The quantum deformed mirror TBA

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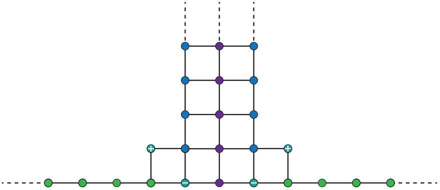
Work done in collaboration with G. Arutyunov and M. de Leeuw, [1208.3478]

Introduction



Arutyunov, Frolov ’09

Gromov, Kazakov, Vieira ’09

Bombardelli, Fioravanti, Tateo ’09

Gromov, Kazakov, Kozak, Vieira ’09

Introduction

The q-deformed mirror TBA

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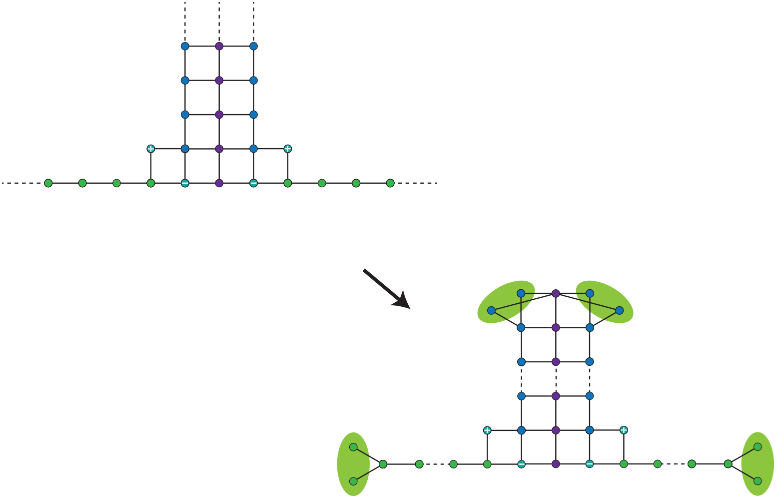
Finite size AdS/CFT

The q-deformed model

q-deformed spin chain TBA

The q-deformed $AdS_5 \times S^5$ mirror TBA

Conclusion



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  - Wider perspective
- Possible (partial) applications to particular deformed backgrounds

Outline

Finite size integrability in AdS_5/CFT_4

- TBA describes finite size string spectrum via a mirror model

Zamolodchikov '89
Ambjorn, Janik and Kristjansen '05
Arutyunov and Frolov '07

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  - Thermodynamic limit: different string hypothesis

q-deformed $\mathfrak{su}(2|2)$

- The $\mathfrak{su}(2|2)$ superalgebra in Chevalley-Serre basis ($3 \times E, F, H$)

$$[H_i, H_j] = 0\,,\;\; [H_i, E/F_j] = \pm A_{ij} E/F_j\,,\;\; [E_i, F_j\} = \delta_{ij} D_i H_i\,,$$

with

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}\,,\quad D = \mathrm{diag}(1, -1, -1)$$

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- We take $q = e^{i\pi/k}$ with integer $k > 2$

q-deformed scattering theory

- q-deformation extends to $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$
  - $\mathfrak{psu}_q(2|2)$ invariant R-matrix

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  - Parametrized by deformed $x^\pm$ variables
  - Natural *definition* of E and p in terms of U and V

Parametrizing the fundamental representation

- Central charges in terms of $x^{\pm}$

$$U^2 = \frac{1}{q}\frac{x^+ + \xi}{x^- + \xi}\,, \quad V^2 = q\,\frac{x^+}{x^-}\frac{x^- + \xi}{x^+ + \xi}$$

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- Then the shortening condition is (equivalent to)

$$\frac{1}{q}\left(x^+ + \frac{1}{x^+}\right) - q\left(x^- + \frac{1}{x^-}\right) = \left(q - \frac{1}{q}\right)\left(\xi + \frac{1}{\xi}\right)$$

with

$$\xi = -\frac{i}{2}\frac{g(q-q^{-1})}{\sqrt{1-\frac{g^2}{4}(q-q^{-1})^2}}$$

The dispersion relation

- To connect smoothly with string theory ($q = 1$) we define:

$$V^2 \equiv q^H \,, \;\; U^2 \equiv e^{ip}$$

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$$\tilde{H} = 2 \,\mathrm{arcsinh} \left(\frac{1}{g} \frac{\sin \frac{\pi}{2k}}{\sin \frac{\pi}{k}} \sqrt{1 + \left(1 + g^2 \sin^2 \frac{\pi}{k}\right) \frac{\sinh^2 \frac{\pi}{2k} \tilde{p}}{\sin^2 \frac{\pi}{2k}}} \right)$$

