor R given H, and then project it into your favorite (bound-state) rep

More than that: LeClair-Smirnov

Solve/Construct the model from universal R-matrix, its intertwining properties, and the representation theory of the associated quantum group

[LeClair-Smirnov '92]

Can we answer Staudacher's last year question:
"What is ultimately the model we are diagonalizing?"
using the representation theory of the Yangian?

Hic sunt leones. **What would Drinfeld do?**

Study perturbation of YBE around identity, and classify

[Belavin-Drinfeld '82]

Suppose

$$R \sim 1 \otimes 1 + \hbar\, r + \mathcal{O}(\hbar^2)$$

$r \in lie \otimes lie$ **(cf. exponential map) is the** *classical r-matrix* **and satisfies simpler equation to study: ClassicalYBE**

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

CLASSICAL r-MATRIX

Classical limit encodes info on quantum

Theorem (Belavin-Drinfeld): if $r(u_1 - u_2) \in lie \otimes lie$ **solves CYBE and [...] has a simple pole in** $u_1 - u_2 = 0$ **with residue** $C_2 =$ **quadratic Casimir of** $lie \otimes lie$**, then**

$\longrightarrow$ **then** r **is unitary, i.e.** $r_{12}(u_1 - u_2) = -r_{21}(u_2 - u_1)$**, meromorphic in plane** $u = u_1 - u_2$**, all poles are simple and form a lattice** Γ

- $dim\,\Gamma = 2$ $\longrightarrow$ **elliptic r-matrix/elliptic q.group**
- $dim\,\Gamma = 1$ $\longrightarrow$ **trigonom. r-matrix/(affine) q.group**
- $dim\,\Gamma = 0$ $\longrightarrow$ **rational r-matrix/Yangian**

How do we see this? *Factorization*:

Yang's example ('67)

$r = \frac{C_2}{u_2 - u_1}$ **solves CYBE (by def** $[C_2, Q^A \otimes 1 + 1 \otimes Q^A] = 0$**)**

$$r = \frac{C_2}{u_2 - u_1} = \frac{Q^A \otimes Q_A}{u_2 - u_1} = \sum_{n \geq 0} Q^A u_1^n \otimes Q_A u_2^{-n-1} = \sum_{n \geq 0} Q_n^A \otimes Q_{A,-n-1}$$

$[Q_n^A, Q_m^B] = i\, f_C^{AB}\, Q_{n+m}^C$: **loop algebra ... = classical Yangian**

Theorem II:

[Belavin-Drinfeld '83]

if r not of difference form, but dual Coxeter number of lie nonzero, $\exists$ change of variables to difference form

REMARK

- **r-matrix controls Poisson brackets of classical L-oper and contains seed of quantization**

$(R \sim 1 \otimes 1 + \hbar\, r + reconstructible)$

WHAT ABOUT AdS/CFT S-MATRIX?

$R \sim 1 \otimes 1 + \hbar r$ **in near BMN limit**

- **$\hbar$ is $1/g$ (with g related to 't Hooft coupling)**
- **r is tree-level string r-matrix**

[Klose-McLoughlin-Roiban-Zarembo '06]

- **Classical representation variable**

$$x^{\pm}(x) = x\left(\sqrt{1 - \frac{1}{g^2(x-\frac{1}{x})^2}} \pm \frac{i}{g(x-\frac{1}{x})}\right) \;\to\; x$$

[Arutyunov-Frolov '06]

LET US TRY

[AT '07]

- **Algebra tends to a limiting centrally-extended $\mathfrak{psu}(2|2)$**

- **Classical r-matrix is not diff-form (Belavin-Drinfeld [2] not applicable),** *nevertheless* **has a simple pole at origin** $x_1 - x_2 = 0$ **(**$dim\Gamma = 0$**, suggests Yangians...)**

- **Th. [easy]: To satisfy CYBE with such pole, residue** *must be* **Casimir of** *lie* $\otimes$ *lie*

  For centr-ext $\mathfrak{psu}(2|2)$ we don't have it...

  How to get away?

  Residue at origin is quadr. Casimir of $\mathfrak{gl}(2|2)$!

  Just on pole, borrow an extra generator B*=diag*$(1, 1, -1, -1)$ *from nearest non-degenerate superalgebra, to respect Th. [easy]*
  Away from residue B is broken

- **Only on pole, $\exists$ change of variables to diff-form (consistent with Bel.-Drinf. Th. II)**

> **Such** *borrowing* **reminds of math prescription for universal R-matrices based on degenerate Cartan matrices**
>
> [Khoroshkin-Tolstoy '91]

[2] Super-version $\longrightarrow$ consult [Leites-Serganova '84]

- **Khoroshkin-Tolstoy prescription: universal R-matrix goes**

$$R = \prod_{roots} e^{E^+ \otimes E^-} \quad e^{a_{ij}^{-1} H^i \otimes H^j} \quad \prod_{roots} e^{E^- \otimes E^+}$$

$E^\pm$ **are roots of** *lie*, H_i **Cartan generators and** a_{ij} **Cartan matrix**

"If degenerate, add H_k's until you can invert it"

- **Expected we had to call in an** $H_4 = B$ **soon or later**

  We may have a chance of factorizing $\longrightarrow$

- **Remark:** $\mathfrak{gl}(2|2)$ **Casimir is well-known from opposite regime** $g = 0$

$$R_{1\,loop} \sim 1 \otimes 1 \; + \frac{C_2^{\mathfrak{gl}(2|2)}}{u_1 - u_2}$$

(up to twists). Yang's *quantum* **R-matrix, prototype for QYBE**

YANGIAN DOUBLES

- **Remember Yang**

$$r = \frac{C_2}{u_2 - u_1} = G^{-1}_{AB}\,\frac{Q^A \otimes Q^B}{u_2 - u_1} = \sum_{n\geq 0} G^{-1}_{AB}\,Q^A_n \otimes Q^B_{-n-1}$$

