

CDT, a quantum theory of geometry

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4d QG regularized by CDT

Main goal (at least in 80ties) for QG

- Obtain the background geometry ($\langle\langle g_{\mu\nu} \rangle\rangle$) we observe
- Study the fluctuations around the background geometry

What lattice gravity (CDT) offers:

- A non-perturbative QFT definition of QG
- A background independent formulation
- An emergent background geometry ($\langle\langle g_{\mu\nu} \rangle\rangle$)
- The possibility to study the quantum fluctuations around this emergent background geometry.

Problems to confront for a lattice theory

- (1) How to face the non-renormalizability of quantum gravity (this is a problem for any field theory of quantum gravity, not only lattice theories)
- (2) Provide evidence of a continuum limit (where the continuum field theory has the desired properties)
- (3) If rotation is performed to Euclidean signature, how does one deal with the unboundedness of the Euclidean Einstein-Hilbert action?
- (4) If there exists no continuum field theory of gravity, can a lattice theory be of any use?

(1) Facing the non-renormalizability of gravity

Effective QFT of gravity

We believe gravity exists as an effective QFT for $E^2 \ll 1/G$.

True for other non-renormalizable theories

Weak interactions $\mathcal{L} = \bar{\psi}\partial\psi + G_F\bar{\psi}(\cdot)\psi\bar{\psi}(\cdot)\psi$

Nonlinear sigma model $\mathcal{L} = (\partial\pi)^2 + \frac{1}{F_\pi^2} \frac{(\pi\partial\pi)^2}{1 - \pi^2/F_\pi^2}$

Good for $E^2 \ll 1/G_F$ and $E^2 \ll F_\pi^2$.

Effective QFT of gravity

Lowest order quantum correction to the gravitational potential of a point particle:

$$\frac{G}{r} \rightarrow \frac{G(r)}{r}, \quad G(r) = G \left(1 - \omega \frac{G}{r^2} + \dots \right), \quad \omega = \frac{167}{30\pi}.$$

The gravitational coupling constant becomes **scale dependent** and transferring from distance to energy we have

$$G(E) = G(1 - \omega GE^2 + \dots) \approx \frac{G}{1 + \omega GE^2}.$$

Effective QFT of the electric charge

Same calculation in QED

$$\frac{e^2}{r} \rightarrow \frac{e^2(r)}{r}, \quad e^2(r) = e^2 \left(1 - \frac{e^2}{6\pi^2} \ln(mr) + \dots \right), \quad mr \ll 1.$$

The electric charge is also scale dependent and has a **Landau pole**

$$e^2(E) = e^2 \left(1 + \frac{e^2}{6\pi^2} \ln(E/m) + \dots \right) \approx \frac{e^2}{1 - \frac{e^2}{6\pi^2} \ln(E/m)}.$$

$$GE^2 \ll 1 \rightarrow G(E)E^2 \ll 1$$

BUT

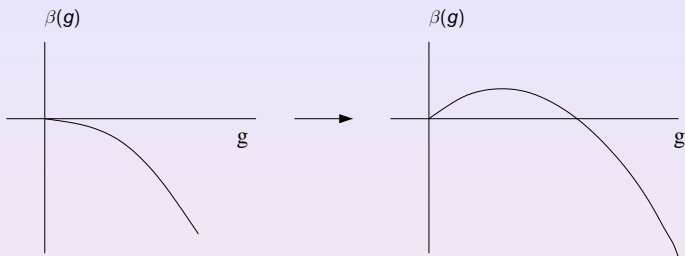
$$G(E)E^2 < 1 \quad (\ll 1 ?).$$

Suddenly seems as if quantum gravity has become an (almost) reliable quantum theory at all energy scales.

The behavior can be described the **β function** for QG. For the dimensionless coupling constant $\tilde{G}(E) = G(E)E^2$

$$E \frac{d\tilde{G}}{dE} = \beta(\tilde{G}), \quad \beta(\tilde{G}) = 2\tilde{G} - 2\omega\tilde{G}^2.$$

Two **fixed points** ($\beta(\tilde{G}) = 0$): $\tilde{G} = 0$ and $\tilde{G} = 1/\omega$.

