Quantum Cosmology & the Very Early Universe

Abhay Ashtekar

Institute for Gravitation and the Cosmos, Penn State

Detailed Review: AA & Singh; Available late June/early July

Understanding emerged from the work of a large body of researchers:

Bojowald, Barrau, Calcagni, Campiglia, Corichi, Henderson, Kaminski, Lewandowski, Martin-Benito, Mena, Pawlowski, Rovelli, Singh, Sloan, Vandersloot, Wilson-Ewing ....

Quantum Theory & Gravitation Conference, June 15th 2011

Organization

- Loop Quantum Gravity (LQG) (Rovelli, Lewandowski, Speziale, Freidel & Barrett talks) has several interfaces: Quantum Geometry, NC Field Theory, Regge Calculus, Group Field Theory, State Sum Models, Cosmology, ...

- Goal: i) Present simple examples of the powerful role of the quantum geometry underlying LQG;
  ii) Illustrate conceptual issues: Issue of Time, Non-perturbative Dynamics, Fate of Singularities; UV-IR Tension, Entropy Bounds...
  iii) Potential for confronting quantum gravity with observations.
  Inflation Issues, Origin of the Bunch Davis Vacuum, Primordial Grav Waves, ...

- Organization:

1. Conceptual Setting
2. Loop Quantum Cosmology: Basic Results
3. Novel features at the Foundation
4. Applications & Extensions
   (Entropy bound; Inflation; QFT on QSTs, Spin Foams, Group Field Theory)
5. Summary & Discussion

1. Conceptual Setting

• In general relativity, the gravitational field encoded is in the very geometry of space-time $\Rightarrow$ space-time itself ends at singularities (also in inflationary scenarios (Borde,Guth Vilenkin)). General expectation: theory is pushed beyond its domain of applicability. Need Quantum Gravity: Singularities are our gateways to physics beyond Einstein.

• Some Long-Standing Questions expected to be answered by Quantum Gravity Theories from first principles:

$\star$ How close to the big-bang does a smooth space-time of GR make sense? (Onset of inflation?)

$\star$ Is the Big-Bang singularity naturally resolved by quantum gravity? (answer is 'No' in the Wheeler-DeWitt theory)

$\star$ Is a new principle/ boundary condition at the Big Bang essential? (e.g. The Hartle-Hawking 'no-boundary proposal'.)

$\star$ Is the quantum evolution across the 'singularity' deterministic? (So far the answer is 'No' e.g. in the Pre-Big-Bang and Ekpyrotic scenarios)

$\star$ What is on the 'other side'? A quantum foam? Another large, classical universe? ...

Older Quantum Cosmology (DeWitt, Misner, Wheeler . . . 70's)

- Since only finite number of DOF $a(t), \phi(t)$, field theoretical difficulties bypassed; analysis reduced to standard quantum mechanics.

- Quantum States: $\Psi(a, \phi)$; $\hat{a}\Psi(a, \phi) = a\Psi(a, \phi)$ etc.

Quantum evolution governed by the Wheeler-DeWitt differential equation

$$\ell_{\rm Pl}^4 \; \frac{\partial^2}{\partial a^2}(f(a)\Psi(a, \phi)) = {\rm const}\, G \; \hat{H}_\phi \, \Psi(a, \phi)$$

Without additional assumptions, e.g. matter violating energy conditions, singularity is not resolved. Precise Statement provided by the consistent histories approach (Craig & Singh).

General belief since the seventies: This is a real impasse because of the von-Neumman's uniqueness theorem.

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General belief since the seventies: This is a real impasse because of the von-Neumman's uniqueness theorem.

- In LQC, situation is very different. How is this possible? If one follows the procedure used in LQG (Lewandowski's talk), one of the assumptions of the von Neumann theorem violated $\Rightarrow$ uniqueness result bypassed.

Inequivalent representations even for mini-superspaces. New quantum mechanics (AA, Bojowald, Lewandowski). Novel features precisely in the deep Planck regime.

Some Long Standing Questions (contd)

$\star$ How does one extract physics from solutions to the Hamiltonian constraint (e.g. WDW equation)? Dynamics from the frozen formalism? Dirac observables? Emergent time? (Scale factor —natural candidate in the Misner parametrization— not single-valued in closed models.)

$\star$ Can one have a deterministic evolution across the singularity and agreement with GR at low curvatures, e.g., recollpase in the closed models? (Background dependent perturbative approaches have difficulty with the first while background independent approaches, with second (Green and Unruh))

In LQC, these issues have been resolved for several minisuperspaces.
(Bojowald; AA, Pawlowski, Singh, Vadersloot, ...)
(Scalar field as internal/emergent time; Physical Hilbert space, Dirac observables, semi-classical states, detailed dynamics.)

The physical sector of the theory can be constructed in detail. GR a good approximation till curvature $\sim 10^{-2} m_{\rm Pl}^2$, but the singularities are resolved. Evolution deterministic. No new principle needed.

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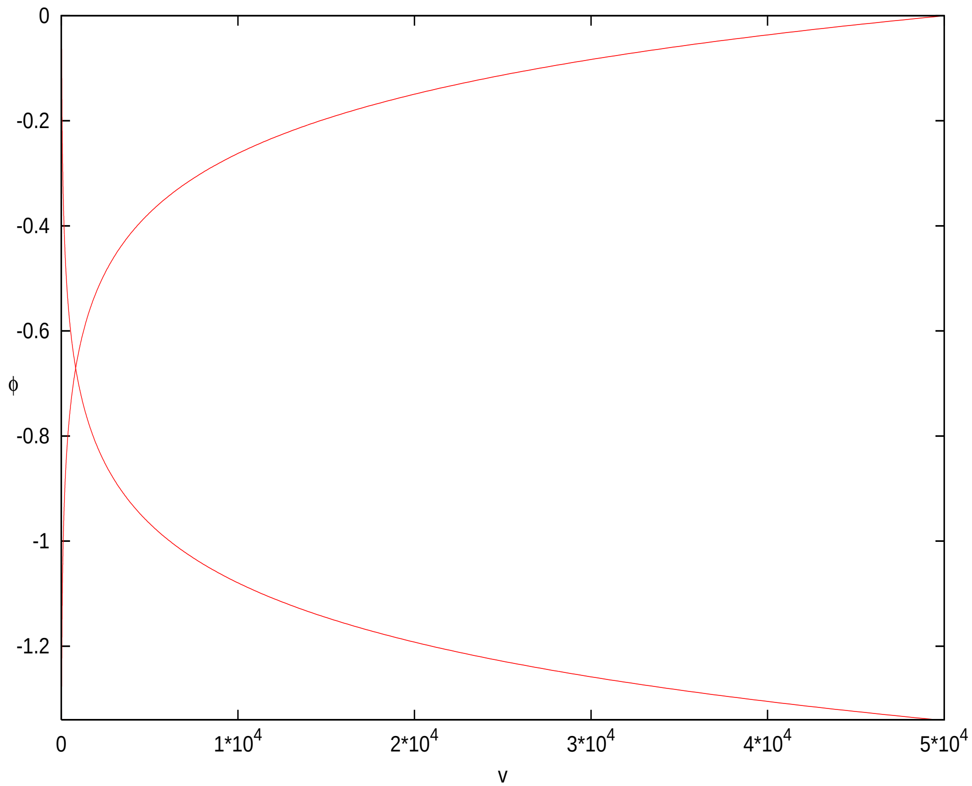
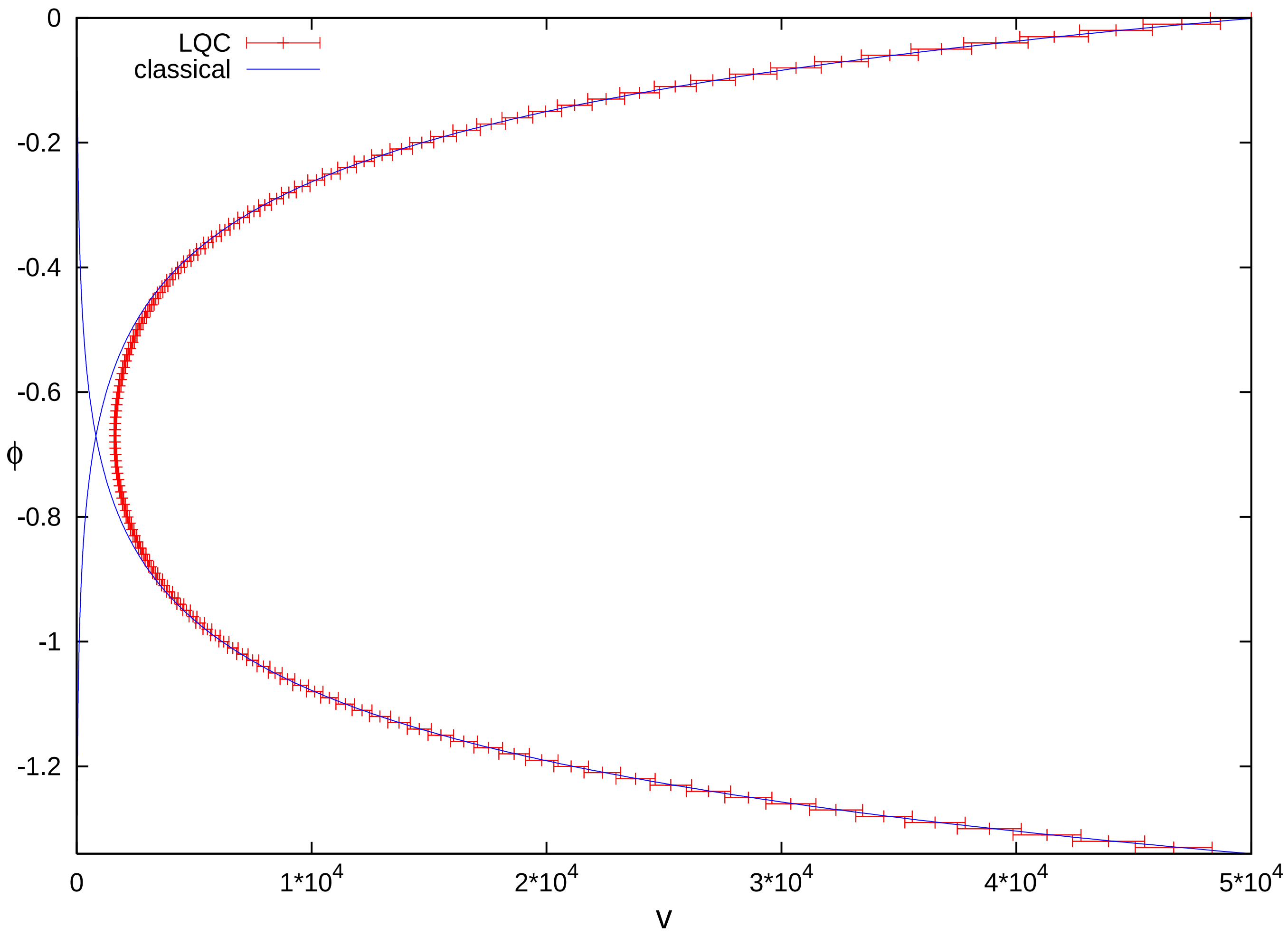
2. Loop Quantum Cosmology: Basic Results