- **$\exists$ way (surpassed by history)**

[Moriyama-AT '07]

$$r = \sum_{n\geq 0} \mathbb{G}^\alpha_{a,n}\otimes\hat{\mathbb{Q}}^a_{\alpha,-n-1} - \mathbb{Q}^a_{\alpha,n}\otimes\hat{\mathbb{G}}^\alpha_{a,-n-1} + \mathbb{H}_n\otimes\hat{\mathbb{B}}_{-n-1} + \mathbb{B}_n\otimes\hat{\mathbb{H}}_{-n-1}$$
$$+ (\mathbb{L}^a_{b,n}\otimes\hat{\mathbb{L}}^b_{a,-n-1} - \mathbb{L}^a_{b,-n-1}\otimes\hat{\mathbb{L}}^b_{a,n}) - (\mathbb{R}^\alpha_{\beta,n}\otimes\hat{\mathbb{R}}^\beta_{\alpha,-n-1} - \mathbb{R}^\alpha_{\beta,-n-1}\otimes\hat{\mathbb{R}}^\beta_{\alpha,n})$$

for an enlarged Cartan matrix $a^{-1}_{ij}H^iH^j = 4\mathbb{H}\mathbb{B} + \mathbb{L}^2 - \mathbb{R}^2$

Bonus: $B_n = \frac{1}{2}(x^n - x^{-n})diag(1,1,-1,-1)$ **vanishes at** $n = 0$

(indeed, $\exists$ no such Lie algebra symmetry)

- **For higher n (higher Yangian generators), suggests $\exists$ of additional (non-local) symmetry of B-type**

Is this additional symmetry confirmed at quantum level?

YES

Matsumoto-Moriyama-AT '07, Beisert-Spill '07

SECRET SYMMETRY

$\exists$ (first level) Yangian symmetry of quantum S-matrix

$$\Delta(\hat{B}) = \hat{B}\otimes 1 \,+\, 1\otimes\hat{B} + \frac{i}{2g}(\mathbb{G}^{\alpha}_{a}\otimes\mathbb{Q}^{a}_{\alpha} \,+\, \mathbb{Q}^{a}_{\alpha}\otimes\mathbb{G}^{\alpha}_{a})$$

$$\Sigma(\hat{B}) \,=\, -\hat{B} \,+\, \frac{2i}{g}\mathbb{H}$$

$$\hat{B} \,=\, \frac{1}{4}(x^{+} + x^{-} - 1/x^{+} - 1/x^{-})\, diag(1,1,-1,-1)$$

[Do not be misled by appearances: formula is exact $\forall g$]

Generates through comm new type of Yangian susys. Consistent with classical limits, both obsolete ($\longleftarrow$) and new

New [Beisert-Spill '07]

$$r \;=\; \frac{\mathcal{T} - \tilde{B}\otimes\mathbb{H} - \mathbb{H}\otimes\tilde{B}}{i(u_1-u_2)} - \frac{\Sigma\otimes\mathbb{H}}{iu_2} + \frac{\mathbb{H}\otimes\Sigma}{iu_1} - \frac{\mathbb{H}\otimes\mathbb{H}}{\frac{2iu_1u_2}{u_1-u_2}}$$

$$\mathcal{T} = 2\left(\mathbb{R}_{\beta}{}^{\alpha}\otimes\mathbb{R}_{\alpha}{}^{\beta} - \mathbb{L}_{b}{}^{a}\otimes\mathbb{L}_{a}{}^{b} + \mathbb{G}_{a}{}^{\alpha}\otimes\mathbb{Q}_{\alpha}{}^{a} - \mathbb{Q}_{\alpha}{}^{a}\otimes\mathbb{G}_{a}{}^{\alpha}\right)$$

$$\tilde{B} = \frac{1}{2}\frac{1}{ad+bc}\; diag(1,1,-1,-1)$$

Nice classical double. Confirmed for bound states

[de Leeuw '08, Arutyunov-de Leeuw-AT '09]

Interesting questions: B_0 appears now explicitly, but how to make it a symmetry? (Does it have to...?)
Connection wt [Dorey-Vicedo '06, Mikhailov - Schaefer-Nameki '08, Magro '08]?

A STUBBORN BOY: THE DIFFERENCE FORM

- $\hat{B} = \frac{u}{\epsilon(p)} \, diag(1, 1, -1, -1)$ **also goes** $B_n \sim u^n B$

- r **has pretty** $\frac{1}{u_1 - u_2}$**'s in nice places**

- **∃ Drinfeld's second realization of the Yangian**

  Evaluation representation is still of type $Q_n \sim (u + y)^n \, Q$

  [Spill-AT '08]

- **Connection with exceptional Lie algebra** $D(2, 1; \alpha)$ **should help localizing** $u_1 - u_2$ **dependence**

  [Beisert '05, Matsumoto-Moriyama '08, '09]

- **Looks like difference form** $u_1 - u_2$ **is almost there, because Yangian calls for it (Th. Also Easy)**

  Upon rep, it is then masked by additional dependence on u **of representation labels** $a(u), b(u), c(u), d(u)$ **entering the S-matrix**

- *Universal R-matrix should disentangle it!*

  Provocative rewriting of fundamental R-matrix [AT '08]
  (like Khoroshkin-Tolstoy's? What has future told?)

THE BOUND STATE S-MATRIX

[Arutyunov-de Leeuw-AT '09]

- **On one hand, desire of complete set of finite dimensional rep S-matrices, to figure out universal R-matrix (dreaming of a group-theoretic solution with nice ensuing math)**

- **On the other hand, we want finite-size. Integrability dictates: finite-size is obtained once we know the *entire* asymptotic data, including *all* scattering matrices**

- **So far, powerful conjectures for transfer matrix eigenvalues based on standard treatments (cf. Bazhanov-Reshetikhin)**

  [Beisert '07, Gromov-Kazakov-Vieira '09]

  and superbe success of Luescher's corrections

  [Fiamberti-Santambrogio-Sieg-Zanon '07, Bajnok-Janik '08, Bajnok-Janik--Lukowski '08]

  have nurtured fascinating constructions

  [Gromov-Kazakov-Vieira '09, Arutyunov-Frolov '09, Bombardelli-Fioravanti--Tateo '09, Frolov-Suzuki '09]

  But centrally-extended $\mathfrak{psu}(2|2)$ is very special, and a complete mathematical proof is still missing... Can we prove these conjectures through alternative path?

