Poisson algebraic geometry and matrix regularizations

Joakim Arnlind
Max-Planck-Institut für Gravitationsphysik

Quantum Theory and Gravitation
Zürich, 14/6 2011

General ideas

For a moment, let us forget about why we study the following question and state the main theme of this talk:

Given a Riemannian manifold with a Poisson structure on the algebra of smooth functions, can one write geometric quantities as Poisson algebraic expressions?

- That is, can one write, for instance, curvature as sums, products and Poisson brackets of smooth functions?
- More specifically, if the manifold is embedded into an ambient space, is it possible to find expressions only involving the embedding coordinates?

General ideas

IF one can obtain such a description, then **WHY**?

- Having any kind of discrete / non-commutative / quantum analogue of the algebra of functions, such that Poisson brackets are approximated by commutators, this allows one to study / define geometric concepts in the non-commutative setting.
- Is it possible to develop Riemannian geometry for arbitrary (commutative) Poisson algebras?
- Can one extend the results to non-commutative Poisson algebras to obtain an approach to non-commutative geometry?

References

The results in this talk are based on the following papers:

- *On the classical geometry of embedded surfaces in terms of Poisson brackets.* J. Arnlind, J. Hoppe and G. Huisken. `arXiv:1001.1604`
- *Discrete curvature and the Gauss-Bonnet theorem.* J. Arnlind, J. Hoppe and G. Huisken. `arXiv:1001.2223`
- *On the classical geometry of embedded surfaces in terms of Nambu brackets.* J. Arnlind, J. Hoppe and G. Huisken. `arXiv:1003.5981`
- *Multi linear formulation of differential geometry and matrix regularizations.* J. Arnlind, J. Hoppe and G. Huisken. `arXiv:1009.4779`
- *On the geometry of Kähler–Poisson structures.* J. Arnlind and G. Huisken. `arXiv:1103.5862`

Matrix regularizations

Let us introduce a particular example which motivates the questions we have been working on.

Let (Σ, ω) be a compact surface together with a symplectic form ω inducing a Poisson bracket $\{\cdot, \cdot\}$ on $C^\infty(\Sigma)$. Moreover, let $\{T^\alpha\}$ be a sequence of linear maps

$$T^\alpha : C^\infty(\Sigma) \to \mathrm{Herm}_{N_\alpha}$$

(Herm_{N_α} = the set of hermitian $N_\alpha \times N_\alpha$ matrices) where $\{N_\alpha\}$ is a sequence of strictly increasing positive integers.

By $\hbar_\alpha = \hbar(N_\alpha)$ we denote a strictly decreasing positive real valued function.

Definition

Let $\{T^\alpha\}$ be a sequence of maps as on the previous slide. If $\{T^\alpha\}$ has the following properties for all $f,h\in C^\infty(\Sigma)$

$$\lim_{\alpha\to\infty}||T^\alpha(f)||<\infty, \tag{1}$$

$$\lim_{\alpha\to\infty}||T^\alpha(fh)-T^\alpha(f)T^\alpha(h)||=0, \tag{2}$$

$$\lim_{\alpha\to\infty}\left|\left|\frac{1}{i\hbar_\alpha}[T^\alpha(f),T^\alpha(h)]-T^\alpha(\{f,h\})\right|\right|=0, \tag{3}$$

$$\lim_{\alpha\to\infty}2\pi\hbar_\alpha\,\mathrm{Tr}\,T^\alpha(f)=\int_\Sigma f\omega, \tag{4}$$

where $||\cdot||$ denotes the operator norm, then we call the pair $(T^\alpha,\hbar_\alpha)$ a *matrix regularization of* (Σ,ω).

Regularized Membrane Theory

These kind of fuzzy spaces have been used in many physical contexts, in particular to regularize "Membrane Theory".

Classical Membrane Theory describes two-surfaces moving in time to sweep out 3-manifolds of vanishing mean curvature in an ambient Minkowski space.

The regularized (bosonic) equations of motion have the form

$$\ddot{X}_i = -\frac{1}{\hbar_\alpha^2}\sum_{j=1}^{m}\big[[X_i, X_j], X_j\big]$$

where $X_1, \ldots, X_m$ are hermitian $N_\alpha \times N_\alpha$ matrices corresponding to the embedding coordinates.