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- As for $q = 1$, this can be uniformized on a torus
  (“torus = space of short reps”)

The dispersion relation on the torus

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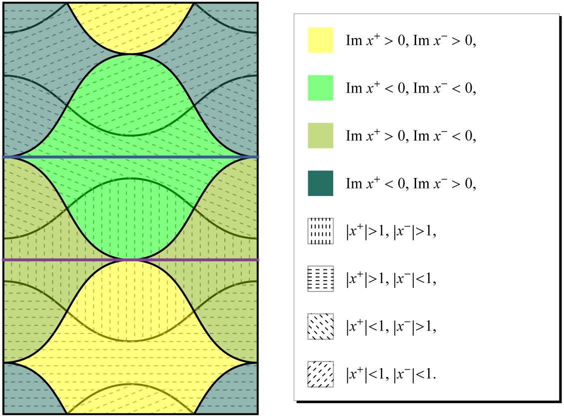

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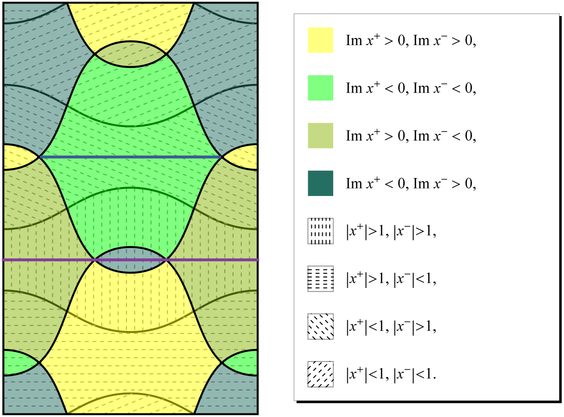

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$$1 = e^{i\tilde{p}_l R} \prod_{i \neq l}^{K^{\mathrm{I}}} \frac{1}{\sigma^2} \frac{x_i^+ - x_l^-}{x_i^- - x_l^+} \frac{1 - \frac{1}{x_i^- x_l^+}}{1 - \frac{1}{x_i^+ x_l^-}} \prod_{\alpha=1}^{2} \prod_{i=1}^{K^{\mathrm{II}}_{(\alpha)}} \sqrt{q} \frac{y_i^{(\alpha)} - x_l^-}{y_i^{(\alpha)} - x_l^+} \sqrt{\frac{x_l^+}{x_l^-}},$$

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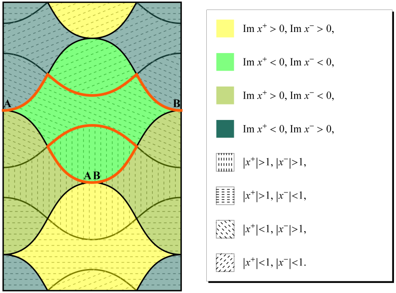

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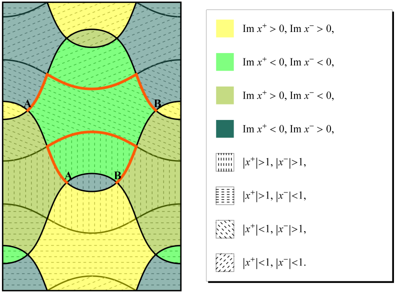

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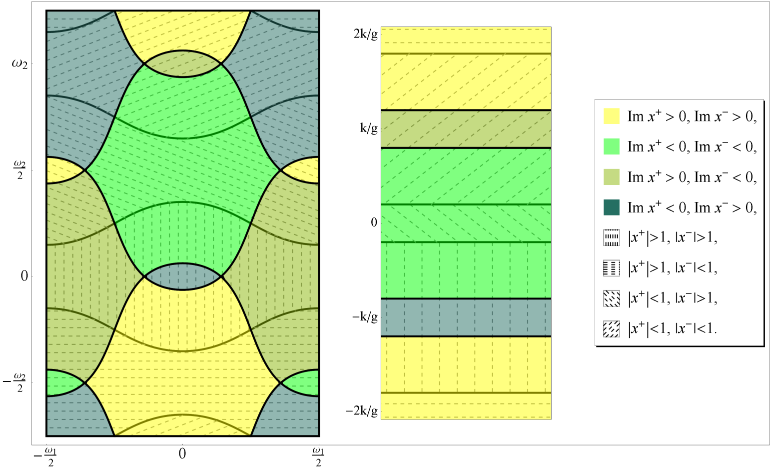