- **Let us follow a direct S-matrix approach $\longrightarrow$**

EXPLICIT CONSTRUCTION

We use superspace formalism (atypical totally symm rep)

[Arutyunov-Frolov '08]

$$
\begin{aligned}
\Phi(w,\theta) &= \sum_{\ell=0}^{\infty} \Phi_\ell(w,\theta) \\
\Phi_\ell &= \phi^{a_1\ldots a_\ell} w_{a_1}\ldots w_{a_\ell} + \phi^{a_1\ldots a_{\ell-1}\alpha} w_{a_1}\ldots w_{a_{\ell-1}}\theta_\alpha + \\
&\quad \phi^{a_1\ldots a_{\ell-2}\alpha\beta} w_{a_1}\ldots w_{a_{\ell-2}}\theta_\alpha\theta_\beta
\end{aligned}
$$

$$
\mathbb{L}_a{}^b = w_a\tfrac{\partial}{\partial w_b} - \tfrac{1}{2}\delta_a^b w_c \tfrac{\partial}{\partial w_c} \qquad \mathbb{R}_\alpha{}^\beta = \theta_\alpha\tfrac{\partial}{\partial\theta_\beta} - \tfrac{1}{2}\delta_\alpha^\beta\theta_\gamma\tfrac{\partial}{\partial\theta_\gamma}
$$

$$
\mathbb{Q}_\alpha{}^a = a\,\theta_\alpha\tfrac{\partial}{\partial w_a} + b\,\epsilon^{ab}\epsilon_{\alpha\beta}w_b\tfrac{\partial}{\partial\theta_\beta} \qquad \mathbb{G}_a{}^\alpha = d\,w_a\tfrac{\partial}{\partial\theta_\alpha} + c\,\epsilon_{ab}\epsilon^{\alpha\beta}\theta_\beta\tfrac{\partial}{\partial w_b}
$$

$$
\mathbb{C} = ab\left(w_a\tfrac{\partial}{\partial w_a} + \theta_\alpha\tfrac{\partial}{\partial\theta_\alpha}\right) \qquad \mathbb{C}^\dagger = cd\left(w_a\tfrac{\partial}{\partial w_a} + \theta_\alpha\tfrac{\partial}{\partial\theta_\alpha}\right),
$$

$$
\mathbb{H} = (ad+bc)\left(w_a\tfrac{\partial}{\partial w_a} + \theta_\alpha\tfrac{\partial}{\partial\theta_\alpha}\right)
$$

$$
a = \sqrt{\tfrac{g}{2\ell}}\eta \qquad b = \sqrt{\tfrac{g}{2\ell}}\tfrac{i\zeta}{\eta}\left(\tfrac{x^+}{x^-} - 1\right)
$$

$$
c = -\sqrt{\tfrac{g}{2\ell}}\tfrac{\eta}{\zeta x^+} \qquad d = \sqrt{\tfrac{g}{2\ell}}\tfrac{x^+}{i\eta}\left(1 - \tfrac{x^-}{x^+}\right)
$$

$$
\eta = e^{i\frac{p}{4}}\sqrt{ix^- - ix^+}
$$

$$
x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2i\ell}{g}
$$

The integer ℓ is the number of bound-state constituents

[Dorey '06, Chen-Dorey-Okamura '06, Roiban '06]

INVARIANT SUBSPACES

Because of $\mathfrak{su}(2)\times\mathfrak{su}(2)$ *invariance, S-matrix is block-diagonal*

Case I a, b**:** $2\times\ell_1\ell_2$ **vectors** $\in V^{\mathrm{I}}$ **(**a, b **for** $\alpha = 3,4$ **resp.)**

$$|k,l\rangle^{\mathrm{I}} \equiv \underbrace{\theta_\alpha w_1^{\ell_1-k-1}w_2^k}_{\text{Space1}}\ \underbrace{\vartheta_\alpha v_1^{\ell_2-l-1}v_2^l}_{\text{Space2}}$$

Case II a, b**:** $2\times 4\ell_1\ell_2$ **vectors** $\in V^{\mathrm{II}}$ **(**a, b **for** $\alpha = 3,4$ **resp.)**

$$\begin{aligned}
|k,l\rangle_1^{\mathrm{II}} &\equiv \underbrace{\theta_\alpha w_1^{\ell_1-k-1}w_2^k}\ \underbrace{v_1^{\ell_2-l}v_2^l}\\
|k,l\rangle_2^{\mathrm{II}} &\equiv \underbrace{w_1^{\ell_1-k}w_2^k}\ \underbrace{\vartheta_\alpha v_1^{\ell_2-l-1}v_2^l}\\
|k,l\rangle_3^{\mathrm{II}} &\equiv \underbrace{\theta_\alpha w_1^{\ell_1-k-1}w_2^k}\ \underbrace{\vartheta_3\vartheta_4 v_1^{\ell_2-l-1}v_2^{l-1}}\\
|k,l\rangle_4^{\mathrm{II}} &\equiv \underbrace{\theta_3\theta_4 w_1^{\ell_1-k-1}w_2^{k-1}}\ \underbrace{\vartheta_\alpha v_1^{\ell_2-l-1}v_2^l}
\end{aligned}$$

Case III: $6\ell_1\ell_2$ **vectors** $\in V^{\mathrm{III}}$

$$\begin{aligned}
|k,l\rangle_1^{\mathrm{III}} &\equiv \underbrace{w_1^{\ell_1-k}w_2^k}\ \underbrace{v_1^{\ell_2-l}v_2^l}\\
|k,l\rangle_2^{\mathrm{III}} &\equiv \underbrace{w_1^{\ell_1-k}w_2^k}\ \underbrace{\vartheta_3\vartheta_4 v_1^{\ell_2-l-1}v_2^{l-1}}\\
|k,l\rangle_3^{\mathrm{III}} &\equiv \underbrace{\theta_3\theta_4 w_1^{\ell_1-k-1}w_2^{k-1}}\ \underbrace{v_1^{\ell_2-l}v_2^l}\\
|k,l\rangle_4^{\mathrm{III}} &\equiv \underbrace{\theta_3\theta_4 w_1^{\ell_1-k-1}w_2^{k-1}}\ \underbrace{\vartheta_3\vartheta_4 v_1^{\ell_2-l-1}v_2^{l-1}}\\
|k,l\rangle_5^{\mathrm{III}} &\equiv \underbrace{\theta_3 w_1^{\ell_1-k-1}w_2^k}\ \underbrace{\vartheta_4 v_1^{\ell_2-l}v_2^{l-1}}\\
|k,l\rangle_6^{\mathrm{III}} &\equiv \underbrace{\theta_4 w_1^{\ell_1-k}w_2^{k-1}}\ \underbrace{\vartheta_3 v_1^{\ell_2-l-1}v_2^l}
\end{aligned}$$

