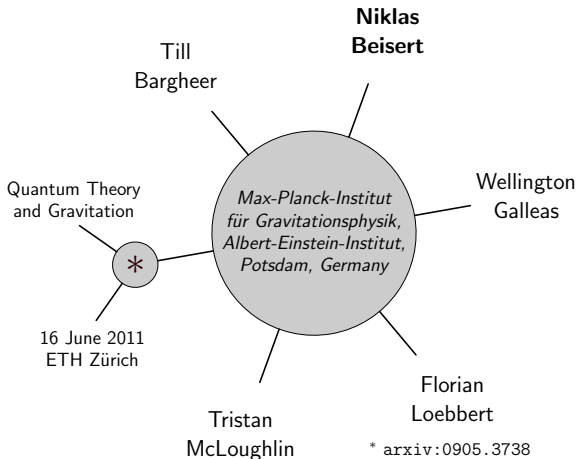


Conformal Symmetry and Integrability for Scattering Amplitudes in $\mathcal{N} = 4$ Super Yang-Mills*



Remarkable $\mathcal{N} = 4$ Super Yang–Mills

$U(N_c)$ gauge field \mathcal{A}_μ , 4 fermions Ψ_α^a , 6 scalars Φ_m

$$S_{\mathcal{N}=4} \sim \frac{N_c}{\lambda} \int \frac{d^4x}{4\pi^2} \text{Tr} \left(\frac{1}{4} (\mathcal{F}_{\mu\nu})^2 + \frac{1}{2} (\mathcal{D}_\mu \Phi_m)^2 - \frac{1}{4} [\Phi_m, \Phi_n]^2 + \dots \right).$$

- Unique action, three unrenormalised couplings λ , N_c , θ_{top} .
- **Exact superconformal symmetry** $\mathfrak{psu}(2, 2|4)$.
- And some mysterious features: AdS/CFT, integrability, dualities, ...

Magic in the Planar Limit:

[NB et al.]
[1012.3982]

- Integrability in the planar limit: $\mathfrak{psu}(2, 2|4)$ Yangian.
- Planar anomalous dimensions (presumably/largely) solved.
- Simplifications for planar scattering, dual conformal symmetry.
- Novel scattering tools: Twistors, CSW/BCF, Graßmannian, TBA, ...

I. Overview:

**Gluon Amplitudes,
AdS/CFT and Integrability**

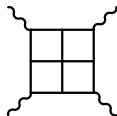
Planar Scattering Amplitudes

Intriguing result in $\mathcal{N} = 4$ SYM in the planar limit $N_c \rightarrow \infty$:

Four-gluon scattering amplitude obeys **BDS relation**

[Anastasiou, Bern] [Bern
Dixon] [Smirnov]

$$A(p, \lambda) \simeq A^{(0)}(p) \exp \left(2D_{\text{cusp}}(\lambda) M^{(1)}(p) + F(p, \lambda) \right).$$



Only required data: • tree level, • one loop, • cusp dimension.

- No finite remainder function $F(p, \lambda) = 0$.

Scattering amplitudes constructible by unitarity and suitable ansatz.

Verified BDS relation at $\mathcal{O}(\lambda^4)$ with

[Bern
Dixon] [Bern, Czakon, Dixon]
[Smirnov] [Kosower, Smirnov]

$$D_{\text{cusp}}(\lambda) = \frac{1}{2} \frac{\lambda}{\pi^2} - \frac{1}{96} \frac{\lambda^2}{\pi^2} + \frac{11}{23040} \frac{\lambda^3}{\pi^2} - \left(\frac{73}{2580480} + \frac{\zeta(3)^2}{1024\pi^6} \right) \frac{\lambda^4}{\pi^2} \pm \dots$$

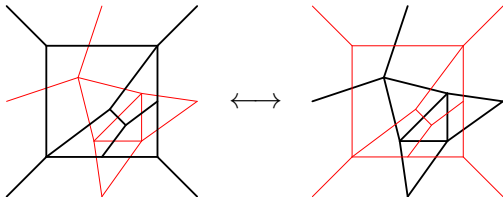
Non-trivial finite remainder for $n \geq 6$.

[Alday
Maldacena] [Drummond, Henn]
[Korchemsky
Sokatchev] [Bartels
Lipatov
Sabio Vera] [Bern, Dixon, Kosower,
Roiban, Spradlin
Vergu, Volovich]

Simplicity and Dual Conformal Symmetry

Planar amplitudes and integrals simpler than expected:

- No bubbles, no triangles, typically scalar boxes (also non-planar).
- Similarity of momentum and position space propagators in $D = 4$.