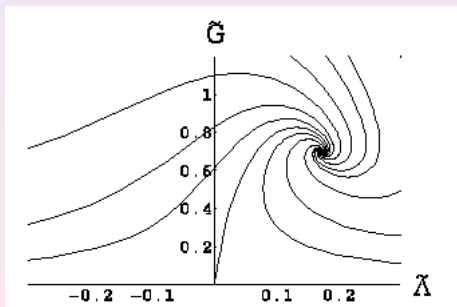


Generic situation for asymptotic free theories in d dimensions, extended to $d + \varepsilon$ dimensions.

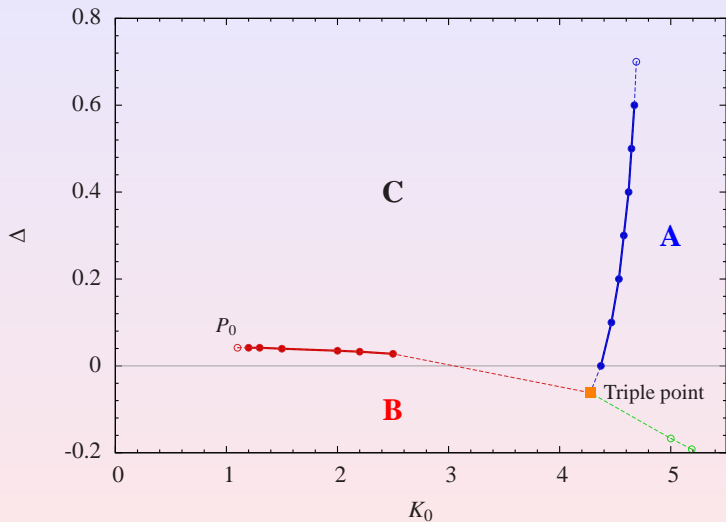
$$\beta(g) \rightarrow \varepsilon g + \beta(g)$$

The four-Fermi action, the nonlinear sigma model and QG are all renormalizable theories in 2d, with a negative β -function and have a $2 + \varepsilon$ expansion. For QG first explored by Kawai et al.

Alternatively one can use the exact renormalization group approach. (Reuter et al., Litim,). Philosophy: asymptotic safety (Weinberg).



(2) Continuum limit ?



Defining the continuum limit in lattice field theory

Let the lattice coordinate be $\mathbf{x}_n = \mathbf{a} n$, \mathbf{a} being the lattice spacing and $\mathcal{O}(\mathbf{x}_n)$ an observable.

$$-\log \langle \mathcal{O}(\mathbf{x}_n) \mathcal{O}(\mathbf{x}_m) \rangle \sim |n - m| / \xi(g_0) + o(|n - m|).$$

$$\xi(g_0) \propto \frac{1}{|g_0 - g_0^c|^\nu}, \quad a(g_0) \propto |g_0 - g_0^c|^\nu.$$

$$m_{ph} a(g_0) = 1 / \xi(g_0), \quad e^{-|n-m|/\xi(g_0)} = e^{-m_{ph} |\mathbf{x}_n - \mathbf{x}_m|}$$

$\langle \mathcal{O}(\mathbf{x}_n) \mathcal{O}(\mathbf{y}_m) \rangle$ falls off exponentially like $e^{-m_{ph} |\mathbf{x}_n - \mathbf{y}_m|}$ for $g_0 \rightarrow g_0^c$ when $|\mathbf{x}_n - \mathbf{y}_m|$, but not $|n - m|$, is kept fixed in the limit $g_0 \rightarrow g_0^c$.

How to define the equivalent of $\langle \mathcal{O}(x_n) \mathcal{O}(y_m) \rangle$ in a diffeomorphism invariant theory

$$\langle \phi \phi(R) \rangle \equiv \int \mathcal{D}[g_{\mu\nu}] e^{-S[g_{\mu\nu}]} \times \\ \iint \sqrt{g(x)} \sqrt{g(y)} \langle \phi(x) \phi(y) \rangle_{matter}^{[g_{\mu\nu}]} \delta(R - d_{g_{\mu\nu}}(x, y)).$$

$\langle \phi(x) \phi(y) \rangle_{matter}^{[g_{\mu\nu}]}$ denotes the correlator of the matter fields calculated for a fixed geometry, defined by the metric $g_{\mu\nu}(x)$.