S-matrix is of block-diagonal form

$$R = \begin{pmatrix} \boxed{\mathscr{X}} & & & & \\ & \boxed{\mathscr{Y}} & & 0 & \\ & & \boxed{\mathscr{Z}} & & \\ & 0 & & \boxed{\mathscr{Y}} & \\ & & & & \boxed{\mathscr{X}} \end{pmatrix}$$

$$\begin{aligned} \mathscr{X} : V^{\mathrm{I}} &\longrightarrow V^{\mathrm{I}} \\ |k,l\rangle^{\mathrm{I}} &\mapsto \sum_{m=0}^{k+l} \mathscr{X}_m^{k,l} |m, k+l-m\rangle^{\mathrm{I}} \end{aligned}$$

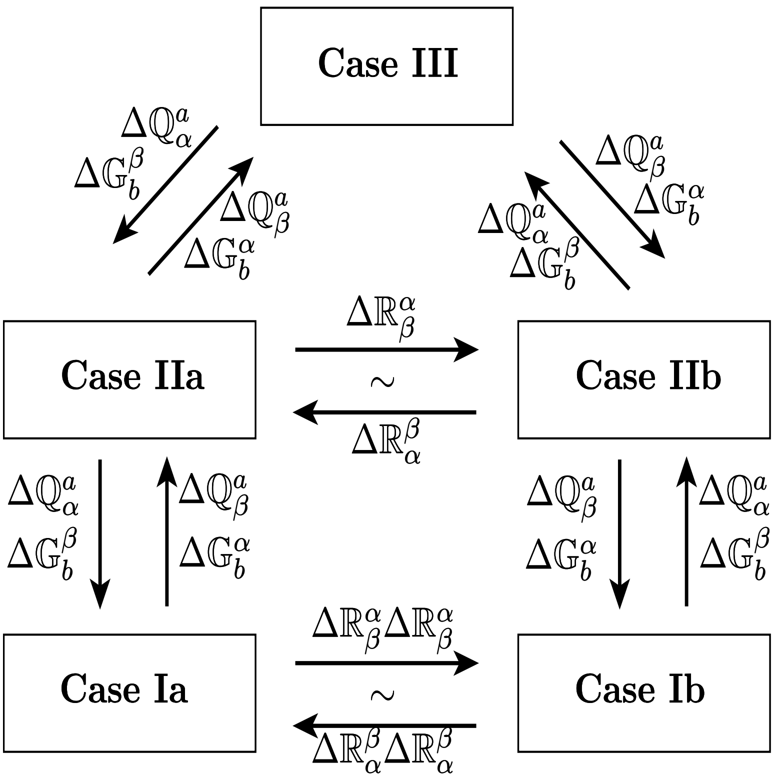
$$\begin{aligned} \mathscr{Y} : V^{\mathrm{II}} &\longrightarrow V^{\mathrm{II}} \\ |k,l\rangle_j^{\mathrm{II}} &\mapsto \sum_{m=0}^{k+l} \sum_{j=1}^{4} \mathscr{Y}_{m;i}^{k,l;j} |m, k+l-m\rangle_j^{\mathrm{II}} \end{aligned}$$

$$\begin{aligned} \mathscr{Z} : V^{\mathrm{III}} &\longrightarrow V^{\mathrm{III}} \\ |k,l\rangle_j^{\mathrm{III}} &\mapsto \sum_{m=0}^{k+l} \sum_{j=1}^{6} \mathscr{Z}_{m;i}^{k,l;j} |m, k+l-m\rangle_j^{\mathrm{III}} \end{aligned}$$

Full S-matrix is *two such copies* **times** *square of*

$$\begin{aligned} S_0(p_1,p_2) &= \left(\frac{x_1^-}{x_1^+}\right)^{\frac{\ell_2}{2}} \left(\frac{x_2^+}{x_2^-}\right)^{\frac{\ell_1}{2}} \sigma(x_1,x_2) \times \\ &\quad \times \sqrt{G(\ell_2-\ell_1)G(\ell_2+\ell_1)} \prod_{q=1}^{\ell_1-1} G(\ell_2-\ell_1+2q) \end{aligned}$$

$$G(\ell) = \frac{u_1-u_2+\frac{\ell}{2}}{u_1-u_2-\frac{\ell}{2}} \qquad u = \frac{g}{4i}\left(x^+ + \frac{1}{x^+} + x^- + \frac{1}{x^-}\right)$$



THE DERIVATION

Case I looks easiest, we start there $\longrightarrow$

We need exact solution for Case I. First, define a 'vacuum'

$$|0\rangle \equiv w_1^{\ell_1}\ v_1^{\ell_2}\ \in V^{\rm III}$$

such that $R|0\rangle = |0\rangle$, **and then use** $\Delta^{op}\,R = R\,\Delta$ **:**

$$\begin{aligned} R|0,0\rangle^{\rm I} &= R\,\frac{\Delta(\mathbb{Q}^1_3)\Delta(\mathbb{G}^4_2)|0\rangle}{(a_2c_1-a_1c_2)\ell_1\ell_2} = \frac{\Delta^{op}(\mathbb{Q}^1_3)\Delta^{op}(\mathbb{G}^4_2)R|0\rangle}{(a_2c_1-a_1c_2)\ell_1\ell_2} \\ &= \frac{x_1^- - x_2^+}{x_1^+ - x_2^-}\frac{e^{i\frac{p_1}{2}}}{e^{i\frac{p_2}{2}}}\,|0,0\rangle^{\rm I} \equiv \mathcal{D}|0,0\rangle^{\rm I} \end{aligned}$$

Bound states are evaluation representations of Yangian, with corresp. bound state parameter u

[de Leeuw '08]