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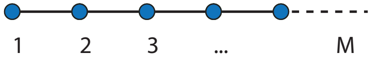

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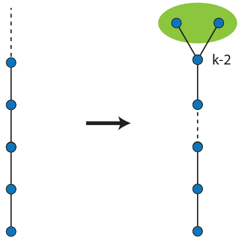

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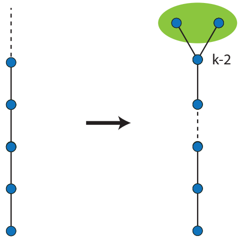

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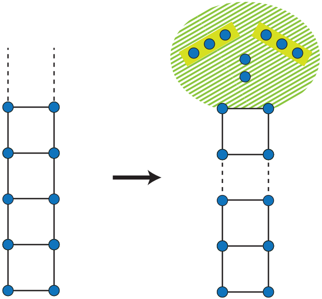

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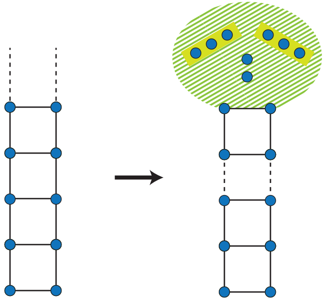

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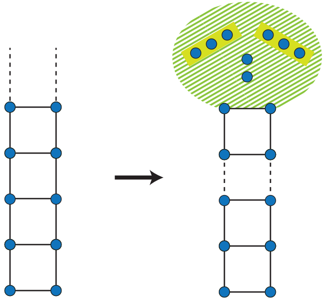

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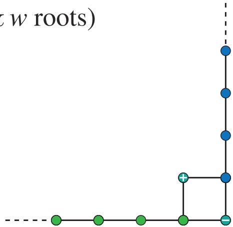

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- Mirror $\mathfrak{psu}(2|2)$: Hubbard model ($y(v)$ & w roots)

- String hypothesis:
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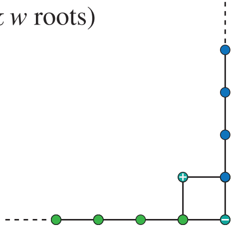
  - $M|w$ strings, any M ($\mathfrak{su}(2)$)
  - $M|vw$ strings, any M ($\mathfrak{su}(2)$)



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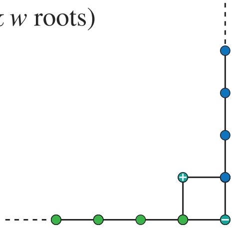

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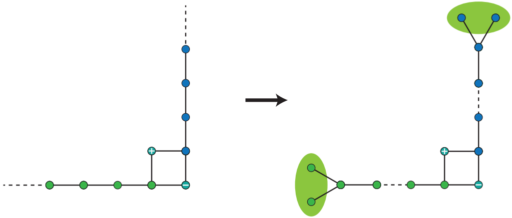

Undeformed Mirror TBA

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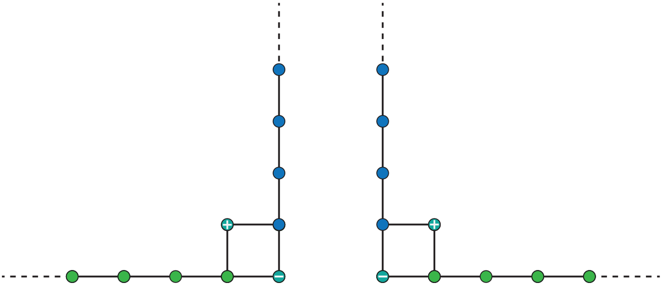
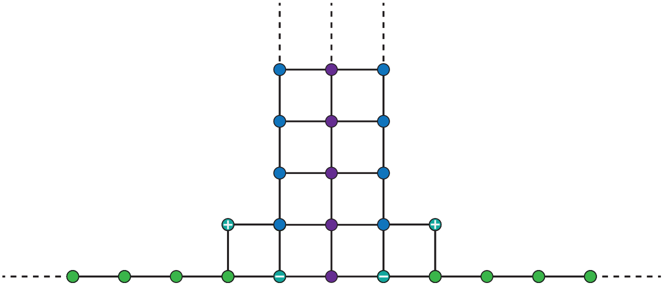
Two Hubbard subsystems