It works in 2d Euclidean QG (Liouville gravity)

(3) Unboundedness of the Euclidean action

Already the discussion about continuum limit of the lattice theories hinted a rotation to Euclidean signature. The Einstein-Hilbert action is unbounded from below, caused by the conformal factor:

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$$

$$S[g, \Lambda, G] = -\frac{1}{16\pi G} \int d^4\xi \sqrt{g} (R - 2\Lambda).$$

$$S[\tilde{g}, \Lambda, G] = -\frac{1}{16\pi G} \int d^4\xi \sqrt{\tilde{g}} (\Omega^2 R + 6\partial^\mu \Omega \partial_\mu \Omega - 2\Lambda \Omega^4).$$

How is this dealt with ?.

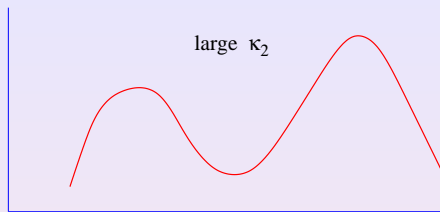
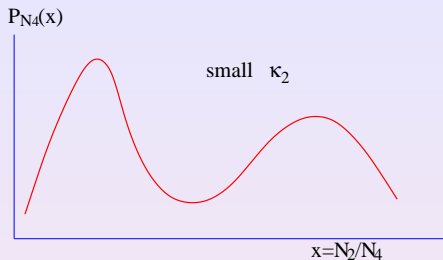
Using the lattice regularization called **dynamical triangulations (DT)** the Euclidean action **is** bounded for a fixed lattice spacing **a** and a fixed four-volume **$V_4 = N_4 a^4$** . However, for $a \rightarrow 0$ the unboundedness re-emerges.

$$S[T] = -\kappa_2 N_2(T) + \kappa_4 N_4(T), \quad c_1 < \frac{N_2}{N_4} (=x) < c_2.$$

The unbounded configurations corresponds to $x \approx c_2$. But are they important in the non-perturbative path integral ?

$$Z = \sum_T e^{-S[T]} = \sum_{N_4} e^{-\kappa_4 N_4} \sum_{N_2} \mathcal{N}(N_2, N_4) e^{\kappa_2 N_2}$$

$$\mathcal{N}(N_2, N_4) e^{\kappa_2 N_2} = P_{N_4}(x), \quad \sum_x P_{N_4}(x) = f(N_4) e^{\kappa_4^c(\kappa_2) N_4}$$

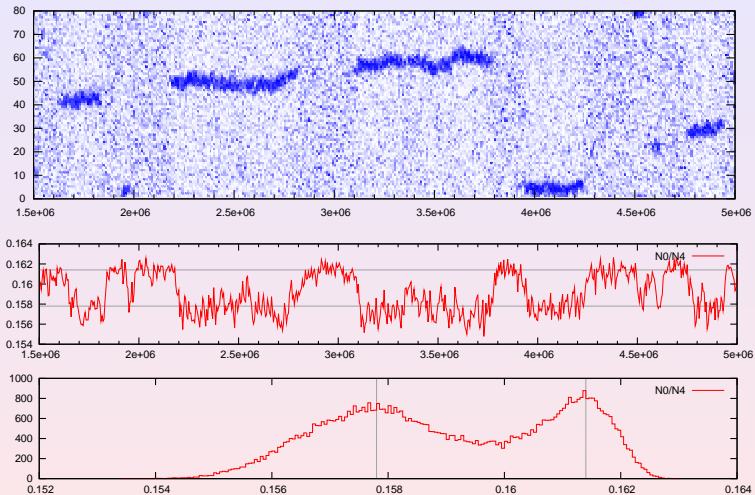


$$P_{N_4}(x) \approx A e^{N_4 \left(\kappa_4^C - \alpha(x-x_0)^2 \right)} + \tilde{A} e^{N_4 \left(\tilde{\kappa}_4^C - \tilde{\alpha}(x-\tilde{x}_0)^2 \right)}.$$

$$\kappa_2 \rightarrow \kappa_2 + \Delta\kappa_2, \quad \kappa_4^C \rightarrow k_4^C + \Delta\kappa_2 x_0, \quad \tilde{\kappa}_4^C \rightarrow \tilde{k}_4^C + \Delta\kappa_2 \tilde{x}_0$$

Phase transition when $\kappa_4^C = \tilde{\kappa}_4^C$.