One generates entire Case I from $|0,0\rangle^{\mathrm{I}}$, using Yangian:

$$|k,l\rangle^{\mathrm{I}} = \frac{\prod_{i=1}^{k}\left[\Delta(\hat{\mathbb{L}}_2^1) + \frac{\ell_1-2u_2-2i+1}{2}\Delta(\mathbb{L}_2^1)\right]\prod_{j=1}^{l}\left[-\Delta(\hat{\mathbb{L}}_2^1) - \frac{1+2j-2u_1-\ell_2}{2}\Delta(\mathbb{L}_2^1)\right]}{\prod_{r=1}^{k}(\ell_1-r)\prod_{p=1}^{l}(\ell_2-p)\prod_{q=1}^{k+l}\left(\delta u+\frac{\ell_1+\ell_2}{2}-q\right)}|0,0\rangle^{\mathrm{I}}$$

from which $\Delta^{op} R = R\Delta$ **gives (for** $\delta u = u_1 - u_2$**)**

$$R|k,l\rangle^{\mathrm{I}} = \mathcal{D}\times \frac{\prod_{i=1}^{k}\left[\Delta^{op}(\hat{\mathbb{L}}_2^1) + \frac{\ell_1-2u_2-2i+1}{2}\Delta^{op}(\mathbb{L}_2^1)\right]\prod_{j=1}^{l}\left[-\Delta^{op}(\hat{\mathbb{L}}_2^1) - \frac{1+2j-2u_1-\ell_2}{2}\Delta^{op}(\mathbb{L}_2^1)\right]}{\prod_{r=1}^{k}(\ell_1-r)\prod_{p=1}^{l}(\ell_2-p)\prod_{q=1}^{k+l}\left(\delta u+\frac{\ell_1+\ell_2}{2}-q\right)}|0,0\rangle^{\mathrm{I}}$$

Explicit computation produces

$$R|k,l\rangle^{\mathrm{I}} = \sum_{n=0}^{k+l}\mathscr{X}_n^{k,l}|n,k+l-n\rangle^{\mathrm{I}}$$

$$\mathscr{X}_n^{k,l} = \mathcal{D}\frac{\prod_{i=1}^{n}(\ell_1-i)\prod_{i=1}^{k+l-n}(\ell_2-i)}{\prod_{r=1}^{k}(\ell_1-r)\prod_{p=1}^{l}(\ell_2-p)\prod_{q=1}^{k+l}(\delta u+\frac{\ell_1+\ell_2}{2}-q)}\times$$
$$\times\sum_{m=0}^{k}\left\{\binom{k}{k-m}\binom{l}{n-m}\prod_{p=1}^{m}\mathfrak{c}_p^{+}\prod_{p=1-m}^{l-n}\mathfrak{c}_p^{-}\prod_{p=1}^{k-m}\mathfrak{d}_{\frac{k-p+2}{2}}\prod_{p=1}^{n-m}\tilde{\mathfrak{d}}_{\frac{k+l-m-p+2}{2}}\right\}$$

$$\mathfrak{c}_m^{\pm} = \delta u \pm \frac{\ell_1-\ell_2}{2} - m + 1 \qquad \tilde{\mathfrak{c}}_m^{\pm} = \delta u \pm \frac{\ell_1+\ell_2}{2} - m + 1$$
$$\mathfrak{d}_i = \ell_1 + 1 - 2i \qquad \tilde{\mathfrak{d}}_i = \ell_2 + 1 - 2i$$

Amplitude is restriction of Hypergeometric:

$$\mathscr{X}_n^{k,l} = (-1)^{k+n}\,\pi D \frac{\sin[(k-\ell_1)\pi]\,\Gamma(l+1)}{\sin[\ell_1\pi]\,\sin[(k+l-\ell_2-n)\pi]\,\Gamma(l-\ell_2+1)\Gamma(n+1)}$$
$$\times \frac{\Gamma(n+1-\ell_1)\Gamma\left(l+\frac{\ell_1-\ell_2}{2}-n-\delta u\right)\Gamma\left(1-\frac{\ell_1+\ell_2}{2}-\delta u\right)}{\Gamma\left(k+l-\frac{\ell_1+\ell_2}{2}-\delta u+1\right)\Gamma\left(\frac{\ell_1-\ell_2}{2}-\delta u\right)} \times$$
$${}_4\tilde{F}_3\left(-k,-n,\delta u+1-\frac{\ell_1-\ell_2}{2},\frac{\ell_2-\ell_1}{2}-\delta u;1-\ell_1,\ell_2-k-l,l-n+1;1\right)$$

where $\quad {}_4\tilde{F}_3(x,y,z,t;r,v,w;\tau) = {}_4F_3(x,y,z,t;r,v,w;\tau)/[\Gamma(r)\Gamma(v)\Gamma(w)]$

LUCKY SITUATION:

Our ${}_4F_3(a_i;b_j;1)$ **is 'balanced':** $\quad \sum a_i - \sum b_j = -1$

$\longrightarrow$ $6j$**-symbol**

$${}_4F_3\left(a_1,a_2,a_3,a_4;b_1,b_2,b_3;1\right) = \frac{(-1)^{b_1+1}\Gamma\left(b_2\right)\Gamma\left(b_3\right)\sqrt{\Gamma\left(1-a_1\right)\Gamma\left(1-a_2\right)\Gamma\left(1-a_3\right)}}{\Gamma\left(1-b_1\right)\sqrt{\Gamma\left(b_2-a_1\right)\Gamma\left(b_2-a_2\right)}} \times$$
$$\frac{\sqrt{\Gamma\left(1-a_4\right)\Gamma\left(a_1-b_1+1\right)\Gamma\left(a_2-b_1+1\right)\Gamma\left(a_3-b_1+1\right)\Gamma\left(a_4-b_1+1\right)}}{\sqrt{\Gamma\left(b_2-a_3\right)\Gamma\left(b_2-a_4\right)\Gamma\left(b_3-a_1\right)\Gamma\left(b_3-a_2\right)\Gamma\left(b_3-a_3\right)\Gamma\left(b_3-a_4\right)}} \times$$
$$\begin{Bmatrix} \frac{1}{2}\left(-a_1-a_4+b_3-1\right) & \frac{1}{2}\left(-a_1-a_3+b_2-1\right) & \frac{1}{2}\left(a_1+a_2-b_1-1\right) \\ \frac{1}{2}\left(-a_2-a_3+b_3-1\right) & \frac{1}{2}\left(-a_2-a_4+b_2-1\right) & \frac{1}{2}\left(a_3+a_4-b_1-1\right) \end{Bmatrix}$$

The relevant $6j$**-symbol** $\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$ **has coefficients**

$$j_1 = \frac{1}{2}\left(k+l-n+\frac{\ell_1-\ell_2}{2}+\delta u\right) \qquad j_2 = \frac{1}{2}\left(\frac{\ell_1+\ell_2}{2}-2-l-\delta u\right)$$
$$j_3 = \frac{1}{2}\left(\ell_1-2-k-n\right) \qquad j_4 = \frac{1}{2}\left(\frac{\ell_1-\ell_2}{2}-1+l-\delta u\right)$$
$$j_5 = \frac{1}{2}\left(\frac{\ell_1+\ell_2}{2}-1-k-l+n+\delta u\right) \qquad j_6 = \frac{1}{2}\left(\ell_2-1\right)$$