FLRW, k=0, $\Lambda = 0$ Model coupled to a massless scalar field ϕ. Instructive because every classical solution is singular. Provides a foundation for more complicated models.



Classical trajectories

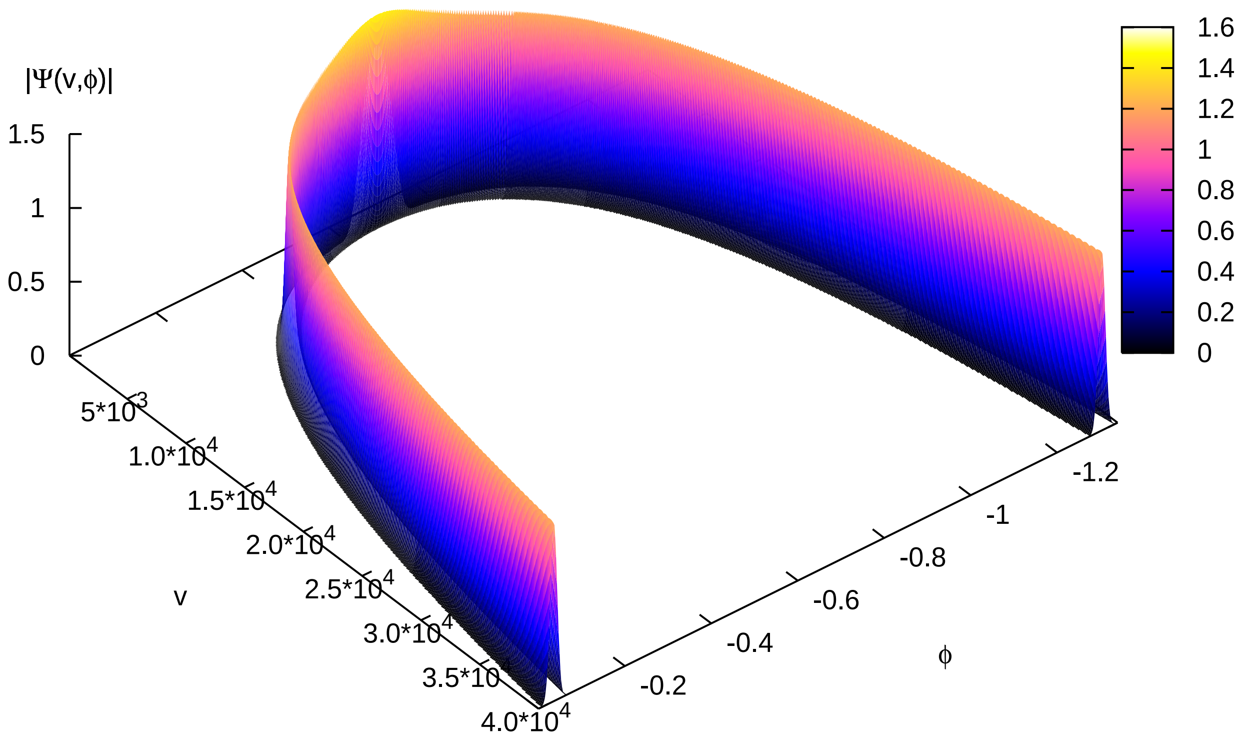
k=0 LQC



Expectations values and dispersions of $\hat{V}|_{\phi}$ & classical trajectories.

(AA, Pawlowski, Singh) Gamow's favorite paradigm realized.

k=0 LQC



Absolute value of the physical state $\Psi(v, \phi)$

(AA, Pawlowski, Singh)

k=0 Results

Assume that the quantum state is semi-classical at a late time and evolve backwards and forward. Then: (AA, Pawlowski, Singh)

• The state remains semi-classical till *very* early and *very* late times, i.e., till $R \sim 10^{-2} m_{\rm Pl}^2$ or $\rho \sim 10^{-3} \rho_{\rm Pl}$. $\Rightarrow$ We know 'from first principles' that space-time can be taken to be classical during the inflationary era (since $\rho \sim 10^{-12} \rho_{\rm Pl}$ at the onset of inflation).

• In the deep Planck regime, semi-classicality fails. But quantum evolution is well-defined through the Planck regime, and remains deterministic unlike in other approaches. No new principle needed. The final quantum space-time is vastly larger than what general relativity had us believe.

• No unphysical matter. All energy conditions satisfied. But the left side of Einstein's equations modified because of quantum geometry effects: Main difference from WDW theory. Finally, Effective equations surprisingly effective!

k=0 Results

- To compare with the standard Friedmann equation, convenient to do an algebraic manipulation and move the quantum geometry effect to the right side. Then the Quantum Corrected, Effective Friedmann Eq is:

$$(\dot{a}/a)^2 = (8\pi G\rho/3)[1 - \rho/\rho_{\rm crit}] \quad \text{where } \rho_{\rm crit} \sim 0.41\rho_{\rm Pl}.$$

Big Bang replaced by a quantum bounce.