The A-C transition



Do we know examples of such **entropy driven** phase transitions? Yes, the **Kosterlitz-Thouless transition** in the XY model. This Abelian 2d spin model has vortices with energy

$$E = \kappa \ln(R/a)$$

Saturating the partition function by single vortex configurations:

$$Z \equiv e^{-F/k_B T} = \sum_{\text{spin configurations}} e^{-E[\text{spin}]/k_B T} \approx \left(\frac{R}{a}\right)^2 e^{-[\kappa \ln(R/a)]/k_B T}.$$

$S = k_B \ln(\text{number of configurations})$ has the same functional form as the vortex energy. Thus

$$F = E - ST = (\kappa - 2k_B T) \ln(R/a)$$

(4) No continuum limit ?

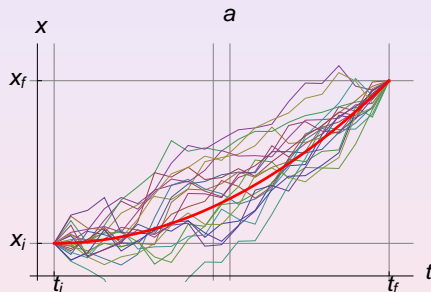
Examples

- Lattice compact $U(1)$ gauge theory in 3 dimensions has confinement for all values of the coupling constant, due to **lattice monopoles**. It describes perfectly the non-perturbative physics of the Georgi-Glashow model, i.e. the physics below the scale of Higgs and the W-particle. The formula for the string tension is the **same** expressed in terms of lattice monopoles masses and continuum monopole masses.
- Lattice compact $U(1)$ gauge theory in 4 dimensions at the phase transition point describes the low energy physics of certain broken $\mathcal{N} = 1, 2$ supersymmetric field theories. In fact, one can use the supersymmetric symmetry breaking technology of Seiberg et al. scale matching to “post-dict” (unfortunately) the lattice critical exponents.

Lattice gravity: causal dynamical triangulations (CDT)

Basic tool: **The path integral**

Text-book example: non-relativistic particle in one dimension.



$$x(t) = \langle x(t) \rangle + y(t)$$
$$\langle |y| \rangle \propto \sqrt{\hbar/m\omega}$$

In QG we want $\langle x(t) \rangle$

$$\langle |y| \rangle \propto \sqrt{\hbar G}$$

Transition amplitude as a weighted sum over all possible trajectories. On the plot: time is **discretized** in steps a , trajectories are piecewise linear.

In a **continuum limit** $a \rightarrow 0$

$$G(\mathbf{x}_i, \mathbf{x}_f, t) := \int_{\text{trajectories: } \mathbf{x}_i \rightarrow \mathbf{x}_f} e^{iS[\mathbf{x}(t)]}$$

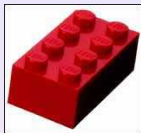
where $S[\mathbf{x}(t)]$ is a classical action.

The QG amplitude between the two geometric states

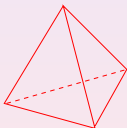
$$G(\mathbf{g}_i, \mathbf{g}_f, t) := \int_{\text{geometries: } \mathbf{g}_i \rightarrow \mathbf{g}_f} e^{iS[\mathbf{g}_{\mu\nu}(t')]}$$

To define this path integral we need a **geometric** cut-off a and a definition of the class of geometries entering.