REMARKS

- **Case I amplitude shows correct poles**

- **Case I states are highest weight w.r.t 'fermionic' $\mathfrak{su}(2)$, and carry a rep of the 'bosonic' $\mathfrak{su}(2)$ Yangian**

- **Case I S-matrix has difference-form (apart overall factor), and is a $6j$-symbol with (half-)integer coefficients on physical poles** *(namely, this block exhibits standard fusion)*

... Wait...

is it the representation of the universal R-matrix of the Yangian of 'bosonic' $\mathfrak{su}(2)$ in *arbitrary* **evaluation representations (times an overall factor)?**

YES

[Arutyunov-de Leeuw-AT '09]

- **Since we are going to generate all other states and S-matrix blocks from case I, it looks like** *one factor* **of the full universal R-matrix is going to be:**

Khoroshkin-Tolstoy's $\mathfrak{su}(2)$-Yangian universal R-matrix, for the $\mathbb{L}$-generators

OTHER CASES

How do we generate the other cases' S-matrix?

General strategy schematically as follows:

- **one one hand**

$$\begin{aligned} R\,\Delta(\mathbb{Q})\,|Case\,\mathrm{II}\rangle_i \;=\; R\,Q_i\,|Case\,\mathrm{I}\rangle &= Q_i\,R\,|Case\,\mathrm{I}\rangle \\ &= Q_i\,\mathscr{X}\,|Case\,\mathrm{I}\rangle \end{aligned}$$

- **on the other hand**

$$\begin{aligned} R\,\Delta(\mathbb{Q})\,|Case\,\mathrm{II}\rangle_i &= \Delta^{op}(\mathbb{Q})\,R\,|Case\,\mathrm{II}\rangle_i \\ &= R_i^j\,\Delta^{op}(\mathbb{Q})\,|Case\,\mathrm{II}\rangle_j \;=\; R_i^j\,Q_j^{op}\,|Case\,\mathrm{I}\rangle \end{aligned}$$

From which

$$R_i^j = Q_i\,\mathscr{X}\,\left([Q^{op}]^{-1}\right)^j$$

Before being more specific, notice:

This construction automatically provides a *'factorizing twist'* **for the concrete S-matrix**

$$R = F_{21}F^{-1} \qquad \text{[Drinfeld '90]}$$

FULL THERAPY - CASE II

Define

$$\mathbb{S}|k,l\rangle_i^{\mathrm{II}} = \sum_{j=1}^{4}\sum_{m=0}^{k+l} \mathscr{Y}_{m;i}^{k,l;j}|m,k+l-m\rangle_j^{\mathrm{II}}$$

and notice that

$$\Delta\mathbb{Q}_3^1|k,l\rangle_j^{\mathrm{II}} = Q_j(k,l)|k,l\rangle^{\mathrm{I}}$$

$$Q_1(k,l) = a_2(l-\ell_2), \qquad Q_2(k,l) = a_1(\ell_1 - k)$$
$$Q_3(k,l) = b_2, \qquad Q_4(k,l) = -b_1$$

Apply general strategy:

$$^{\mathrm{I}}\langle n,N-n|\ \Delta^{op}\mathbb{Q}_3^1 R\ |k,l\rangle_i^{\mathrm{II}} = \sum_{j=1}^{4}\sum_{m=0}^{k+l} \mathscr{Y}_{m;i}^{k,l;j}\ ^{\mathrm{I}}\langle n,N-n|\ \Delta^{op}\mathbb{Q}_3^1\ |m,N-m\rangle_j^{\mathrm{II}}$$

$$= \sum_{j=1}^{4}\sum_{m=0}^{k+l} \mathscr{Y}_{m;i}^{k,l;j} Q_j^{op}(m,N-m)\ ^{\mathrm{I}}\langle n,N-n|m,N-m\rangle^{\mathrm{I}}$$

$$= \sum_{j=1}^{4} \mathscr{Y}_{n;i}^{k,l;j} Q_j^{op}(n,N-n)$$

$$^{\mathrm{I}}\langle n,N-n|\ \Delta^{op}\mathbb{Q}_3^1 R\ |k,l\rangle_i^{\mathrm{II}} =^{\mathrm{I}} \langle n,N-n|\ R\Delta\mathbb{Q}_3^1\ |k,l\rangle_i^{\mathrm{II}}$$

$$= Q_i(k,l)^{\mathrm{I}}\langle n,N-n|\ R\ |k,l\rangle^{\mathrm{II}} = Q_i(k,l)\sum_{m=0}^{N} \mathscr{X}_m^{k,l}\ ^{\mathrm{I}}\langle n,N-n|m,N-m\rangle^{\mathrm{I}}$$

$$= Q_i(k,l)\mathscr{X}_n^{k,l}$$

This gives four linear equations. Similarly, using $\Delta^{op}\mathbb{G}_2^4$ gives other four. Not enough, need Yangian

$$\Lambda_1 = \Delta(\hat{\mathbb{Q}}_3^1) + \frac{2\Delta\hat{\mathbb{L}}_2^1\Delta(\mathbb{Q}_3^2)}{\ell_1+\ell_2-2(N+1+\delta u)} - \frac{\ell_1-\ell_2+2(N-2n+u_1+u_2)}{2(\ell_1+\ell_2)-4(N+1+\delta u)}\Delta\mathbb{L}_2^1\Delta(\mathbb{Q}_3^2)$$

$$\Lambda_2 = \Delta(\hat{\mathbb{G}}_2^4) + \frac{2\Delta\hat{\mathbb{L}}_2^1\Delta(\mathbb{G}_1^4)}{\ell_1+\ell_2-2(N+1+\delta u)} + \frac{\ell_1-\ell_2+2(N-2n+u_1+u_2)}{2(\ell_1+\ell_2)-4(N+1+\delta u)}\Delta\mathbb{L}_2^1\Delta(\mathbb{G}_1^4)$$

where $N = k + l$. These operators satisfy

$$^{\mathrm{I}}\langle n, N-n|\ \Lambda_a^{op}\, R\ |k,l\rangle_i^{\mathrm{II}} = \sum_{j=1}^{4} \mathscr{Y}_{n;i}^{k,l;j}\, Q_{a,j}^{op}(n, N-n)$$

$$^{\mathrm{I}}\langle n, N-n|\ \Lambda_a\, R\ |k,l\rangle_i^{\mathrm{II}} = \sum_{j=1}^{4} \mathscr{Y}_{n;i}^{k,l;j} Q_{a,j}(n,N-n) + \mathscr{Y}_{n+1;i}^{k,l;j} Q_{a,j}^{+}(n,N-n) + \mathscr{Y}_{n-1;i}^{k,l;j} Q_{a,j}^{-}(n,N-n)$$