- Dual amplitudes and integrals are conformal.
- Dual superconformal symmetry of superspace amplitude.

[Drummond
Korchemsky] . . .
[Sokatchev
Drummond, Henn
Korchemsky
Sokatchev]

Strings:

- Self-duality of superstrings requires also fermionic T-duality.
- Dual superconformal symmetry $\hat{=}$ symmetry of T-dual model.
- Dual superconformal symmetry allows $F(p, \lambda)$ only for $n \geq 6$ legs.

[Berkovits
Maldacena]

Sketch of Scattering Amplitudes

Structure of S-matrix elements (in regularised theory):

$$\mathcal{A} \simeq \sum \text{Colour} \times \text{Polarisation} \times \text{Scalar Loop Integrals.}$$

Scalar factor contains IR divergences (massless particles)

$$\text{Scalar Factor} \simeq \exp(\text{IR Singularity} + \text{Finite Remainder}).$$

Same cusp dimension $D_{\text{cusp}}(\lambda)$ in IR singularity & integrable system:

- How to apply integrability to planar scattering amplitudes?
- Can one also compute remainder function $F(p, \lambda)$?
- Relation between (dual) superconformal symmetry and integrability?

Modern Efficient Tools related to integrability:

- Polarisation: Graßmannian. [Arkani-Hamed, Cachazo, Cheung, Kaplan] [Bullimore, Mason, Skinner] [Kaplan, 0912.0957] . . . [Arkani-Hamed, Bourjaily, Cachazo, Trnka]
- Scalar Factor: TBA. [Alday, Maldacena] [Alday, Gaiotto, Maldacena] [Alday, Maldacena, Sever, Vieira]

Graßmannian

Twistor space representation simplifies complexified amplitudes

[Witten
hep-th/0312171]

$$p^{\beta\dot{\alpha}} = \lambda^{\beta}\tilde{\lambda}^{\dot{\alpha}}, \quad f(\mu, \tilde{\lambda}) = \int d^2\lambda e^{i\langle\lambda, \mu\rangle} f(\lambda, \tilde{\lambda}).$$

Twistor $Z = (\mu, \tilde{\lambda})$ transforms a vector under $SU(2, 2)$ conformal group.

Polarisation structure of N^{k-2} MHV planar scattering amplitudes expressed through Graßmannian integral (C : $k \times n$ matrix, M_j : $k \times k$ minors of consecutive columns in C)

[Arkani-Hamed, Cachazo
Cheung, Kaplan]

$$A_{n,k}(Z) = \int \frac{d^{k(n-k)}C \delta^{k(4|4)}(CZ)}{M_1 \cdots M_n}$$

- Higher-loop integrands constructible.
- Can isolate finite quantities (remainder function)
- Integrals correspond to volumes of AdS polytopes.
- Integration contours? For trees?!

[Arkani-Hamed, Bourjaily
Cachazo, Caron-Huot, Trnka]

[Arkani-Hamed, Bourjaily
Cachazo, Trnka]

Outline

Symmetries of Scattering Amplitudes (S-matrix):

- Understand symmetries of S-matrix.
- Apply symmetries to (fully?) constrain S-matrix.

How to treat (super)conformal symmetry of the S-matrix in $\mathcal{N} = 4$ SYM?

Concretely

- Structure of symmetries: superconformal and Yangian
- Free symmetries
- Symmetries at tree level, [one loop], ...

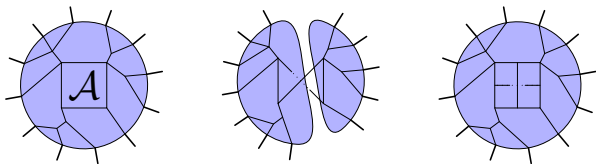
Ultimately

- Perturbative symmetries increasingly messy. Not so useful.
- Symmetries are **non-perturbative!**(?) Exploit them there!