- The matter density operator $\hat{\rho} = \frac{1}{2}\,(\hat{V}_\phi)^{-1}\,\hat{p}^2_{(\phi)}\,(\hat{V}_\phi)^{-1}$ has an absolute upper bound on the physical Hilbert space (AA, Corichi, Singh):

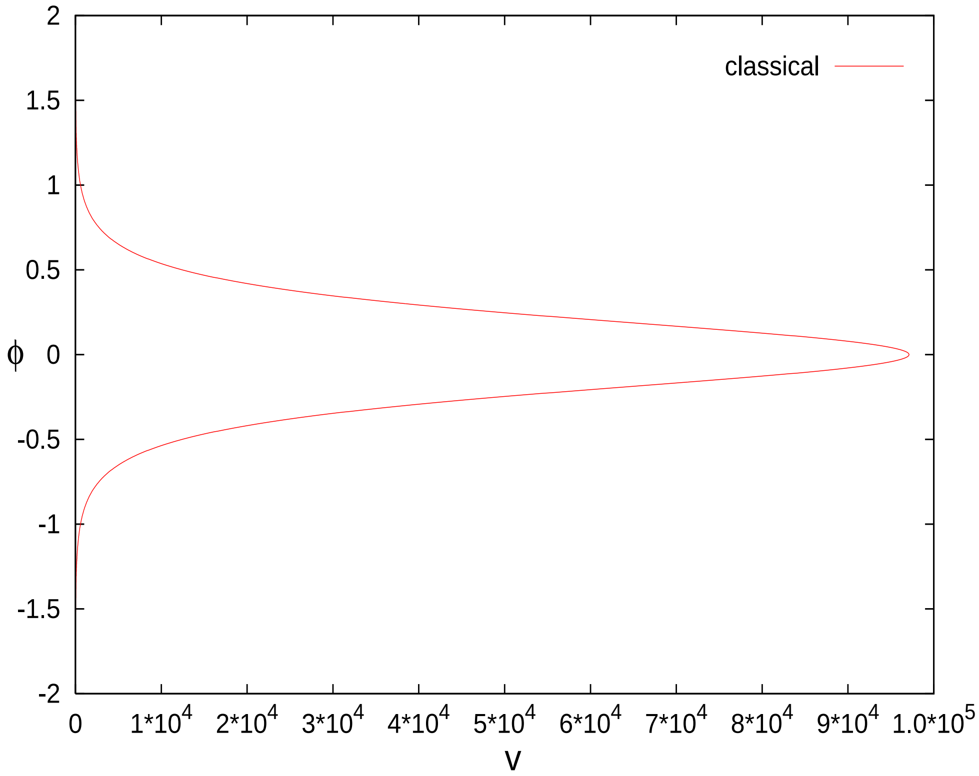
$$\rho_{\rm sup} = \sqrt{3}/16\pi^2\gamma^3 G^2\hbar \approx 0.41\rho_{\rm Pl}!$$

Provides a precise sense in which the singularity is resolved. (Brunnemann & Thiemann)

- Quantum geometry creates a brand new repulsive force in the Planck regime, replacing the big-bang by a quantum bounce. Repulsive forces due to quantum matter are familiar: Fermi degeneracy pressure in Neutron stars. Difference: Quantum nature of geometry rather than matter. Rises and dies extremely rapidly but strong enough to resolve the singularity.

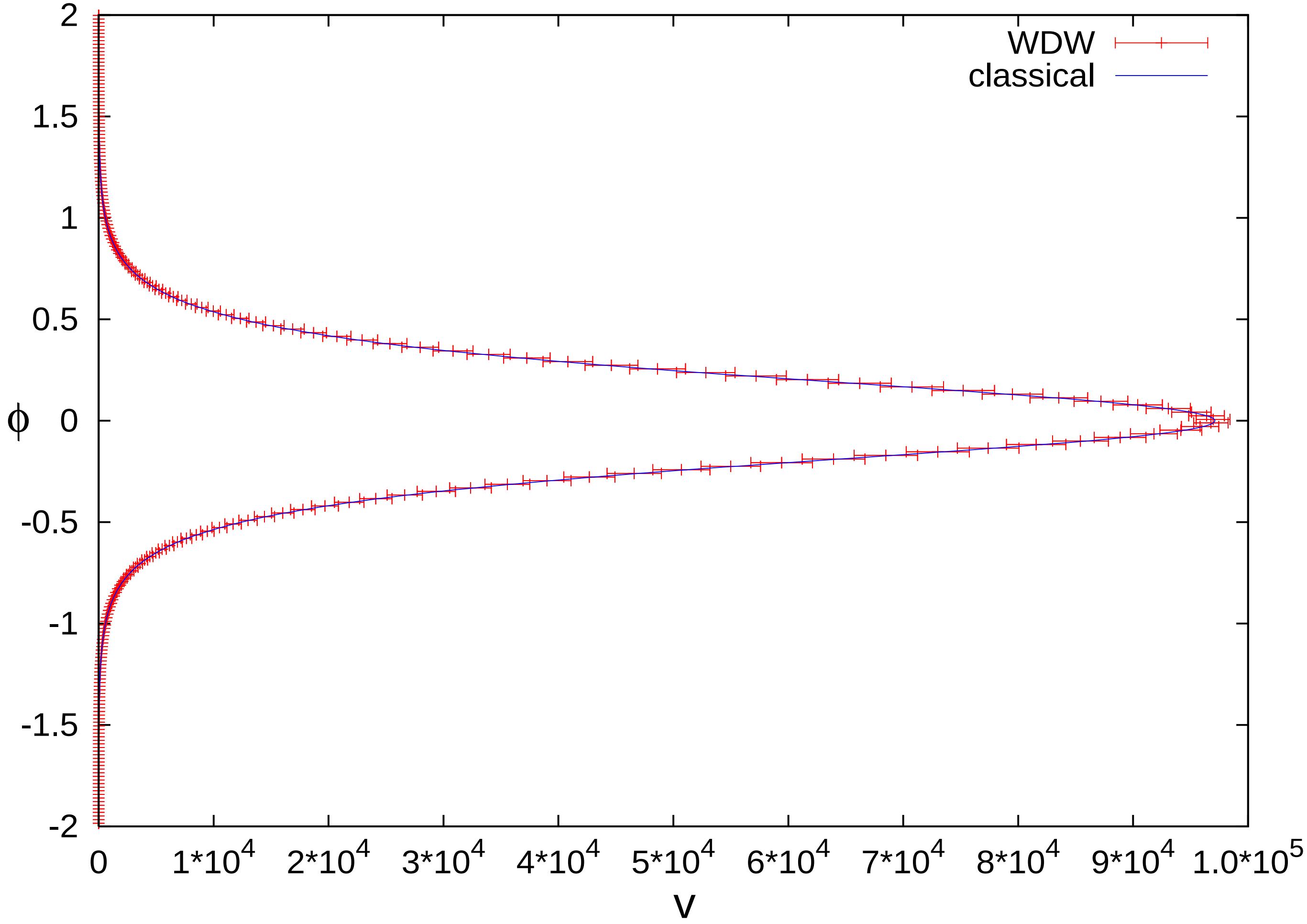
The Closed Model: Bouncing/Phoenix Universes.

Another Example: k=1 FLRW model with a massless scalar field ϕ. Instructive because again every classical solution is singular; scale factor not a good global clock; More stringent tests because of the classical re-collapse. (Tolman, Sakharov, Dicke,...)