Geometry and topology in matrix sequences?

Solutions to the equations of motion are expected to contain matrix regularizations of surfaces of arbitrary genus. Thus, given a particular sequence of matrix solutions, how can one recognize the topology of the surface?

If one has an expression for the scalar curvature in terms of Poisson brackets of embedding functions, then one can compute the corresponding discrete expression (by replacing Poisson brackets by commutators) and take the limit as $N_\alpha \to \infty$. In this way one can simply compute the topology of the surface.

In general, how are geometric properties of the manifold reflected in the sequence of matrix algebras?

Before going into the general setup, let me just display an example showing that the Gaussian curvature of a surface embedded in $\mathbb{R}^m$ can be written in terms of Poisson brackets. That is

$$K = \frac{1}{\gamma^4} \sum_{j,k,l=1}^{m} \Big(\frac{1}{2}\{\{x^j, x^k\}, x^k\}\{\{x^j, x^l\}, x^l\} - \frac{1}{4}\{\{x^j, x^k\}, x^l\}\{\{x^j, x^k\}, x^l\} \Big),$$

where

$$\gamma^2 = \frac{1}{2} \sum_{i,k=1}^{m} \{x^i, x^k\}^2.$$

Here, x^i are the embedding coordinates of the surface in $\mathbb{R}^m$.

Higher dimensional manifolds?

In fact, for surfaces, everything can be written in terms of Poisson brackets (curvature, complex structure, Gauss equations, Codazzi equations, etc.).

What about higher dimensional manifolds? It turns out that there are two approaches to the problem. For a general n-dimensional Riemannian submanifold, the geometry can be expressed in terms of a n-ary multi linear bracket (a “Nambu-bracket”), much in the same way as for surfaces.

Is there a particular class of higher dimensional manifolds for which one can still use the Poisson bracket?
It turns out that **almost Kähler manifolds** provide the right framework for studying this question.

Kähler–Poisson structures

A Poisson bivector θ is such that $\{f, h\} \equiv \theta^{ab}(\partial_a f)(\partial_a h)$ defines a Poisson bracket.

Definition

Let (Σ, g) be a Riemannian manifold. A *Kähler–Poisson structure* on (Σ, g) is a Poisson bivector θ such that

$$\gamma^2 g^{ab} = \theta^{ap}\theta^{bq}g_{pq}.$$

for some $\gamma \in C^\infty(\Sigma)$.

Almost Kähler manifolds

Let Σ be an (almost) hermitian manifold with complex structure J and hermitian metric g, i.e. $g\big(J(X), J(Y)\big) = g(X, Y)$ for all vector fields X, Y.
The *Kähler form* ω is defined as $\omega(X, Y) = g\big(J(X), Y\big)$ for all vector fields X, Y. The manifold is called a (almost) Kähler manifold if the Kähler form is closed, i.e. $d\omega = 0$. Since ω is closed, its inverse defines a Poisson bracket on $C^\infty(\Sigma)$. It is easy to check that this is a Kähler–Poisson structure with $\gamma = 1$. In fact, one has the following result.

Proposition

Let (Σ, g) be a Riemannian manifold. A Kähler–Poisson structure exists on (Σ, g) if and only if it is an almost Kähler manifold.

Geometric setup

Let M be a m-dimensional Riemannian manifold with metric η, and let Σ be a n-dimensional submanifold, via the embedding functions $x^1, \ldots, x^m$, with induced metric g. Moreover, assume there exists a Kähler–Poisson structure θ on Σ.

Geometric quantities in the ambient space M will be denoted by a "bar", such as the covariant derivative $\bar{\nabla}$ and the Christoffel symbols $\bar{\Gamma}^i_{jk}$. Indices $i, j, k, l, \ldots$ run from 1 to m.

Local coordinates on Σ are denoted by u^a, and indices $a, b, c, d, \ldots$ run from 1 to n. We shall consider the x^i's as functions of the u^a's.

The projection operator

The Poisson bracket on $C^\infty(\Sigma)$ is then defined as

$$\{f,h\} = \theta^{ab}\big(\partial_a f\big)\big(\partial_b g\big).$$

A simple but important observation is the following: Define

$$\mathcal{D}^{ij} = \frac{1}{\gamma^2}\{x^i,x^k\}\{x^j,x^l\}\eta_{kl},$$

and let it act on TM as $\mathcal{D}(X)^i = \mathcal{D}^{ij}\eta_{jk}X^k$.