Yangian makes the matrix equation invertible:

$$\mathscr{Y}_n^{k,l} \equiv \begin{pmatrix} \mathscr{Y}_{n;1}^{k,l;1} & \mathscr{Y}_{n;2}^{k,l;1} & \mathscr{Y}_{n;3}^{k,l;1} & \mathscr{Y}_{n;4}^{k,l;1} \\ \mathscr{Y}_{n;1}^{k,l;2} & \mathscr{Y}_{n;2}^{k,l;2} & \mathscr{Y}_{n;3}^{k,l;2} & \mathscr{Y}_{n;4}^{k,l;2} \\ \mathscr{Y}_{n;1}^{k,l;3} & \mathscr{Y}_{n;2}^{k,l;3} & \mathscr{Y}_{n;3}^{k,l;3} & \mathscr{Y}_{n;4}^{k,l;3} \\ \mathscr{Y}_{n;1}^{k,l;4} & \mathscr{Y}_{n;2}^{k,l;4} & \mathscr{Y}_{n;3}^{k,l;4} & \mathscr{Y}_{n;4}^{k,l;4} \end{pmatrix}$$

$$\begin{pmatrix} a_4 & a_3 & 0 & 0 \\ c_4 & c_3 & 0 & 0 \\ 0 & 0 & a_4 & a_3 \\ 0 & 0 & c_4 & c_3 \end{pmatrix} A\ \mathscr{Y}_n^{k,l} = \begin{pmatrix} a_2 & a_1 & 0 & 0 \\ c_2 & c_1 & 0 & 0 \\ 0 & 0 & a_2 & a_1 \\ 0 & 0 & c_2 & c_1 \end{pmatrix} \left\{ B^{+}\mathscr{X}_n^{k+1,l-1} + B^{-}\mathscr{X}_n^{k-1,l+1} + B\mathscr{X}_n^{k,l} \right\}$$

$$A = \begin{pmatrix} N-n-\ell_2 & 0 & \frac{\mathscr{I}_{34}}{\mathscr{Q}_{34}} & \frac{1}{\mathscr{Q}_{43}} \\ 0 & \ell_1-n & \frac{1}{\mathscr{Q}_{43}} & \frac{\mathscr{I}_{43}}{\mathscr{Q}_{34}} \\ (N-n-\ell_2)(M-\delta u) & (n-\ell_1)\ell_2\mathscr{I}_{34} & \frac{(\delta u-M+\ell_2)\mathscr{I}_{34}}{\mathscr{Q}_{43}} & \frac{\delta u+M+\ell_1-\ell_2\mathscr{Q}_{34}\overline{\mathscr{Q}}_{34}}{\mathscr{Q}_{43}} \\ (N-n-\ell_2)(\ell_1\mathscr{I}_{43}) & (\ell_1-n)(\delta u+M) & \frac{M-\delta u-\ell_2+\ell_1\mathscr{Q}_{34}\overline{\mathscr{Q}}_{34}}{\mathscr{Q}_{43}} & \frac{(\delta u+M+\ell_1)\mathscr{I}_{43}}{\mathscr{Q}_{34}} \end{pmatrix}$$

$$B^+ = \frac{2(\ell_1 - k - 1)\mathfrak{c}^-_{l-n}}{\tilde{\mathfrak{c}}^-_{-N}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ l & 0 & \frac{\mathscr{I}_{12}}{\mathscr{Q}_{12}} & 0 \\ 0 & 0 & \frac{1}{\mathscr{Q}_{21}} & 0 \end{pmatrix} \qquad B^- = \frac{2(\ell_2 - l - 1)\mathfrak{c}^+_{n-l}}{\tilde{\mathfrak{c}}^-_{-N}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\mathscr{Q}_{12}} \\ 0 & k & 0 & \frac{\mathscr{I}_{21}}{\mathscr{Q}_{21}} \end{pmatrix}$$

$$B = \begin{pmatrix} l-\ell_2 & 0 & \frac{\mathscr{I}_{12}}{\mathscr{Q}_{12}} & \frac{1}{\mathscr{Q}_{21}} \\ 0 & \ell_1-k & \frac{1}{\mathscr{Q}_{21}} & \frac{\mathscr{I}_{21}}{\mathscr{Q}_{12}} \\ (l-\ell_2)(N-\delta u) & (\ell_1-k)\ell_2\mathscr{I}_{12} & \frac{(N-\delta u-\ell_2)\mathscr{I}_{12}}{\mathscr{Q}_{12}} & \frac{N-\delta u-\ell_1-\ell_2\mathscr{Q}_{12}\overline{\mathscr{Q}}_{12}}{\mathscr{Q}_{12}} \\ (\ell_2-l)(\ell_1\mathscr{I}_{21}) & (\ell_1-k)(\delta u-N) & \frac{\delta u-N+\ell_1\mathscr{Q}_{12}\overline{\mathscr{Q}}_{12}+\ell_2}{\mathscr{Q}_{12}} & \frac{(\delta u-N+\ell_1)\mathscr{I}_{21}}{\mathscr{Q}_{12}} \end{pmatrix}$$

$$-2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{l(1+n+k-\ell_1)(l-\ell_2)}{\tilde{\mathfrak{c}}^-_{-N}} & 0 & \frac{(l-\ell_2)(1+n+k-\ell_1)\mathscr{I}_{12}}{\tilde{\mathfrak{c}}^-_{-N}\mathscr{Q}_{12}} & \frac{(\ell_1-k)(1+N-n+l-\ell_2)}{\tilde{\mathfrak{c}}^-_{-N}\mathscr{Q}_{21}} \\ 0 & \frac{k(1+N-n+l-\ell_2)(k-\ell_1)}{\tilde{\mathfrak{c}}^-_{-N}} & \frac{(l-\ell_2)(1+n+k-\ell_1)}{\tilde{\mathfrak{c}}^-_{-N}\mathscr{Q}_{21}} & \frac{(\ell_1-k)(1+N-n+l-\ell_2)\mathscr{I}_{21}}{\tilde{\mathfrak{c}}^-_{-N}\mathscr{Q}_{12}} \end{pmatrix}$$

where we introduced

$$\begin{aligned} M &= k + l - 2n \\ \mathscr{Q}_{ij} &= a_i c_j - a_j c_i \\ \overline{\mathscr{Q}}_{ij} &= b_i d_j - d_j b_i \\ \mathscr{I}_{ij} &= a_i d_j - b_j c_i \end{aligned}$$