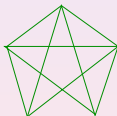
showcasing **piecewise linear geometries** via **building blocks**:



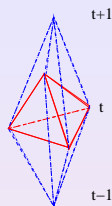
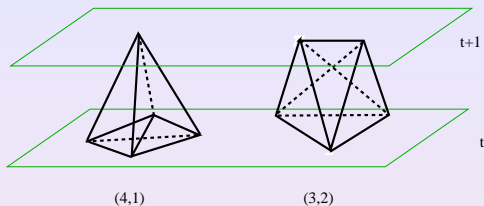
2d



3d



4d



CDT slicing in proper time. Topology of space preserved.

$$a_t^2 = -\alpha a_s^2, \quad iS_L[\alpha] = -S_E[-\alpha]$$

$$S_E[-\alpha] = -(\kappa_0 + 6\Delta)N_0 + \kappa_4 \left(N_4^{(2,3)} + N_4^{(1,4)} \right) + \Delta \left(N_4^{(2,3)} + 2N_4^{(1,4)} \right)$$

$$\begin{aligned}
 G(\mathbf{g}_i, \mathbf{g}_f, t) &:= \int_{\text{geometries: } \mathbf{g}_i \rightarrow \mathbf{g}_f} e^{iS[\mathbf{g}_{\mu\nu}(t')]} \\
 &= \lim_{a \rightarrow 0} \sum_{T: T_i^{(3)} \rightarrow T_f^{(3)}} \frac{1}{C_T} e^{iS_T}
 \end{aligned}$$

$$G_E(\mathbf{g}_i, \mathbf{g}_f, t, \kappa_0, \kappa_4, \Delta) = \lim_{a \rightarrow 0} \sum_{T: T_i^{(3)} \rightarrow T_f^{(3)}} \frac{1}{C_T} e^{-S_E[T]}$$

$$\langle \mathbf{x}_f | e^{i\hat{H}t} | \mathbf{x}_i \rangle \rightarrow \langle \mathbf{x}_f | e^{-\hat{H}\tau} | \mathbf{x}_i \rangle$$

Scaling in the IR limit?

$$Z(\kappa_0, \kappa_4) = \sum_{N_4} e^{-\kappa_4 N_4} Z_{N_4}(\kappa_0),$$

where $Z_{N_4}(\kappa_0)$ is the partition function for a fixed number N_4 of four-simplices (we ignore Δ for simplicity), namely,

$$Z_{N_4}(\kappa_0) = e^{k_4^c N_4} f(N_4, \kappa_0)$$

We want to consider the limit $N_4 \rightarrow \infty$, and fine-tune $\kappa_4 \rightarrow \kappa_4^c$ for fixed κ_0 . We expect the **physical** cosmological constant Λ to be defined by the **approach** to the critical point according to

$$\kappa_4 = \kappa_4^c + \frac{\Lambda}{16\pi G} a^4, \quad (\kappa_4 - \kappa_4^c) N_4 = \frac{\Lambda}{16\pi G} V_4, \quad V_4 = N_4 a^4,$$

How can one imagine obtaining an interesting continuum behavior as a function of κ_0 ? Assume $f(N_4, \kappa_0)$ has the form (numerical evidence)

$$f(N_4, \kappa_0) = e^{k_1(\kappa_0)\sqrt{N_4}}, \quad \left\langle e^{-\frac{1}{G} \int_{V_4} \sqrt{g} R} \right\rangle = e^{c \frac{\sqrt{V_4}}{G}}.$$

$$Z(\kappa_4, \kappa_0) = \sum_{N_4} e^{-(\kappa_4 - \kappa_4^c)N_4 + k_1(\kappa_0)\sqrt{N_4}}.$$

Search for κ_0^c with $k_1(\kappa_0^c) = 0$, with the approach to this point governed by

$$k_1(\kappa_0) \propto \frac{a^2}{G}, \quad \text{i.e.} \quad k_1(\kappa_0)\sqrt{N_4} \propto \frac{\sqrt{V_4}}{G}.$$

$$Z(\kappa_4, \kappa_0) \approx \exp\left(\frac{k_1^2(\kappa_0)}{4(\kappa_4 - \kappa_4^c)}\right) = \exp\left(\frac{c}{G\Lambda}\right),$$

as one would naïvely expect from Einstein's equations, with the partition function being dominated by a typical instanton contribution, for a suitable constant c .

UV scaling limit?