Proposition

The map $\mathcal{D} : TM \to TM$ is the orthogonal projection onto $T\Sigma$.

▸ Proof

By $\Pi = \mathbb{1} - \mathcal{D}$ we denote the projection onto the normal space.

Covariant derivatives

Let $\bar{\nabla}$ denote the covariant derivative on M. In local coordinates one writes

$$\bar{\nabla}_X Y^i = X^k \partial_k Y^i + \bar{\Gamma}^i_{jk} X^j Y^k.$$

Assuming $X, Y \in T\Sigma$ one can write $X^i = \mathcal{D}^{ij} X_j$ which gives

$$\begin{aligned}\bar{\nabla}_X Y^i &= \mathcal{D}^{kl} X_l \partial_k Y^i + \bar{\Gamma}^i_{jk} X^j Y^k \\ &= \frac{1}{\gamma^2}\{Y^i, x^j\}\{x^l, x^m\}\eta_{jm} X_l + \bar{\Gamma}^i_{jk} X^j Y^k,\end{aligned}$$

where all derivatives reside in Poisson brackets. Hence, the covariant derivative on Σ can be expressed in terms of Poisson brackets as

$$\nabla_X Y = \mathcal{D}\big(\bar{\nabla}_X Y\big)$$

for all $X, Y \in T\Sigma$.

Curvature

Having the projection operator and the covariant derivative in the ambient space in terms of Poisson brackets implies that all of submanifold theory can be expressed as Poisson brackets.

For instance, denoting

$$\hat{\nabla}_i = \mathcal{D}_i^k \bar{\nabla}_k$$

one derives Gauss' equation, expressing the curvature of Σ as

$$X^i Y^j Z^k V^l \Big[\bar{R}_{ijkl} + (\hat{\nabla}_k \Pi_{im})(\hat{\nabla}_l \Pi_j^m) - (\hat{\nabla}_l \Pi_{im})(\hat{\nabla}_k \Pi_j^m) \Big]$$

for $X, Y, Z, V \in T\Sigma$, where $\bar{R}_{ijkl}$ is the curvature tensor of M.

Theorems in Differential Geometry

As one is able to express geometry in terms of Poisson brackets, standard theorems in Riemannian geometry yield relations between Poisson brackets.

In case one has a matrix regularization of a manifold, these statements are mapped in to statements about commutators of matrices, giving non-trivial relations in the sequence of matrix algebras.

Let us consider a particular example. On a compact closed manifold, a lower bound on the Ricci curvature induces a lower bound on the eigenvalues of the Laplace operator.

Proposition

Let (Σ, g) be a compact closed surface with Gaussian curvature K and let $-\lambda$ be an eigenvalue of the Laplace operator. If $K \geq \kappa$ for some $\kappa \in \mathbb{R}$, then $\lambda \geq 2\kappa$.

Is there a corresponding theorem for matrix regularizations?

Let us assume that we have a matrix regularization of a surface embedded in $\mathbb{R}^m$, with $X^1, \ldots, X^m$ being matrices converging to the embedding coordinates.

Discrete Curvature and the Laplace Operator

Motivated by the Poisson bracket expressions for the Gaussian curvature and the Laplace operator, one defines

$$\hat{K} = \frac{1}{\hbar^4} \sum_{j,k,l=1}^{m} \bigg(\frac{1}{2} \big[[X^j, X^k], X^k\big] \big[[X^j, X^l], X^l\big] - \frac{1}{4} \big[[X^j, X^k], X^l\big] \big[[X^j, X^k], X^l\big] \bigg)$$

$$\hat{\Delta}(Y) = -\frac{1}{\hbar^2} \sum_{i=1}^{m} \big[[Y, X^i], X^i\big],$$

where Y is an arbitrary matrix. Moreover, one defines eigenvalues and eigenmatrices of $\hat{\Delta}$ in the standard way.