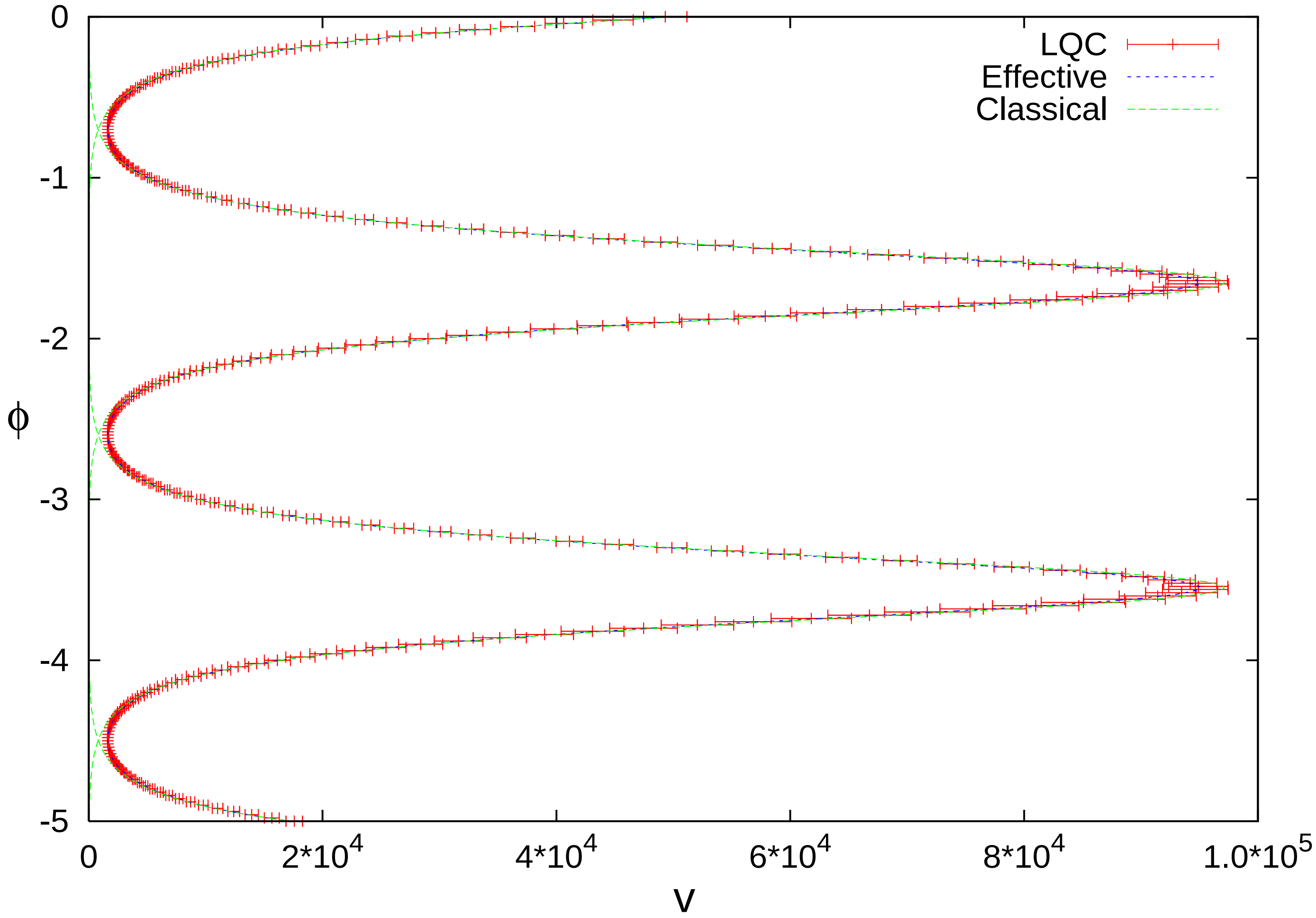
Classical Solutions

k=1 Model: WDW Theory



Expectations values and dispersions of $\hat{V}|_\phi$.

k=1 Model: LQC



Expectations values and dispersions of $\hat{V}|_\phi$ & classical trajectories.

(AA, Pawlowski, Singh, Vandersloot)

k=1: Domain of validity of classical GR

(AA, Pawlowski, Singh, Vandersloot)

- Classical Re-collapse: The infrared issue.

$$\rho_{\rm min} = (3/8\pi G a_{\rm max}^2)\left(1 + O(\ell_{\rm Pl}^4/a_{\rm max}^4)\right)$$

So, even for a very small universe, $a_{\rm max} \approx 23\ell_{\rm Pl}$, agreement with the classical Friedmann formula to one part in 10^5. Classical GR an excellent approximation for $a > 10\ell_{\rm Pl}$. For macroscopic universes, LQC prediction on recollapse indistinguishable from the classical Friedmann formula.

- Quantum Bounces: The ultra-violet issue

For a universe which attains $v_{\rm max} \approx 1\,{\rm Gpc}^3$,

$$v_{\rm min} \approx 6 \times 10^{18}{\rm cm}^3 \approx 10^{117}\ell_{\rm Pl}^3\text{:}\ 6km \times 18km \times 54km \text{ Mountain!}$$

What matters is curvature, which enters Planck regime at this volume.

Generalizations

- Inclusion of Λ (A B P): $\surd$ (Infrared limit trickier)
  Inclusion of a $m^2\phi^2$ inflationary potential (A P S): $\surd$

- More general singularities: At finite proper time, scale factor may blow up, along with similar behavior of density or pressure (Big rip) or curvature or their derivatives diverge at finite values of scale factor (sudden death). Quantum geometry resolves all strong singularities in homogeneous isotropic models with $p = p(\rho)$ matter (Singh).

- Beyond Isotropy and Homogeneity:
  Bianchi Models (A W-E): $\surd$ (Anisotropies & Grav Waves)
  The Gowdy model (G M-B M W-E): $\surd$ (Inhom and Grav Waves.)

These results by AA, Bentevigna, Garay, Martin-Benito, Mena, Pawlowski, Singh, Vandersloot, Wilson-Ewing, ... show that the singularity resolution is quite robust. Anytime a physical observable reaches the Planck regime, the repulsive effect from quantum geometry effect becomes dominant and dilutes it.

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3. Novel Features at the Foundation

• Why was LQC able to resolve the Big Bang singularity when the WDW theory had failed in these models?

• In the WDW quantum cosmology, one did not have guidance from a full quantum gravity theory. Therefore, in quantum cosmology, one just followed standard QM and constructed the Schrödinger representation of the fundamental Weyl algebra.

• By contrast, quantum kinematics of LQG has been rigorously developed. Background independence $\Rightarrow$ unique representation of the kinematic algebra (Lewandowski, Okolow, Sahlmann, Thiemann; Fleishhack) Provides the arena to formulate quantum Einstein equations.

• In LQC we could mimic this framework step by step. One of the assumptions of the von Neumann uniqueness theorem for quantum mechanics is bypassed. In LQC we are led to an inequivalent representation of the Weyl algebra; i.e., new quantum mechanics. WDW theory and LQC are distinct already kinematically!

LQC Kinematics

- The canonically conjugate variables of LQG:

A_a^i, SU(2) gravitational connections and, E_i^a, orthonormal triads.
Spatial homogeneity and isotropy implies

$\star \quad A_a = c \underbrace{\mathring{\omega}_a^i \sigma_i}_{\text{fixed}}, \quad E^a = p \underbrace{\mathring{e}_i^a \sigma^i}_{\text{fixed}} \quad c \sim \dot{a}; \quad |p| = a^2$

$\star$ holonomy: $h_e(c) = \cos\mu c\, \mathbf{1} + \sin\mu c\, \dot{e}^a \mathring{\omega}_a^i \sigma_i$
(Almost periodic in c)

$\star$ Canonically conjugate pairs:
c, p for gravity $\qquad \phi, p_\phi$ for matter

- In full LQG: Generalized connections $\mathcal{A} \to \bar{\mathcal{A}}$;

$\mathcal{H} = L^2(\bar{\mathcal{A}}, \mathrm{d}\mu_o)$; Holonomy operators well-defined; but not connection operators ! Quantum geometry emerges in this representation.

- Following the procedure in full LQG, we are led to:

$c \in \mathbb{R} \;\; \rightarrow \;\; \bar{c} \in \bar{\mathbb{R}}_{\rm Bohr}$ and $\mathcal{H} = L^2(\bar{\mathbb{R}}_{\rm Bohr}, \mathrm{d}\mu_o)$;
Holonomy operators $\hat{h}_\mu$ well-defined on $\mathcal{H}$.
But fail to be continuous in $\mu \;\; \Rightarrow$ no connection operator $\hat{c}$!

Dynamics

• The LQC kinematics cannot support the WDW dynamics. The Hamiltonian constraint involves the field strength F_{ab} of the gravitational connection $A_a = c\, \mathring{\omega}_a^i \sigma_i$. In LQC, the corresponding operator $\hat{F}_{ab}$ is constructed from holonomies around closed loops (that enclose minimum non-zero area). Classical, local F_{ab} recovered only if we coarse grain to ignore the area gap.

• As a result, the dynamical WDW differential equation is replaced by a difference equation.

$$\partial_\phi^2 \Psi(v,\phi) \,=\, C^+(v)\, \Psi(v+4,\, \phi) + C^o(v)\, \Psi(v,\phi) + C^-(v)\Psi(v-4,\, \phi)$$

where the step size is governed by the ‘area gap’ of quantum geometry.

• Good agreement with the WDW equation at low curvatures **but drastic departures in the Planck regime** precisely because the WDW theory ignores quantum geometry. Non-triviality: LQC, based on the new kinematic arena and quantum geometry of LQG has good UV **as well as** good IR properties.

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4. Applications: I. Inflation

- Inflationary scenarios (k=0, FLRW with a scalar field) have had tremendous success with the 7year WMAP data & structure formation. Natural question: How generic is the necessary slow roll inflationary phase?

- Even if a theory allows for inflation, a sufficiently long slow roll may need extreme fine tuning. To test this, we need a measure on the space $\mathcal{S}$ of solutions to the equations. Elegant solution: Use the Liouville measure to calculate a priori probabilities (Gibbons, Hawking, Page, ...). They are useful, if extremely low or extremely high.

- Controversy in the literature. For the $m^2\phi^2$ potential, answers from probability close to 1 (Kauffman, Linde, Mukhanov) to e^{-165} (Gibbons, Turok)! Main Reason: The question is ill posed in general relativity.

- Problem: The Liouville volume of $\mathcal{S}$ is infinite! But the infinity is a gauge artifact associated with the $a \to \lambda\, a$ rescaling freedom. (The Hamiltonian vector field is mapped to itself but the symplectic structure is rescaled.) To extract a finite total measure on $\mathcal{S}$ requires a choice of time instant. No natural choice in GR except the Big Bang but everything diverges there!