If we are close to the UV fixed point, we know that G will not be constant when we change scale, but $\hat{G}(a)$ will. Writing $G(a) = a^2 \hat{G}(a) \approx a^2 \hat{G}^*$,

$$\kappa_4 - \kappa_4^c = \frac{\Lambda}{G(a)} a^4 \approx \frac{\Lambda}{\hat{G}^*} a^2,$$

$$k_1(\kappa_0^c) = \frac{a^2}{G(a)} \approx \frac{1}{\hat{G}^*}.$$

The first of these relations now looks two-dimensional because of the **anomalous scaling** of $G(a)$! Nevertheless, the expectation value of the four-volume is still finite:

$$\langle V_4 \rangle = \langle N_4 \rangle a^4 \propto \frac{\kappa_1^2(\kappa_0^c)}{(\kappa_4 - \kappa_4^c)^2} a^4$$

Relation to asymptotic freedom

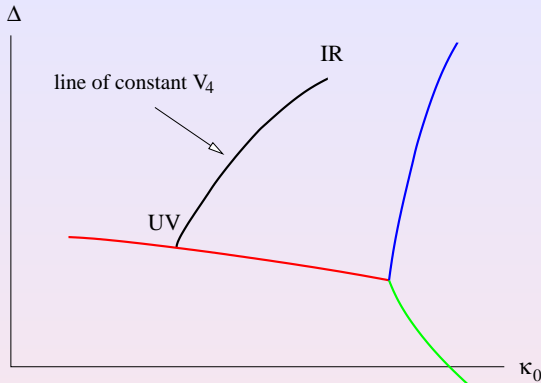
Assume now that we have a fixed point for gravity. The gravitational coupling constant is dimensionful, and we can write for the bare coupling constant

$$G(a) = a^2 \hat{G}(a), \quad a \frac{d\hat{G}}{da} = -\beta(\hat{G}), \quad \beta(\hat{G}) = 2\hat{G} - c\hat{G}^2 + \dots$$

The putative non-Gaussian fixed point corresponds to $\hat{G} \rightarrow \hat{G}^*$, i.e. $G(a) \rightarrow \hat{G}^* a^2$. In our case it is tempting to identify our dimensionless constant k_1 with $1/\hat{G}$, up to the constant of proportionality. Close to the UV fixed point we have

$$\hat{G}(a) = \hat{G}^* - Ka^{\tilde{c}}, \quad k_1 = k_1^* + Ka^{\tilde{c}}, \quad \tilde{c} = -\beta'(\hat{G}^*).$$

Usually one relates the lattice spacing near the fixed point to the bare coupling constants with the help of some correlation length ξ .

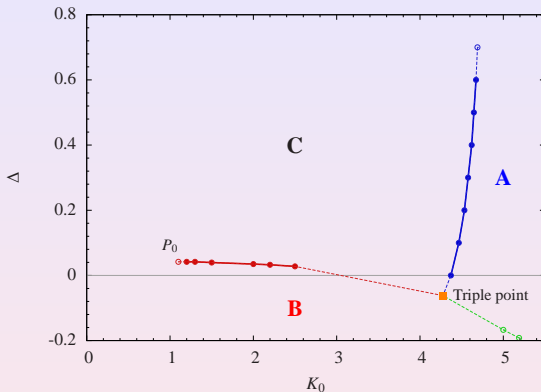


Consider $V_4 = N_4 a^4$ as fixed. It requires the fine-tuning of coupling constants.

$$k_1(N_4) = k_1^c - \tilde{K} N_4^{-\tilde{c}/4}.$$

How to determine $k_1(N_4)$?

Phase diagram of CDT



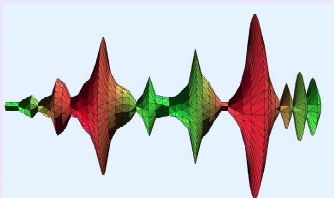
Lifshitz-like diagram....

Phase C: constant magnetization (constant 4d geometry)

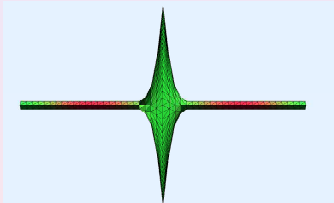
Phase B: zero magnetization (no 4d geometry)

Phase A: oscillating magnetization (conformal mode ?)