Define

$$\begin{aligned} \mathfrak{d}\mathfrak{a} \equiv \det A \frac{\mathscr{Q}_{34}^2}{(n-\ell_1)(N-n-\ell_2)} &= -4\delta u^2 + (\ell_1 - \ell_2)^2 + 4\ell_1\ell_2\mathscr{I}_{34}\mathscr{I}_{43} \\ &= -4\mathfrak{c}_1^+\mathfrak{c}_1^- + 4\ell_1\ell_2\mathscr{I}_{34}\mathscr{I}_{43} \end{aligned}$$

then

$$A^{-1} = \frac{2}{\mathfrak{d}\mathfrak{a}} \begin{pmatrix} \frac{1}{N-n-\ell_2} & 0 & 0 & 0 \\ 0 & \frac{1}{n-\ell_1} & 0 & 0 \\ 0 & 0 & \mathscr{Q}_{43} & 0 \\ 0 & 0 & 0 & \mathscr{Q}_{43} \end{pmatrix} \times$$

$$\times \begin{pmatrix} \frac{\mathfrak{d}\mathfrak{a}}{4} - \left[M+\frac{\ell_1-\ell_2}{2}\right]\left[\mathfrak{c}_1^- + \ell_1\mathscr{I}_{34}\mathscr{I}_{43}\right] & \mathscr{I}_{34}\left(\frac{\mathfrak{d}\mathfrak{a}}{4} - \left[M+\frac{\ell_1-\ell_2}{2}\right]\tilde{\mathfrak{c}}_1^+\right) & \mathfrak{c}_1^- + \ell_1\mathscr{I}_{34}\mathscr{I}_{43} & \tilde{\mathfrak{c}}_1^+\mathscr{I}_{34} \\ -\mathscr{I}_{43}\left(\frac{\mathfrak{d}\mathfrak{a}}{4} - \left[M+\frac{\ell_1-\ell_2}{2}\right]\tilde{\mathfrak{c}}_1^+\right) & -\left[M+\frac{\ell_1-\ell_2}{2}\right]\left[\mathfrak{c}_1^- + \ell_2\mathscr{I}_{34}\mathscr{I}_{43}\right] - \frac{\mathfrak{d}\mathfrak{a}}{4} & \tilde{\mathfrak{c}}_1^+\mathscr{I}_{43} & \mathfrak{c}_1^+ + \ell_2\mathscr{I}_{34}\mathscr{I}_{43} \\ -\ell_1\left[M+\frac{\ell_1-\ell_2}{2}\right]\mathscr{I}_{43} & \frac{\mathfrak{d}\mathfrak{a}}{4} - \mathfrak{c}_1^+\left[M+\frac{\ell_1-\ell_2}{2}\right] & \ell_1\mathscr{I}_{43} & \mathfrak{c}_1^+ \\ \frac{\mathfrak{d}\mathfrak{a}}{4} + \mathfrak{c}_1^-\left[M+\frac{\ell_1-\ell_2}{2}\right] & \ell_2\left[M+\frac{\ell_1-\ell_2}{2}\right]\mathscr{I}_{34} & -\mathfrak{c}_1^- & -\ell_2\mathscr{I}_{34} \end{pmatrix}$$

Therefore, final result

$$\mathscr{Y}_n^{k,l} = A^{-1} \begin{pmatrix} \frac{\mathscr{Q}_{32}}{\mathscr{Q}_{34}} & \frac{\mathscr{Q}_{31}}{\mathscr{Q}_{34}} & 0 & 0 \\ \frac{\mathscr{Q}_{42}}{\mathscr{Q}_{43}} & \frac{\mathscr{Q}_{41}}{\mathscr{Q}_{43}} & 0 & 0 \\ 0 & 0 & \frac{\mathscr{Q}_{32}}{\mathscr{Q}_{34}} & \frac{\mathscr{Q}_{31}}{\mathscr{Q}_{34}} \\ 0 & 0 & \frac{\mathscr{Q}_{42}}{\mathscr{Q}_{43}} & \frac{\mathscr{Q}_{41}}{\mathscr{Q}_{43}} \end{pmatrix} \left\{ \mathscr{X}_n^{k+1,l-1} B^+ + \mathscr{X}_n^{k-1,l+1} B^- + \mathscr{X}_n^{k,l} B \right\}$$

REMARKS

- **(Apart perhaps from overall factor) final result purely depends only on δu, $\mathscr{Q}_{ij}, \overline{\mathscr{Q}}_{ij}, \mathcal{H}_{ij}$ and combinatorial factors involving integer bound-state components**

  [Still, putting this in a universal formula remains hard]

  [but you never know]

- **Case III is similarly generated from Case II. S-matrix is *uniquely* determined**

- **We reproduce known S-matrices in the limit of small bound state numbers [Beisert '05, Arutyunov-Frolov '08, Bajnok-Janik '08]**

- **One can compute transfer matrix eigenvalues in arbitrary bound state representations *via* Algebraic Bethe Ansatz $\longrightarrow$ by restriction, conjectures of [Beisert '07] on quantum characteristic function are nicely confirmed!**

[Arutyunov-de Leeuw-Suzuki-AT arXiv:0906.4783]

CONCLUSIONS

- **A deep mathematical structure is there, in some aspects almost reducible to standard, in some others seemingly so much harder**

- **Nevertheless, blooming of developments allowed to unveil some of the most useful bits of it**

- **More progress expected as one digs deeper and deeper.**

  Role of *secret symmetry*, **derivation of** *quantum double*, **maybe one day** *universal R-matrix* **?**

- **Fascinating connections with Yangian and dual superconformal symmetries of scattering amplitudes await to be fully investigated**

  [Talk by Jan ⟶ Thursday]

...Thank You