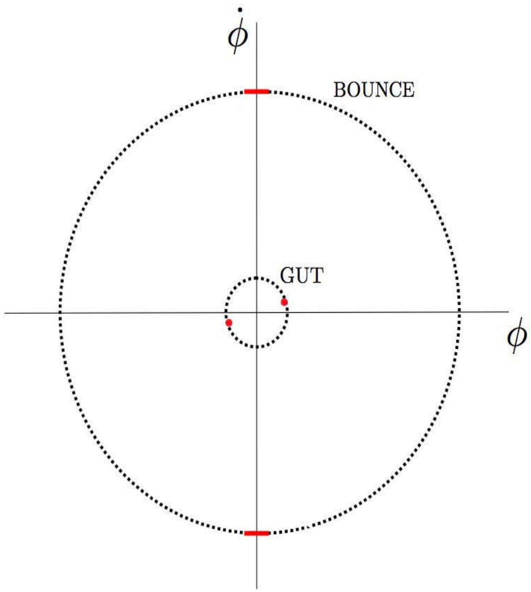
Probability of the WMAP slow roll in LQC

• In LQC, the Big Bang is replaced by the Big Bounce where the effective geometry and matter fields are all smooth. So, it is natural to use that surface to induce a finite measure on the space $\mathcal{S}$ of solutions of effective equations of LQC.

• Start with generic data at the bounce. Evolve. Will it enter slow roll at the $\sim$ GUT energy scale determined by the 7 year WMAP data ($\rho \approx 7.32 \times 10^{-12} m_{\rm Pl}^4$) ? Note: 11 orders of magnitude from the bounce to the onset of the desired slow roll!

• Answer yes, except for an **extremely** tiny part $\mathcal{R}$ of the space of initial data! Probability of NOT achieving the slow roll compatible with WMAP data, in particular with ~ 63 e-foldings:

$$P_{\mathcal{R}} = [\int_{\mathcal{R}} \mathrm{d}\mu_{\rm L}]/[\int_{\mathcal{R}} \mathrm{d}\mu_{\rm L}] \ < 3 \ \times 10^{-6}$$



• This is only an a priori probability. But because it is so high, it would be heavy burden on additional inputs to change them significantly.

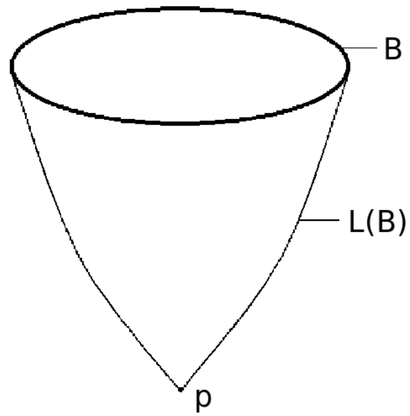
4. Applications: III. Bousso's Covariant Entropy Bound

- Conjecture (Simplest Version): The matter entropy flux across $\mathcal{L}(\mathcal{B})$ is bounded by

$$S := \int_{\mathcal{L}(\mathcal{B})} S^a dA_a \leq \ (A_{\mathcal{B}}/4\ell_{\rm Pl}^2).$$

- Curious features:

i) Requires a notion of entropy current;
ii) Refers to quantum gravity;
iii) Requires a classical geometry.
Consequently, quite difficult to test in practice!



- In classical GR:

If we consider k=0 FLRW models filled with radiation,

$$\frac{S}{(A_{\mathcal{B}}/4\ell_{\rm Pl}^2)} = \frac{2}{3}\ (\frac{2}{45\pi})^{1/4}\ \frac{\sqrt{\ell_{\rm Pl}}}{\sqrt{\tau_f}}\ \left(1 - \sqrt{\frac{\tau_i}{\tau_f}}\right)$$

For round $\mathcal{B}$, the bound holds if $\rho_f < 8.3\rho_{\rm Pl}$ but
arbitrarily large violations
in the deep Planck regime near the singularity.

- LQC provides an ideal arena:

i) Singularity is resolved by quantum gravity;
ii) The wave function is sharply peaked about a mean metric, a smooth field (although coefficients involve $\hbar$).

- Answer: $\frac{S}{(A_{\mathcal{B}}/4\ell_{\rm Pl}^2)} < 0.976$ (AA, Wilson-Ewing)

The bound is satisfied in LQC!

- Illustrates that the entropy bound neednot be a fundamental ingredient in the construction of the theory. It can simply arise in suitable regimes because of other fundamental considerations such as quantum geometry.

5. Discussion: Merits and Limitations of QC

One's first reaction: Symmetry reduction gives only toy models! Full theory much richer and much more complicated. But examples can be powerful.

- Full QED versus Dirac's hydrogen atom.
- Singularity theorems versus first discoveries in simple models.
- BKL behavior: homogeneous Bianchi models.

Do *not* imply that behavior found in examples is necessarily generic. Rather, they can reveal important aspects of the full theory and should not be dismissed a priori.

One can work one's way up by considering more and more complicated cases. (e.g. the Gowdy models have infinite degrees of freedom). At each step, models provide important physical checks well beyond formal mathematics. Can have strong lessons for the full theory.

5. Summary

• Quantum geometry creates a brand new repulsive force in the Planck regime, replacing the big-bang by a quantum bounce. Repulsive force rises and dies *very* quickly but makes dramatic changes to classical dynamics. ('Origin': quantum corrections to Einstein's equations due to area gap.) Physics does not end at singularities.

• A large number of cosmological models have been analyzed; all strong curvature singularities are removed in LQC. Emerging scenario: Anytime a curvature scalar threatens to diverge, quantum gravity repulsion kicks in and cures the UV problem of GR. Yet agreement with GR in the IR regime.

• Detailed analysis in specific models. But the BKL conjecture on the nature of space-like strong curvature singularities in general relativity suggests that the present results may suffice to imply general singularity resolution theorems for space-like singularities of GR (AA, Henderson, Sloan).

Further applications: Examples

- Cosmological Spin Foams (AA, Campiglia, Henderson, Nelson, Rovelli, Vidotto, Wilson-Ewing) Very significant recent advances in Spin Foam Models and Group Field Theory in full LQG. But several important issues remain.

  In cosmological models, these issues have been addressed rigorously by recasting the well-defined Hamiltonian theory as a sum over histories. Answers provide clear support for the spin-foam paradigm and provides concrete hints for further work.

- QFT in cosmological Quantum space-times (AA, Kaminski, Lewandowski). Apparent tension because underlying structures are so different.

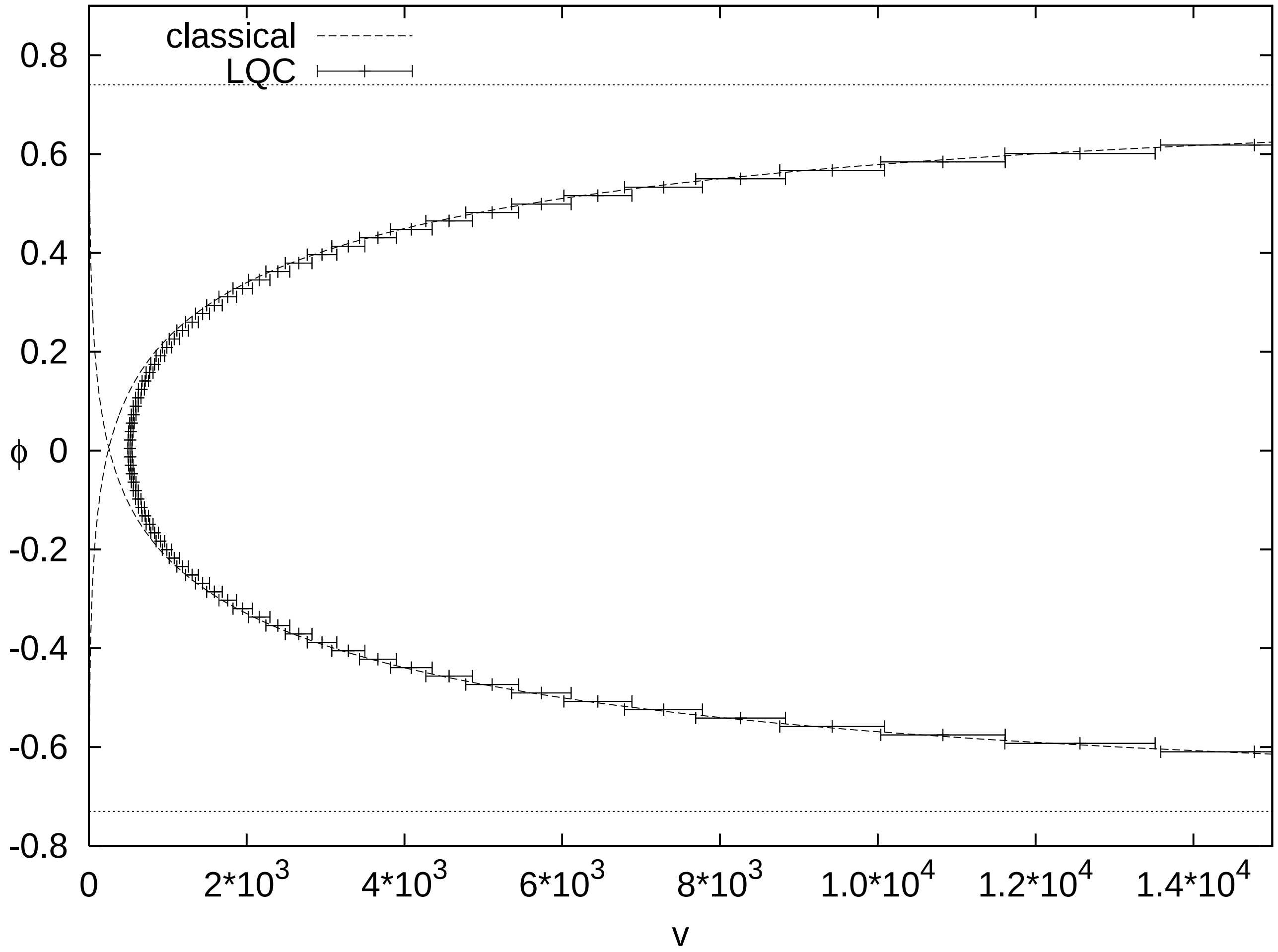
  Yet, through systematic approximations, one arrives at the QFT in CST as practised by cosmologists starting from QFT on QST. Arena well suited for studying cosmological perturbations from the bounce to the onset of inflation. Phenomenological ramifications are being studied. Ex: New avenues to Non-Gaussianity.