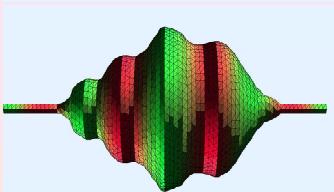
Volume distribution in (imaginary) time



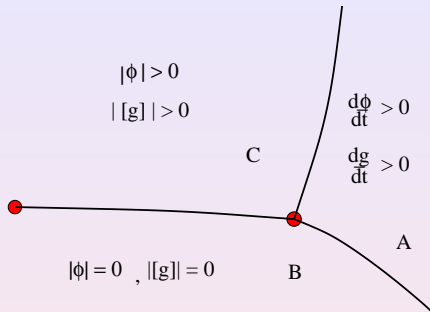
- **Phase A.** The universe “oscillating” in time direction. The oscillation maybe reflecting the dominance of the conformal mode.



- **Phase B.** Compactification into a 3d Euclidean DT. Only minimal extension in the time direction.



- **Phase C.** Extended **de Sitter** phase. $d_H = 4$.



$$S_{\text{Lifshitz}}[\phi] = \int d^D x \left(\mu (\partial_i \phi)^2 + (\Delta \phi)^2 + \cdots + \nu \phi^2 + \phi^4 + \cdots \right)$$

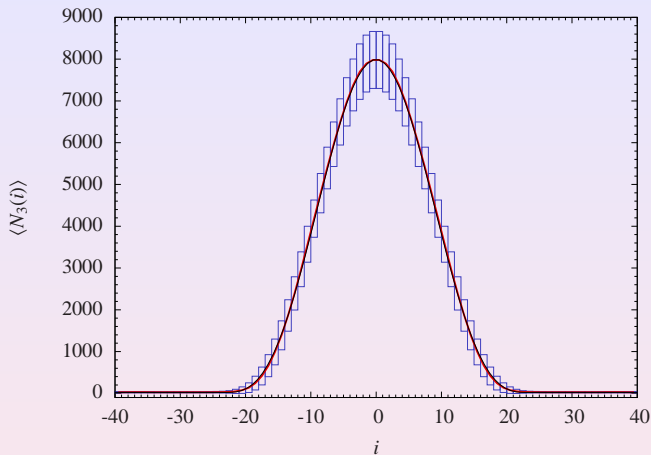
Snapshot of a typical configuration



A typical configuration. Distribution of a spatial volume $N_3(t)$ as a function of (imaginary) time t .

Quantum fluctuation around a **semiclassical background**?

Configuration consists of a “stalk” of the cut-off size and a “blob”. Center of the blob can shift. **We fix the “center of mass” to be at zero time.**



$$\langle N_3(i) \rangle \propto N_4^{3/4} \cos^3 \left(\frac{i}{s_0 N_4^{1/4}} \right)$$

Minisuperspace model

The semiclassical distribution can be obtained from the **minisuperspace effective action** of Hartle and Hawking

$$S_{\text{eff}} = \frac{1}{24\pi G} \int dt \sqrt{g_{tt}} \left(\frac{g^{tt} \dot{V}_3^2(t)}{V_3(t)} + k_2 V_3^{1/3}(t) - \lambda V_3(t) \right),$$

The discretization of this action is (**and we have reconstructed it from the data (the 3-volume–3-volume correlations)**)

$$S_{\text{discr}} = k_1 \sum_i \left(\frac{(N_3(i+1) - N_3(i))^2}{N_3(i)} + \tilde{k}_2 N_3^{1/3}(i) - \tilde{\lambda} N_3(i) \right),$$

$$G = \frac{a^2}{k_1} \frac{\sqrt{C_4} s_0^2}{3\sqrt{6}}.$$

Quantum fluctuations

The classical solution to the minisuperspace action is

$$\sqrt{g_{tt}} V_3^{cl}(t) = V_4 \frac{3}{4B} \cos^3 \left(\frac{t}{B} \right)$$

where $\tau = \sqrt{g_{tt}} t$, $V_4 = 8\pi^2 R^4/3$ and $\sqrt{g_{tt}} = R/B$.