II. Free Symmetries

Scattering Amplitudes

Colour-ordered scattering amplitudes (1-trace, 2-trace, genus-1):



Legs: Field Ω combines on-shell gluons Γ , fermions Ψ & scalars Φ :

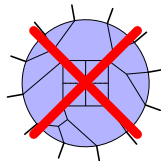
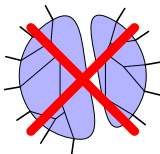
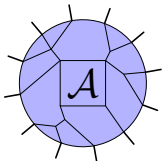
$$\Omega(\lambda, \tilde{\lambda}, \bar{\eta}) = \Gamma(\lambda, \tilde{\lambda}) + \bar{\eta}^a \Psi_a(\lambda, \tilde{\lambda}) + \frac{1}{2} \bar{\eta}^a \bar{\eta}^b \Phi_{ab}(\lambda, \tilde{\lambda}) + \dots$$

Amplitude $\mathcal{A}(A_1, \dots, A_n)$ on spinor helicity superspace $A = (\lambda, \tilde{\lambda}, \bar{\eta})$

$$p^{\beta\dot{\alpha}} = \lambda^\beta \tilde{\lambda}^{\dot{\alpha}}, \quad q^{\alpha b} = \lambda^\beta \bar{\eta}^a.$$

Dual Superconformal Symmetry

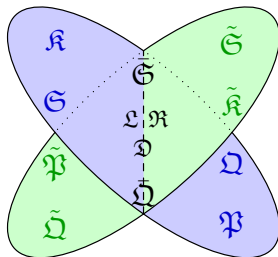
Remarkable features of disk amplitudes (single-trace, large- N_c):



- Simplifications: BDS formula.
- Only particular integrals appear.
- **Dual superconformal** symmetry!
- Another $\mathfrak{psu}(2, 2|4)$:

$\mathfrak{L}, \mathfrak{M}, \mathfrak{D}, \tilde{\mathfrak{D}}, \tilde{\mathfrak{S}}$: shared with **conventional**,
 $\tilde{\mathfrak{P}}, \tilde{\mathfrak{Q}}$: trivial new generators,
 $\tilde{\mathfrak{K}}, \tilde{\mathfrak{S}}$: non-trivial new generators.

[Drummond
Korchemsky
Sokatchev]
[Drummond, Henn
Korchemsky
Sokatchev]



T-Self-Duality in String Theory

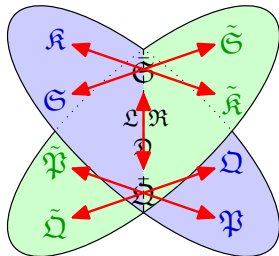
Detour: Consider string theory picture at strong coupling.

- Strings propagate on $AdS_5 \times S^5$ superspace.
- Background is coset space $PSU(2, 2|4)/Sp(2) \times Sp(1, 1)$.
- Isometries of background are Noether symmetries: $\mathfrak{psu}(2, 2|4)$.

T-duality transformation:

[Alday
Maldacena] [Berkovits
Maldacena]

- 4 bosonic + 8 fermionic T-dualities.
- terms at worldsheet boundaries: planar!
- maps $AdS_5 \times S^5$ string model to itself:
self-duality!
- maps **isometries** to **dual isometries**:
dual superconformal symmetry



String Theory Integrability

Integrability enhances conserved charges:

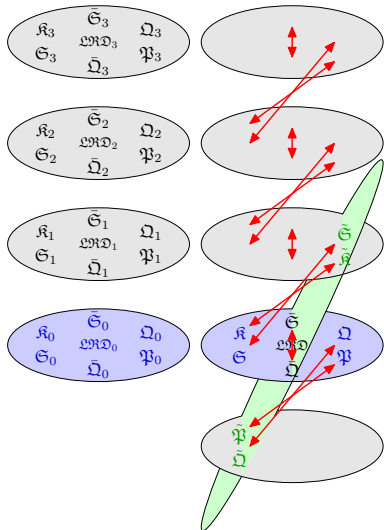
$$Q(z) = zQ_0 + z^2Q_1 + z^3Q_2 + \dots$$

- Q_0 are Noether charges.
- Q_k form (half) loop algebra.
- ∞ -dimensional hidden symmetries.

T-self-duality

[NB, Ricci] [Tseytlin, Wolf] [Berkovits] [Maldacena] [NB 0903.0609]

- maps loop algebra to itself.
- It shows that conventional & dual superconformal symmetry close into loop algebra.
- Quantum algebra called Yangian.