An Analogue Theorem for Matrix Regularizations

There are several subtleties when one considers sequences of matrices converging to functions. However, modulo these technical considerations, the analogue theorem can be stated as follows:

Proposition

Let $(T^\alpha, \hbar_\alpha)$ *be a matrix regularization of the surface* Σ *and let* $\{u_\alpha\}$ *be a convergent eigenmatrix sequence of* $\hat{\Delta}$ *with eigenvalues* $\{-\lambda_\alpha\}$. *If* $\hat{K} \geq \kappa \mathbb{1}$ *for some* $\kappa \in \mathbb{R}$ *then* $\lim_{\alpha \to \infty} \lambda_\alpha \geq 2\kappa$.

Dependence on an underlying manifold?

Apart from technical details about convergence, the proof of the previous theorem is immediate from the standard proof in differential geometry.

On the other hand, the theorem relates different matrix expressions in a non-trivial way. Can one prove such a theorem without assuming that the matrix algebras are regularizations of a manifold? Is there an intrinsic definition of matrix regularizations (without assuming any manifold) for which such theorems can be proven? Matrix algebras which then provide a reasonable non-commutative geometry?

Let us start by considering the case of commutative Poisson algebras and see if one can achieve this goal.

Algebraic abstraction

Having expressed differential geometry in terms of Poisson algebras, one may wonder if standard geometrical results (now written as Poisson algebra statements) hold for general Poisson algebras?

As expected, the class of *all* Poisson algebras is too large, and one has to find a suitable subclass that mimics a function algebra on a manifold.

Thus, we need to encode the condition $\gamma^2 g^{ab} = \theta^{ap}\theta^{bq}g_{pq}$ in the ambient space function algebra.

Kähler–Poisson algebras (with $M = \mathbb{R}^m$)

As we have seen, a simple consequence of the Kähler–Poisson structure condition is that $\mathcal{D}^{ij}$ is a projection. By denoting $\mathcal{P}^{ij} = \{x^i, x^j\}$ we formulate it in the following way:

Definition (Kähler–Poisson algebra)

Let $(\mathcal{A}, \{\cdot,\cdot\}$ be the field of fractions of the polynomial algebra $\mathbb{C}[x^1, \ldots, x^m]$ together with a Poisson structure $\{\cdot,\cdot\}$. The pair $(\mathcal{A}, \{\cdot,\cdot\})$ is called an *almost Kähler–Poisson algebra* if there exists $\gamma^2 \in \mathcal{A}$ such that

$$\mathcal{P}^i{}_j \mathcal{P}^j{}_k \mathcal{P}^k{}_l = -\gamma^2 \mathcal{P}^i{}_l \qquad (*)$$

where repeated indices are summed over from 1 to m. (Note that there is no difference between upper and lower indices.)

Tangent space and normal space

Let us consider the space of derivations $\mathrm{Der}(\mathcal{A})$, spanned by ∂_i, and let $\mathcal{P}$ act on $X = X^i\partial_i$ as

$$\mathcal{P}(X) = \mathcal{P}^i{}_j X^j \partial_i.$$

Condition $(*)$ implies that $\mathcal{D}^{ij} = \gamma^{-2}\mathcal{P}^i{}_k\mathcal{P}^{jk}$ is a projector, i.e. $\mathcal{D}^2 = \mathcal{D}$, which allows for a very natural definition of the tangent space of the "submanifold" as a projective module.

$$\mathcal{X}(\mathcal{A}) = \{\mathcal{D}(X) : X \in \mathrm{Der}(\mathcal{A})\}$$

The dimension of $\mathcal{X}(\mathcal{A})$ is called the *geometric dimension of $\mathcal{A}$*. By writing $\Pi = \mathbb{1} - \mathcal{D}$ we also obtain the normal space as

$$\mathcal{N}(\mathcal{A}) = \{\Pi(X) : X \in \mathrm{Der}(\mathcal{A})\}.$$

We also set $(X, Y) = X^i Y_i$.