Application of Loop Quantum Gravity to Cosmological Settings has provided fresh insights into many long standing conceptual questions of QG and Cosmology. In addition, the field has begun to provide phenomenological results for confronting quantum gravity with observations.

APPENDIX

This is supplementary material that complements and completes what I discussed in my talk at the conference.

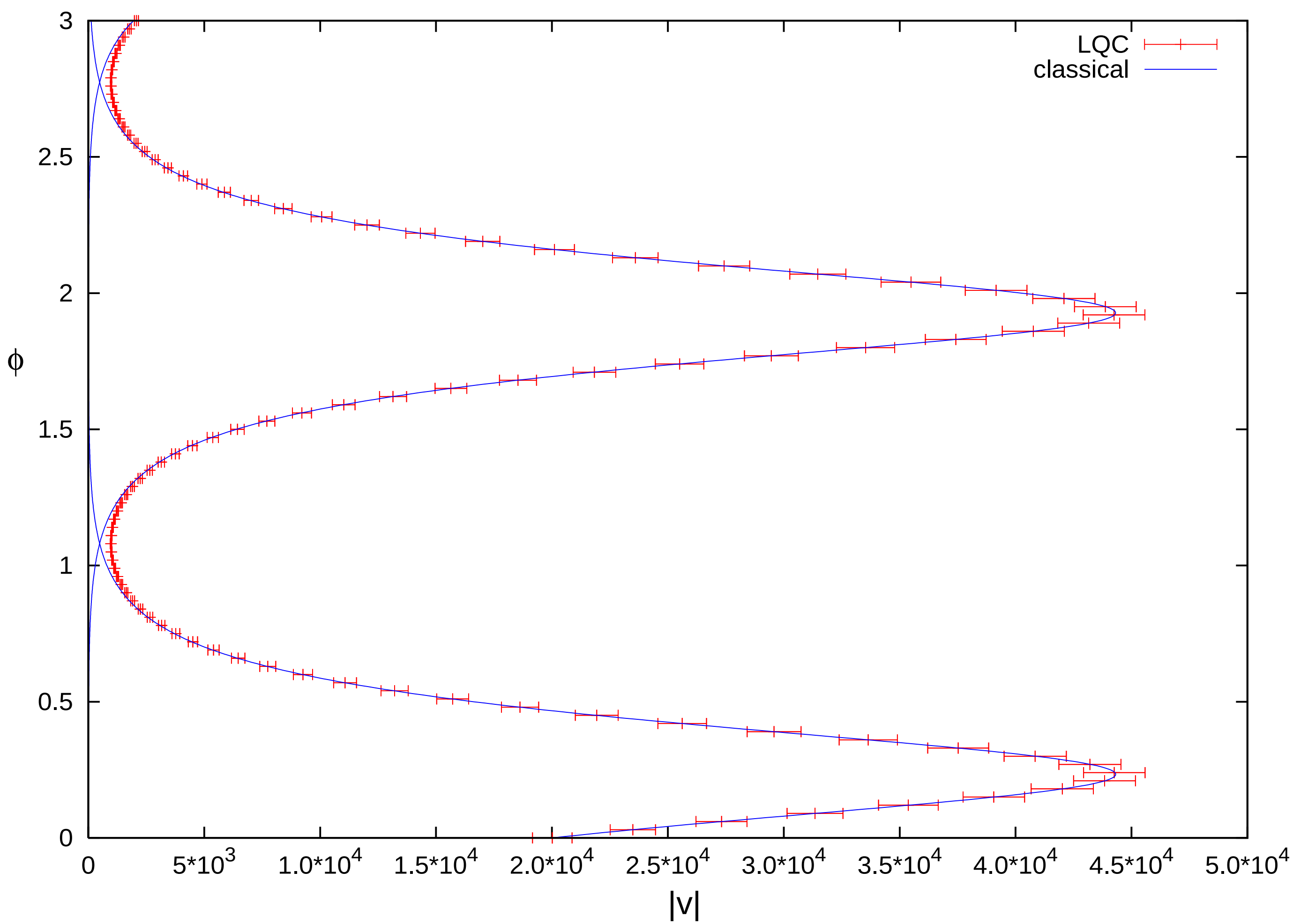
k=0 Model with Positive Λ



Expectations values and dispersions of $\hat{V}|_{\phi}$ & classical trajectories.

(AA, Pawlowski)