Yangian Symmetry

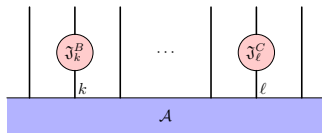
Back to scattering amplitudes in $\mathcal{N} = 4$ SYM:

Free representation $\mathfrak{J} = \mathfrak{J}_0$, $\widehat{\mathfrak{J}} = \mathfrak{J}_1$ of $\mathfrak{psu}(2, 2|4)$ Yangian:

[Drummond
Henn
Plefka]

$$\mathfrak{J}^A = \sum_{k=1}^n \text{Diagram}$$

$$\widehat{\mathfrak{J}}^A = F_{BC}^A \sum_{k < \ell = 1}^n \text{Diagram}$$



- Yangian Symmetry:

Amplitudes are invariant under \mathfrak{J} , $\widehat{\mathfrak{J}}$. [Drummond, Henn, Korchemsky, Sokatchev]

Graßmannian integral invariant. [Drummond, Ferro] [Drummond, Ferro]

- compatible with cyclic structure (exceptional)!

- \mathfrak{J} & $\widehat{\mathfrak{J}}$ all one needs to know.

In fact, only \mathfrak{J} and $\widehat{\mathfrak{P}} = \widehat{\mathfrak{K}}$ sufficient.

- Representation requires planar limit & ordering; depends on colour structure.

- Planar amplitudes are integrable!



Meaning of Integrability

Wait a minute! Integrability makes scattering in $D > 2$ trivial!?!

- Only if integrability refers to local conserved charges.
- Here conserved charges are local in (planar) colour space: String!

Integrability is symmetry enhancement. Properties of S-matrix:

- unitarity
- (super) Poincare invariance
- (super) conformal invariance
- Yangian: Infinite-dimensional algebra (one additional constraint).

Integrability means (pragmatic definition):

- hidden symmetry constrains S-matrix **uniquely**.
- can calculate S-matrix **efficiently**: Graßmannian, TBA.

III. Superconformal Anomaly (?) at Tree Level

Nitpicking

Graßmannian generates free Yangian invariants.
Tree level S-matrix is suitable linear combination. Which?

[Drummond
Ferro] [Korchemsky
Sokatchev]
[Drummond
Henn]

Invariants?

- Individual invariants have spurious singularities.
- Individual invariants have wrong collinear behaviour.
- “Invariants” actually not exactly invariant.
Free symmetries have distributional anomalies!
- Ignore at tree level \Rightarrow hits you hard at loops.
Anomaly smeared by loop integration.
- Repair anomaly by deformed representation.

[Hodges
0905.1473]
[Korchemsky
Sokatchev]
[Cachazo
Svrcek
Witten]

[Bargheer, NB, Galleas] [NB, Henn
Loebbert, McLoughlin] [McLoughlin, Plefka]

Invariant!

- There can be only one invariant: the S-matrix.
- S-matrix assembled from almost-invariants.

Collinear Anomaly

Tree amplitude has poles when particle momenta become collinear

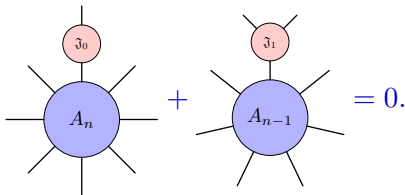
$$\mathcal{A} \sim \langle k, k+1 \rangle^{-1}.$$

Conformal symmetry sensitive to poles. Distributional anomaly

$$\tilde{\mathcal{J}}\mathcal{A} \sim \delta^2(\langle k, k+1 \rangle).$$

Compensate by deformation of conformal representation

[Bargheer, NB, Galleas]
[Loebbert, McLoughlin]



Only complete S-matrix (not individual amplitudes) invariant!

Classical Representation

We find the following corrections for representation of \mathfrak{G} , $\tilde{\mathfrak{G}}$ and \mathfrak{K}

$$\tilde{\mathfrak{G}} = \mathfrak{G}_0 + \mathfrak{G}_+, \quad \mathfrak{G} = \mathfrak{G}_0 + \mathfrak{G}_-, \quad \mathfrak{K} = \mathfrak{K}_0 + \mathfrak{K}_+ + \mathfrak{K}_- + \mathfrak{K}_{+-}.$$

Similar deformations for classical Yangian representation, e.g.