Undeformed Mirror TBA

Two Hubbard subsystems coupled via (∞) Q-particles



The q-deformed mirror TBA

Stijn J. van Tongeren

Finite size AdS/CFT

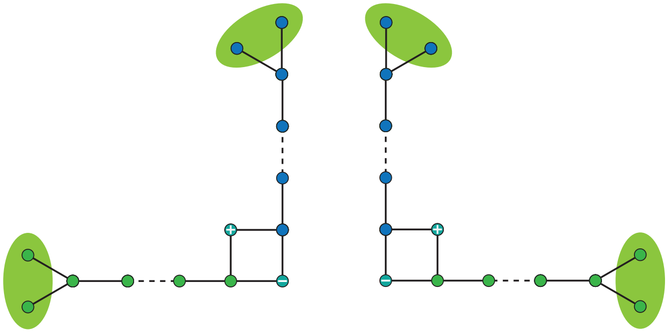
The q-deformed model

q-deformed spin chain TBA

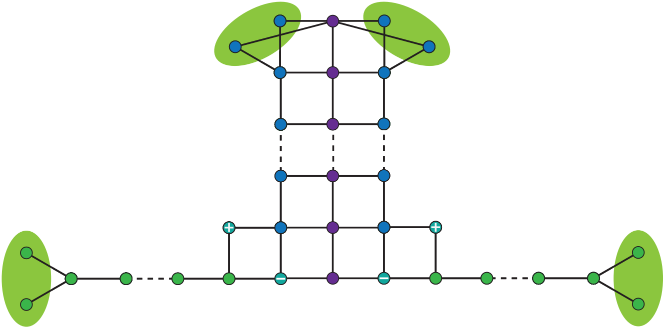
The q-deformed $AdS_5 \times S^5$ mirror TBA

Conclusion

Two q-Hubbard subsystems



Two q-Hubbard subsystems coupled via k Q-particles



q-deformed Mirror TBA equations

$$\log Y_{M|vw} = \log\left(1 + Y_{M+1|vw}\right)\left(1 + Y_{M-1|vw}\right) \star s - \log\left(1 + Y_{M+1}\right) \star s + \delta_{M,1} \log\left(\frac{1 - Y_-}{1 - Y_+}\right) \hat{\star} s$$

$$\log Y_{k-2|vw} = \log\left(1 + Y_{k-3|vw}\right)\left(1 + Y_{k-1|vw}\right)^2 \star s - \log\left(1 + Y_{k-1}\right) \star s\,,$$

$$\log Y_{k-1|vw} = \log\left(1 + Y_{k-2|vw}\right) \star s - \log\left(1 + Y_k\right) \star s\,,$$

$$\log Y_{M|w} = \log\left(1 + Y_{M+1|w}\right)\left(1 + Y_{M-1|w}\right) \star s + \delta_{M,1} \log\left(\frac{1 - Y_-^{-1}}{1 - Y_+^{-1}}\right) \hat{\star} s\,,$$

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$$\log Y_{\pm} = -\log\left(1 + Y_Q\right) \star K^{Qy}_{\pm} + \log \frac{1 + Y^{-1}_{M|vw}}{1 + Y^{-1}_{M|w}} \star K_M + \log \frac{\left(1 + Y_{k-1|vw}\right)}{\left(1 + Y_{k-1|w}\right)} \star K_{k-1}\,.$$

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$$\log Y_k = {\color{red}2}\log Y_{k-1}\star s - \log(1+Y_{k-1})\star s + \log\left(1+Y_{k-1|vw}^{-1}\right)^{\color{red}4}\star s\,.$$

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we can *'derive'* the crossing equation!

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