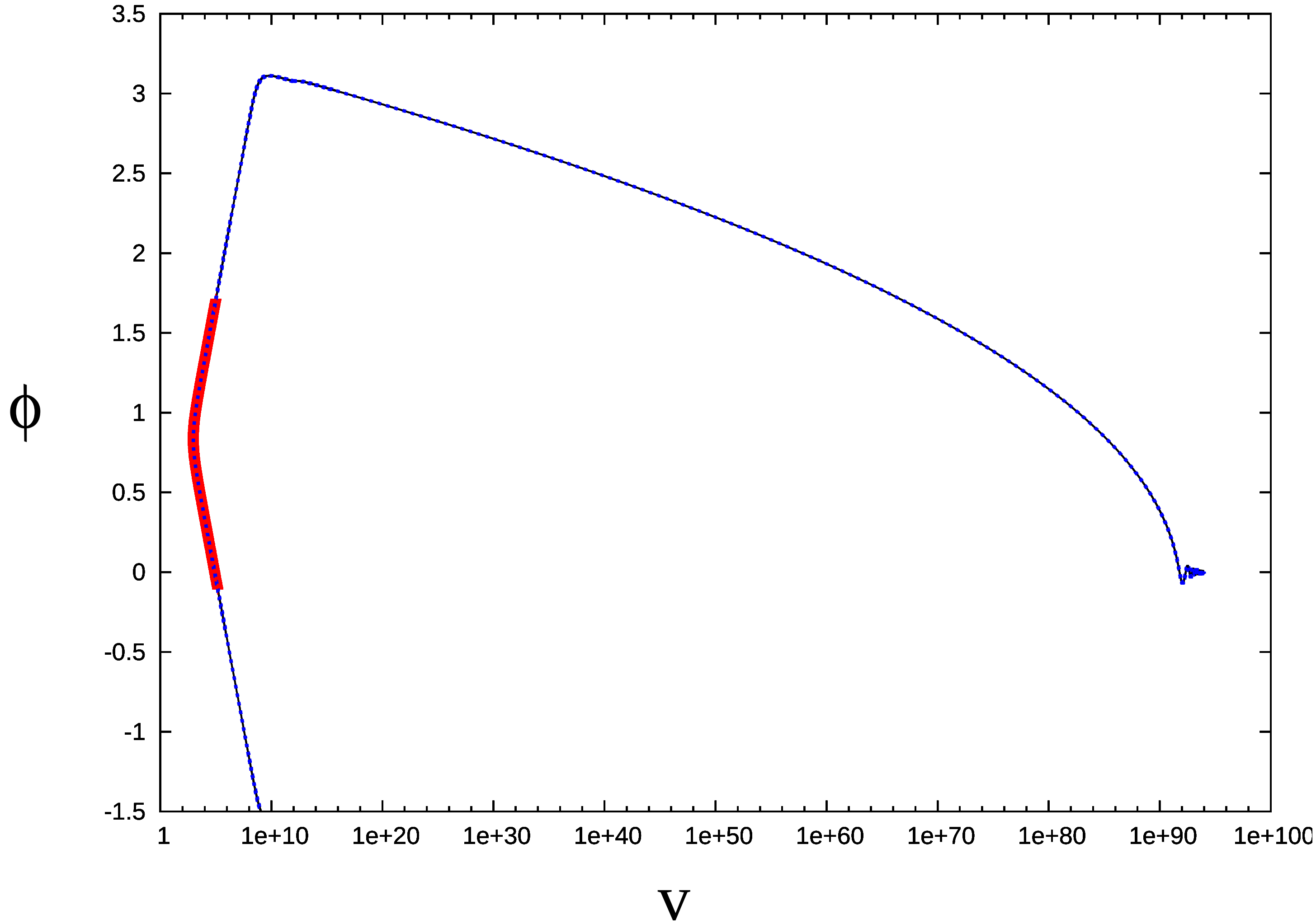
k=0 Model with Negative Λ



Expectations values and dispersions of $\hat{V}|_\phi$ & classical trajectories.

(Bentevigna, Pawlowski)

Inflation



Expectations values and dispersions of $\hat{V}|_\phi$ for a massive inflaton ϕ with phenomenologically preferred parameters (AA, Pawlowski, Singh).

Path Integrals: 3 slides

• Apparent Tension: Major departures from Einstein's theory near the big bang seem surprising at first in the path integral approach where quantum corrections normally become significant only when the action is comparable to $\hbar$!

• But if one starts from the Hamiltonian theory,
the path integral measure not always dictated by $e^{iS_{\rm Cl}}$
Ex: Free non-relativistic particle moving on a Riemannian manifold (DeWitt).

With $H = -(\hbar^2/2m) g^{ab} D_a D_b$, the Feynman procedure leads to

$\langle q_f, t_f | q_i, t_i \rangle = \int D[q(t)]\; e^{iS}$ where $S[q(t)] = (1/2) \int {\rm d}t\, m\, g_{ab}\, \dot{q}^a\, \dot{q}^b + \hbar^2 (R/6m)$

• In GR: Additional complication. No external time!
Result: transition amplitudes replaced by Extraction amplitudes that determine the dynamical content of the theory: In the FLRW models with a scalar field:

$$\mathcal{E}(v_f, \phi_f;\, v_i, \phi_i) = \int {\rm d}\alpha \langle {\rm v_f}, \phi_{\rm f} |\, {\rm e}^{{\rm i}\alpha \hat{\rm C}} |\, {\rm v_i}, \phi_{\rm i} \rangle$$

so that

$\Psi_{\rm phys}(v, \phi) = \sum_{v'} \int {\rm d}\phi\, \mathcal{E}({\rm v}, \phi; {\rm v}', \phi')\, \Psi_{\rm kin}({\rm v}, \phi')$, and

$(\Phi_{\rm phys},\, \Psi_{\rm phys}) := \sum_{v,\, v'} \int {\rm d}\phi\, {\rm d}\phi'\, \bar{\Phi}_{\rm kin}({\rm v}, \phi)\, \mathcal{E}({\rm v}, \phi; {\rm v}', \phi') \Psi_{\rm kin}({\rm v}', \phi')$

From the Hamiltonian Theory to Path Integrals

- Start with: $\mathcal{E}(v_f, \phi_f; v_i, \phi_i) = \int \mathrm{d}\alpha \langle \mathrm{v_f}, \phi_\mathrm{f} \,|\mathrm{e}^{\mathrm{i}\alpha\hat{\mathrm{C}}}|\, \mathrm{v_i}, \phi_\mathrm{i}\rangle$ Treat $\alpha\hat{C}$ as a fictitious Hamiltonian and the mathematical 'evolution' it generates for $\Delta t = 1$. Then follow Feynman to write $e^{i\alpha\hat{C}} = [e^{i\epsilon\alpha\hat{C}}]^N$ with $\epsilon = 1/N$; insert a complete basis between each exponential to rewrite $\mathcal{E}$ as a sum over quantum paths in the phase space:

$$\mathcal{E}(v_f, \phi_f; v_i, \phi_i) = \int \mathrm{d}\alpha \;\int [\mathcal{D}\mathrm{v_q}(\tau)]\; [\mathcal{D}\mathrm{b_q}(\tau)]\; [\mathcal{D}\mathrm{p}(\tau)]\; [\mathcal{D}\phi(\tau)]\; \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\bar{\mathrm{S}}}$$

- Quantum paths $\Rightarrow$ Sum involved paths only with $v \in 4n\ell_o$ and $b \in (0, \pi/\ell_o)$, where $\ell_o^2 =$ Area gap. None of these paths passes through the classical singularity ($b = \infty$)! $\Rightarrow$ Singularity Resolution.

- Can address the tension more directly by using a trick from the path integral framework of a particle on a circle. Can simply rewrite the path integral as an integral over all phase space paths. Then,

$$\mathcal{E}(v_f, \phi_f; v_i, \phi_i) = \int \mathrm{d}\alpha \;\int [\mathcal{D}\mathrm{v}(\tau)]\; [\mathcal{D}\mathrm{b}(\tau)]\; [\mathcal{D}\mathrm{p}(\tau)]\, [\mathcal{D}\phi(\tau)]\; \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathrm{S}},$$

where

$$S = \int_0^1 \mathrm{d}\tau \left(\mathrm{p}\dot{\phi} - \tfrac{1}{2}\mathrm{b}\dot{\mathrm{v}} - \alpha\left(\mathrm{p}^2 - 3\pi\mathrm{G}\mathrm{v}^2\,\frac{\sin^2 \ell_\mathrm{o}\mathrm{b}}{\ell_\mathrm{o}^2}\right)\right) \;\neq \mathrm{S_{EH}}.$$

- Now all paths are allowed but weighted by a quantum corrected action. Captures quantum geometry effects, as it must.

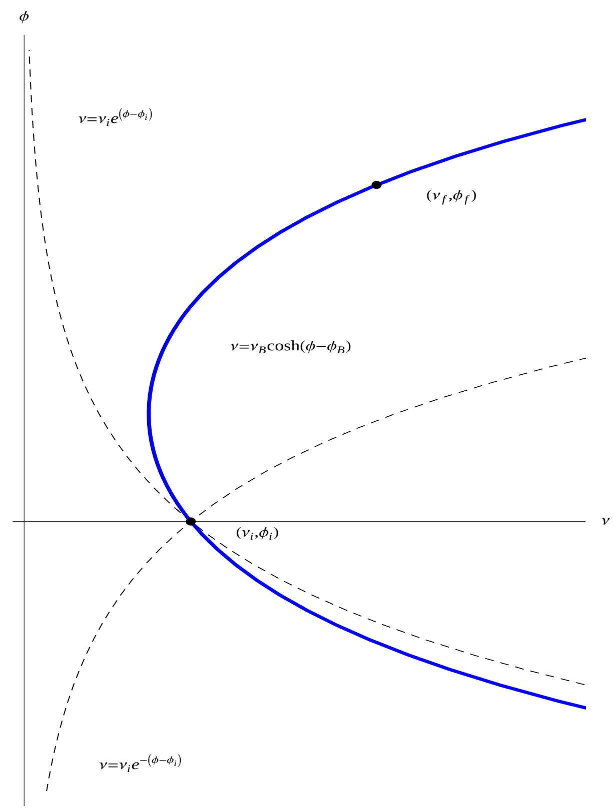
Steepest Descent and WKB

- Subtlety in using the WKB approximation: Now the action has a $\hbar$-dependent term because of ℓ_o. So, the standard (Rovelli, $\mathcal{S}3.2$, $\mathcal{S}5.2$), $\hbar$-expansion acquires subtleties.

$$\hbar \to 0, \gamma \to \infty \text{ such that } \ell_o \sim \sqrt{\gamma^3 \hbar G} \text{ is kept fixed.}$$

Then we obtain a well-defined WKB expansion.

- Non-trivial check: The leading order WKB term yields an excellent approximation to the (numerically computed) exact result away from the 'classically' forbidden region.