[NB
Schwab
to appear]

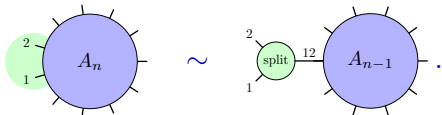
$$\hat{\mathfrak{Q}} = \dots + \sum_{k < j=1}^n \dots + \dots$$

To be done:

- Does the deformation annihilate all tree amplitudes? Yes.
- Is it a consistent representation of superconformal symmetry? Yes.
- What does it mean? Conformal symmetry & physical asymptotic states.

Invariance of Tree Amplitudes

Collinear limit is universal for all (tree) amplitudes: splitting function

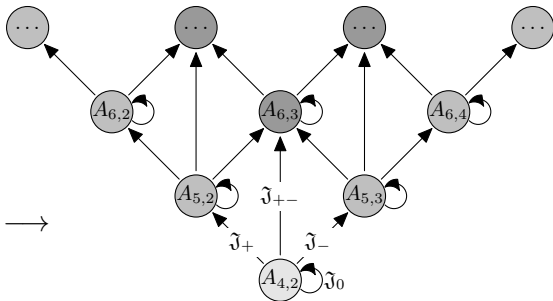


Follows e.g. from recursion relation by inheritance.

Exact invariance of all tree amplitudes:

[Bargheer, NB, Galleas
Loebbert, McLoughlin]

- Collinear singularities universal, same as for MHV.
- No anomalies from multi-particle singularities.
- Structure of cancellations \longrightarrow
 $A_{n,k}$: n -leg N^{k-2} MHV.



IV. Implications

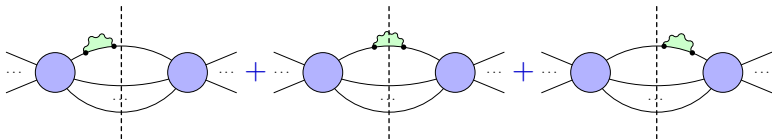
Massless Asymptotic States

Reconsider conceptual problems of massless scattering:

- No mass gap: Massless particles can “decay” into particle showers



- Particle number not well-defined in asymptotic region.
- But: Shower particles are strictly collinear.
Single massless particle physically indistinguishable from shower.
- Overcounting of collinear states leads to IR divergencies at loop level.
- Can cancel IR divergencies in cross sections:



Finite, but scattering amplitudes are more convenient.

Symmetry for Massless Asymptotic States

Asymptotic space:

- Fock space is too large; overcounts collinear states.
- Should project out collinear states: Conceptually hard!
- Rather embed asymptotic space in larger Fock space.
Keep collinear issues in mind.

Our results are in line with the above:

- Conformal anomaly precisely where asymptotic particles overcount.
- Deformation makes superconformal representation compatible with embedding of asymptotic space into Fock space. (?)
- Exact conformal invariance incompatible with fixed particle number.
Have to consider generating functional \mathcal{A} instead of amplitude(s) A .
Scattering operator for asymptotic space instead of scattering matrix.

Uniqueness

All tree amplitudes have been constructed by recursion relations

[Drummond
Henn]

$$A_{n,k}(p) = A_n^{\text{MHV}}(p) \sum_{\alpha} c_{\alpha} R_{\alpha}(p).$$

Each R is (almost) invariant under the free Yangian representation.
How to obtain the correct physical linear combination $c_{\alpha} = 1$?

- Demand absence of spurious singularities or equivalently
- demand correct collinear behaviour.

[Hodges
0905.1473]

[Korchemsky
Sokatchev]

Deformed representation ensures correct collinear behaviour.

[Bargheer, NB, Galleas
Loebbert, McLoughlin]

Therefore **symmetry alone** fixes correct linear combination $c_{\alpha} = 1$!

Very important for construction by symmetry at higher loops:

- Adding any invariant respects symmetry: **ambiguity!**
- Adding tree level amounts to an overall factor; **okay.**

Can symmetry fix planar amplitude completely (non)perturbatively?

V. Conclusions

Conclusions

★ Superconformal Symmetry at Tree Level

- Tree amplitudes almost invariant under free superconformal symmetry.
- Invariance violated for singular configurations: Collinear momenta.
- Transformations can be corrected to make trees fully invariant.
- Dynamic corrections requires, changes number of legs.
- Superconformal algebra closes onto gauge transformations.
- Yangian appears to lead to a unique invariant \Rightarrow the S-matrix.

★ Superconformal Symmetry at One Loop

- Transformations can be corrected to make loops fully invariant.

★ Open Problems

- How does the algebra at loop level close?
- What's new at two loops?
- What about conformal inversions?