Covariant derivative

We learned from differential geometry that the covariant derivative on the submanifold can be written as (in the case of $M = \mathbb{R}^m$)

$$\nabla_X Y^i = \mathcal{D}\big(\hat{\nabla}_X Y\big)^i = \mathcal{D}^{ij} X^k \mathcal{D}_k(Y_j),$$

where $\mathcal{D}_k(u) = \mathcal{D}^l_k \partial_l u = \gamma^{-2}\{u, x^l\}\mathcal{P}_{kl}$. Let us take this as a definition for derivations $X, Y \in \mathcal{X}(\mathcal{A})$.

$$\nabla_X Y^i = \frac{1}{\gamma^4} X^k \{x^i, x^l\}\{x^j, x_l\}\{Y_j, x^m\}\{x_k, x_m\}.$$

Affine connection

Proposition

Let $\mathcal{A}$ be an almost Kähler–Poisson algebra. For all $X, Y, Z \in \mathcal{X}(\mathcal{A})$ and $u \in \mathcal{A}$, the covariant derivative has the following properties

1. $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z,$
2. $\nabla_{(X+Y)} Z = \nabla_X Z + \nabla_Y Z,$
3. $\nabla_{(uX)} Y = u\nabla_X Y,$
4. $\nabla_X(uY) = \nabla_X(u)Y + u\nabla_X Y,$

where $\nabla_X(u) = X^k \mathcal{D}_k(u).$

Torsion-free metric connection

Proposition

The covariant derivative in an almost Kähler–Poisson algebra has no torsion, i.e. $\nabla_X Y - \nabla_Y X - [X, Y] = 0$ *for all* $X, Y \in \mathcal{X}(\mathcal{A})$.

Proposition

In an almost Kähler–Poisson algebra it holds that $(\nabla_X \mathcal{D})(Y, Z) = 0$ *for all* $X, Y, Z \in \mathcal{X}(\mathcal{A})$.

Bianchi identities

By introducing

$$R(X,Y,Z) \equiv R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$$

one can prove the Bianchi identities.

Proposition

Let $\mathcal{A}$ be an almost Kähler–Poisson algebra and let R be the curvature tensor of $\mathcal{A}$. For all $X, Y, Z, V \in \mathcal{X}(\mathcal{A})$ it holds that

$$R(X,Y,Z) + R(Z,X,Y) + R(Y,Z,X) = 0$$
$$(\nabla_X R)(Y,Z,V) + (\nabla_Y R)(Z,X,V) + (\nabla_Z R)(X,Y,V) = 0.$$

Sectional curvature

Introduce the sectional curvature with respect to $X, Y \in \mathcal{X}(\mathcal{A})$

$$K(X, Y) = \frac{R(X, Y, X, Y)}{(X, X)(Y, Y) - (X, Y)^2}.$$

Proposition

Let $\mathcal{A}$ be an almost Kähler–Poisson algebra with curvature tensor R and geometric dimension $n \geq 3$. If $K(X, Y) = k \in \mathcal{A}$ for all $X, Y \in \mathcal{X}(\mathcal{A})$ then $\{k, u\} = 0$ for all $u \in \mathcal{A}$.

Bounding eigenvalues of the algebraic Laplacian?

Can one prove a slightly more non-trivial theorem for almost Kähler–Poisson algebras? Let us consider the example of bounds on the eigenvalues of the Laplacian.

Differential geometric proof

Let us recall the proof. One rewrites

$$\int_\Sigma (\Delta u)^2 = -\lambda \int_\Sigma u\Delta u = \lambda \int_\Sigma |\nabla u|^2$$

On the other hand

$$\begin{aligned}\int_\Sigma (\Delta u)^2 &= \int_\Sigma \nabla_i\nabla^i(u)\nabla_k\nabla^k(u) = -\int_\Sigma \nabla^i(u)\nabla_i\nabla_k\nabla^k(u)\\ &= -\int_\Sigma \Big(\nabla^i(u)\nabla_k\nabla_i\nabla^k(u) - R_{ik}\nabla^i(u)\nabla^k(u)\Big)\\ &\geq \frac{1}{n}\int_\Sigma (\Delta u)^2 + \kappa\int_\Sigma |\nabla u|^2 = \Big(\frac{\lambda}{n}+\kappa\Big)\int_\Sigma |\nabla u|^2\end{aligned}$$

Comparing the two calculations gives $\lambda \geq n\kappa/(n-1)$.

What do we actually use?

- Relation between covariant derivatives and curvature
- Partial integration
- Cauchy-Schwartz inequality.

Let us assume that $\mathcal{A}$ is a $*$-algebra, i.e. there exists a map $*:\mathcal{A}\to\mathcal{A}$ such that $(a^*)^* = a$, $(a+b)^* = a^* + b^*$, $(ab)^* = b^*a^*$ and $(\lambda a)^* = \bar{\lambda}a^*$ for $a\in\mathcal{A}$ and $\lambda\in\mathbb{C}$.