- Summary: there is no tension between the path integral and Hamiltonian frameworks. LQC Perspective: Incorrect to start with the Einstein-Hilbert action on classical geometries. Rather, to correctly handle uv issues, have to keep track of quantum geometries. Then the weight associated with the classically singular paths is negligible; bouncing solutions of effective LQC equations dominate.

Precise relation between LQC and the WDW Theory

Question analyzed in detail for the k=0 model. (Corichi, Singh, AA). Expect the answer to be the same for others.

Start with the ‘same physical state at time $\phi = \phi_o$’ and evolve using LQC or WDW theory. Then:

- Certain predictions of LQC approach those of the WDW theory as the area gap λ goes to zero:
  Given a semi-infinite ‘time’ interval $\Delta\phi$ and $\epsilon > 0$, there exists a $\delta > 0$ such that $\forall \lambda < \delta$, ‘physical predictions of the two theories are within ϵ of each other.’
- However, approximation is *not* uniform. The WDW theory is *not* the limit of sLQC:
  Given $N > 0$ however large, there exists a ϕ such that
  $\langle \hat{V}_\phi \rangle_{\rm sLQC} - \langle \hat{V}_\phi \rangle_{\rm WDW} > N$.
  LQC is *fundamentally* discrete.

SUPPLEMENT

1. How is the Hamiltonian constraint handled in LQC?

• Form of the constraint $C_{\rm H} \sim \underbrace{(\epsilon^{ij}{}_k E^a_i E^b_j/\sqrt{q})}_{\rm Thiemann} \quad \underbrace{F^k_{ab}}_{\rm holonomy}$

• Classically: $F^k_{ab} = -2 \lim_{\mathrm{Ar}_\Box \to 0} \left(\mathrm{Tr}(h_{\Box_{ab}} - 1)\tau^k/\mathrm{Ar}_\Box\right)$
Quantum Theory: Limit does not exist because there is no local operator corresponding to the connection or curvature. Different from full LQG: Diff constraint handled by gauge fixing.

• LQC View (Bojowald, Lewandowski, AA): Quantum geometry $\Rightarrow$ should not shrink the loop to zero but only till the area enclosed $\mathrm{Ar}\Box$ w.r.t. the fiducial metric equals the lowest eigenvalue $\Delta = 2\sqrt{3}\pi\gamma\ell^2_{\rm Pl}$ of the area operator. So, the fundamental operator has Planck scale non-locality; Familiar local expression emerges only in the classical limit. (μ_o-Scheme)

• Singularity resolved. But the resulting quantum Hamiltonian constraint had a serious limitation: Predicted deviations from the classical theory even in certain ‘tame’ situations. (More later). Physically motivated, improved constraint remedies this drawback while retaining all desirable features.

SUPPLEMENT

• New idea (Pawlowski, Singh, AA): Do this with Physical area of $\square$ (which is state dependent). The resulting operator mimics certain features of the full theory. Idea subtle to implement but important physical consequences: Overcomes problems with the older LQC dynamics. ($\bar{\mu}$-Scheme). (more later)

• Hamiltonian constraint: Use a representation in which geometry (i.e. $\hat{V} \sim \hat{a}^3$) and matter field (i.e., $\hat{\phi}$) are diagonal : $\Psi(v, \phi)$

Then the Wheeler DeWitt equation is replaced by a difference equation:

$$C^+(v)\,\Psi(v+4,\,\phi) + C^o(v)\,\Psi(v,\phi) + C^-(v)\Psi(v-4,\,\phi)\;=\;\hat{H}_\phi\Psi(v,\phi)$$

Fundamentally, a constraint equation. Selects physical states. However, this equation also dictates quantum dynamics.

• The 'lattice' has uniform spacing in $v \sim a^3$ (not p or μ which $\sim a^2$). Dynamics cannot be supported by a Vehlino type quantum kinematics.

SUPPLEMENT

2.How do you extract dynamics/physics from the ‘frozen formalism’?

To extract physics, we need to:

- Isolate ‘time’ to give meaning to ‘evolution’.

- Solutions to the constraint: Physical states. Introduce a physical inner product and suitable Dirac observables.

- Construct states which represent the actual universe at late time. ‘Evolve back’ towards the big bang.

- Is the classical singularity ‘resolved’? In what sense? (Brunnemann and Thiemann) ‘Wave function vanishes at the singularity’ not enough; Physical inner product may be non-local. Need to analyze the behavior of the Dirac observables.

- What is on the ‘other side’ of the classical big-bang? (Quantum foam?? Another classical universe??)

SUPPLEMENT

- The quantum Hamiltonian constraint takes the form:

$$-\Theta\,\Psi(v,\phi) = \partial_\phi^2 \Psi(v,\phi) \qquad (\star)$$

where Θ is a positive, self-adjoint difference operator independent of ϕ :
$\Theta\,\Psi(v,\phi) = \; C^+(v)\,\Psi(v+4,\,\phi) + C^o(v)\,\Psi(v,\phi) + C^-(v)\,\Psi(v-4,\,\phi).$
Suggests ϕ could be used as 'emergent time' also in the quantum theory. Relational dynamics.

- Physical states: solutions to ($\star$), invariant under $v \to -v$. Observables: $\hat{p_\phi}$ and $\hat{V}|_{\phi=\phi_o}$. Inner product: Makes these self-adjoint or, equivalently, use group averaging. Analogy with KG equation in a static space-time. Semi-classical states: Generalized coherent states.

- Physical states:

$$\Psi(v,\phi) \text{ satisfying } \quad -i\hbar\partial_\phi \Psi(v,\phi) = \sqrt{\Theta}\,\Psi(v,\phi)$$

Dirac observables:

$$\hat{p}_{(\phi)}\Psi(v,\phi) = -i\hbar\partial_\phi\,\Psi(v,\phi) \equiv \sqrt{\Theta}\Psi(v,\phi)$$

$$\hat{V}|_\phi\,\Psi(v,\phi) = e^{i\sqrt{\Theta}(\phi-\phi_o)}\,|v|\Psi(v,\phi_o).\ \text{ Similarly } \hat{\rho}|_\phi.$$

SUPPLEMENT

What are the differences between the older, μ_o evolution of (Bojowald, Lewandowski, AA) and the $\bar{\mu}$ framework (Pawlowski, Singh, AA) in these models?

Differences are very significant with lessons for full LQG.

- In the k=0 model on $\mathbb{R}^3$, scale factor a refers to a fiducial metric: $q_{ab} = a^2(t)\, q^o_{ab}$. If $q^o_{ab} \to \alpha^2\, q^o_{ab}$, $a \to \alpha^{-1} a$. Physics should not depend on q^o_{ab} or the value of $a(t)$. (So, claims such as quantum effects are important for $a < a^\star$ in the older literature (based on the spectrum of $\widehat{1/V}$) are physically unsound.).

- Further, in this case *every* quantization requires an additional structure: An elementary Cell $\mathcal{C}$. We absorb factors of the volume V_o of $\mathcal{C}$ w.r.t. q^o_{ab} in the definition of canonical variables c, p so that the symplectic structure is independent of the q^o_{ab} choice. So, the classical Hamiltonian theory depends only $\mathcal{C}$ and not on q^o_{ab}. Same is true of quantum kinematics. Thus, e.g., $p^{3/2}$ is the physical volume of $\mathcal{C}$.

- i) In μ_o quantization, the Hamiltonian constraint operator depends on q_o^{ab} again. In the $\bar{\mu}$ quantization, it does not.

SUPPLEMENT

- ii) For each choice of $\mathcal{C}$ we get a quantum theory. In the μ_o evolution, the density at the bounce point goes as: $\rho_b \propto 1/p_\phi$. So, a Gaussian peaked at a classical phase space point can bounce with $\rho_b =$ density of water! Major departures from the classical theory also away from the bounce: in presence of a cosmological constant, large deviations occur when $\Lambda a^2 \geq 1$ although the space-time curvature is low. In $\bar\mu$ evolution, $\rho_b \approx 0.41\rho_{\rm pl}$ always. No departures from GR at low curvatures.

- iii) *Physical results* should be independent of the choice of $\mathcal{C}$. In $\bar\mu$ evolution they are. Not in the μ_o scheme. Ex: Given a classical solution $(a(t), \phi(t))$ when do quantum effects become important? Answer in the μ_o scheme depends on the choice of the cell! Answer not 'gauge invariant'. In the $\bar\mu$ scheme it is.

- Lessons:

a) LQC: Although it seems natural at first, detailed considerations show that the μ_o quantization of the Hamiltonian constraint is physically incorrect;
b) LQG: Whether a quantization of the Hamiltonian constraint has a 'good infrared behavior' is likely to be very subtle.