Writing $V_3(t) = V_3^{cl}(t) + x(t)$ we can expand the action around this solution

$$S(V_3) = S(V_3^{cl}) + \frac{1}{18\pi G} \frac{B}{V_4} \int dt x(t) \hat{H} x(t).$$

where the Hermitian operator \hat{H} is:

$$\hat{H} = -\frac{d}{dt} \frac{1}{\cos^3(t/B)} \frac{d}{dt} - \frac{4}{B^2 \cos^5(t/B)},$$

In the **quadratic approximation** the volume fluctuations are:

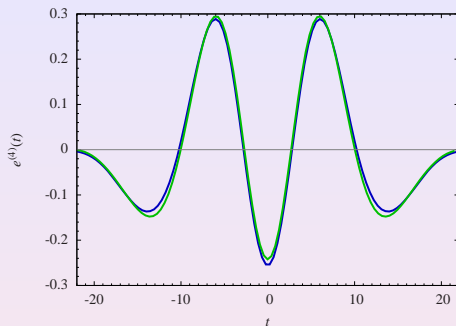
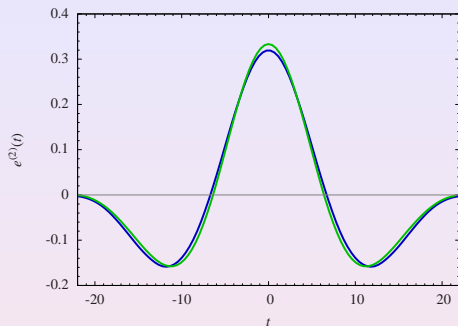
$$C(t, t') := \langle x(t)x(t') \rangle \sim \hat{H}^{-1}(t, t').$$

\hat{C} and \hat{H} have the same eigenfunctions.

$C(t, t')$ can be **measured** as

$$C(i, i') = \left\langle \left(N_3(i) - \langle N_3(i) \rangle \right) \left(N_3(i') - \langle N_3(i') \rangle \right) \right\rangle,$$

and its eigenfunctions can be found and compared to the ones **calculated** from \hat{H} .



No parameters are put in ! (expect $t_i/B = i/s_0 N_4^{1/4}$)

We conclude that the quadratic approximation to the minisuperspace action describes the measured quantum fluctuations well.

Size of our Quantum universe

For a specific value of the bare coupling constants ($\kappa_0 = 2.2$, $\Delta = 0.6$) we have high-statistics measurements for N_4 ranging from 45.500 to 362.000 four-simplices.

Largest universe corresponds to approx. 10^4 hyper-cubes.

We have $G = \text{const. } a^2/k_1$ and we have measured k_1 .

$$G \approx 0.23a^2, \quad \ell_P \approx 0.48a, \quad \ell_P \equiv \sqrt{G}.$$

From $V_4 = 8\pi^2 R^4/3 = C_4 N_4 a^4$, we obtain that

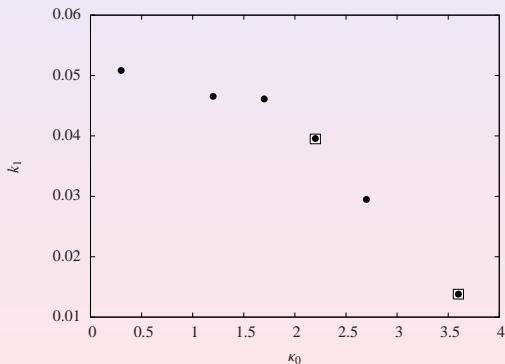
$$R = 3.1a$$

The linear size πR of the quantum de Sitter universes studied here lies in the range of 12-21 ℓ_P for the N_4 used.

Trans-Planckian ?

$$\ell_P = \sqrt{G} \propto \frac{a}{\sqrt{k_1(\kappa_0, \Delta)}} \quad \text{i.e.} \quad k_1(\kappa_0, \Delta) \rightarrow 0.$$

BUT IS IT POSSIBLE ?



Summary and perspectives

- We have obtained the (Euclidean) minisuperspace action from first principles. (**The self-organized de Sitter space**)
 - We have an effective field theory of (something we call) QG down to a few Planck scales.
-
- Investigate a possible UV fixed point (points, the B-C line). Possibly Hořava-Lifshitz gravity.
 - couple matter to the system and investigate cosmological implications.
 - Measure the wave function of the universe

$$\langle x | e^{-t\hat{H}} | y \rangle \rightarrow \psi_0(y) \psi_0(x) e^{-tE_0}$$