Positive Linear Functionals – Partial Integration

A linear functional μ is a $\mathbb{C}$-linear map $\mu : \mathcal{A} \to \mathbb{C}$ such that $\mu(a^*) = \overline{\mu(a)}$. Furthermore, μ is called *positive* if $\mu(a^* a) \geq 0$ for all $a \in \mathcal{A}$.

We shall assume that μ fulfills the divergence theorem, i.e.

$$\mu\Big(\operatorname{div}(X)\Big) \equiv \mu\Big(\nabla_i X^i\Big) = 0$$

for all $X \in \mathcal{X}(\mathcal{A})$.

In a matrix regularization, the ordinary matrix trace satisfies the divergence theorem due to $\mathrm{Tr}[A, B] = 0$.

The Positive Cone and Inner Products

In a general $*$-algebra, one introduces the *positive cone* $\mathcal{A}_+$ consisting of elements of the form

$$a_1^* a_1 + a_2^* a_2 + \cdots + a_n^* a_n.$$

An element in $a \in \mathcal{A}_+$ is called *positive* and we write $a \geq 0$, and $a \geq b$ if $a - b \geq 0$.

It is easy to check that the $\mathcal{A}$-valued bilinear form on $\mathcal{X}(\mathcal{A})$ defined as $(X, Y) = X^i Y_i$ satisfies

$$(X^*, X) \geq 0,$$

where X^* is defined via $X^*(a) = \big(X(a^*)\big)^*$, which implies a Cauchy-Schwartz inequality.

A Bound on the Eigenvalues of the Laplacian

Modulo a few technical details, one can now provide a completely algebraic proof of the following theorem.

Proposition

Let $\mathcal{A}$ be an almost Kähler–Poisson algebra with $\dim(\mathcal{X}(\mathcal{A})) = n$ *and a positive linear functional satisfying the divergence theorem. Let $-\lambda$ be an eigenvalue of the Laplace operator. If there exists $\kappa \in \mathbb{R}$ such that $R(X^*, X) \geq \kappa(X^*, X)$ for all $X \in \mathcal{X}(\mathcal{A})$ then $\lambda \geq n\kappa/(n-1)$.*

Summary

- We have shown that the differential geometry of an almost Kähler submanifold can be expressed as Poisson brackets of the embedding coordinates.
- Consequently, we defined almost Kähler–Poisson algebras, as algebraic analogues of function algebras.
- Almost Kähler–Poisson algebras have natural concepts of tangent and normal space, as well as a nice theory of curvature.
- The connection has all the properties one wants, like being torsion free and metric as well as satisfying the Bianchi identities.
- We have illustrated the usefulness of these algebras by proving algebraic counterparts of several classical theorems in differential geometry.

Outlook

- How far can one push the analogy with differential geometry? Can we prove more theorems? A theory of Chern classes?
- What is the natural algebraic generalization of submanifolds of curved spaces?
- Can one choose more general types of algebras (and fields) in the definition of almost Kähler–Poisson algebras?
- Non-commutative Kähler–Poisson algebras? Might provide an interesting concept of non-commutative geometry.

Proof

Let X be a vector in $T\Sigma$ and write $X = X^i\partial_i = X^a\partial_a$. One then computes

$$
\begin{aligned}
\mathcal{D}^{ij}X_j &= \frac{1}{\gamma^2}\theta^{ab}(\partial_a x^i)(\partial_b x^k)X_j\theta^{pq}(\partial_p x^j)(\partial_q x^l)\eta_{kl} \\
&= \frac{1}{\gamma^2}\theta^{ab}\theta^{pq}g_{bq}(\partial_a x^i)(\partial_p x^j)X_j \\
&= g^{ap}(\partial_a x^i)(\partial_p x^j)X^c(\partial_c x^k)\eta_{jk} \\
&= g^{ap}g_{pc}X^c(\partial_a x^i) = X^a(\partial_a x^i) = X^i.
\end{aligned}
$$

Since $(\partial_a x^i)N_i = 0$ for any vector normal to the submanifold, it follows that $\mathcal{D}^{ij}N_j = 0$.

Back