FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG
NATURWISSENSCHAFTLICHE FAKULTÄT

Towards Loop Quantum Supergravity

Norbert Bodendorfer
(work by NB, Thomas Thiemann, Andreas Thurn [arXiv:1106.1103])

FAU Erlangen-Nürnberg

ETH Zürich, 23.06.2011

Plan of the talk

1. Why Loop Quantum Supergravity?
2. Vacuum Loop Quantum Gravity in $3+1$ Dimensions
3. New Variables for Arbitrary Dimensions
   - Hamiltonian Point of View
   - Lagrangian Point of View
   - Quantum Theory
   - Generalisations
4. Possible Applications of the New Variables
   - Black Hole Entropy
   - Cosmology
5. Conclusion

Outline

Why Loop Quantum Supergravity?

Quantum Gravity

- perturbative: Superstring theory / M-theory
- non-perturbative: Loop Quantum Gravity

Why Loop Quantum Supergravity?

Quantum Gravity

- perturbative: Superstring theory / M-theory
- non-perturbative: Loop Quantum Gravity

$\rightarrow$ Compare results?

Why Loop Quantum Supergravity?

Quantum Gravity

- perturbative: Superstring theory / M-theory
- non-perturbative: Loop Quantum Gravity

→ Compare results?

- Problem: Superstring / M-theory require
  - higher dimensions
  - supersymmetry
  - additional particles

Why Loop Quantum Supergravity?

Quantum Gravity

- perturbative: Superstring theory / M-theory
- non-perturbative: Loop Quantum Gravity

$\rightarrow$ Compare results?

- Problem: Superstring / M-theory require
  - higher dimensions
  - supersymmetry
  - additional particles
- However: Higher dimensional Supergravities
  - are considered as the low-energy limits of Superstring / M-theory
  - have action of the type S_{GR} + more

Why Loop Quantum Supergravity?

Quantum Gravity

- perturbative: Superstring theory / M-theory
- non-perturbative: Loop Quantum Gravity

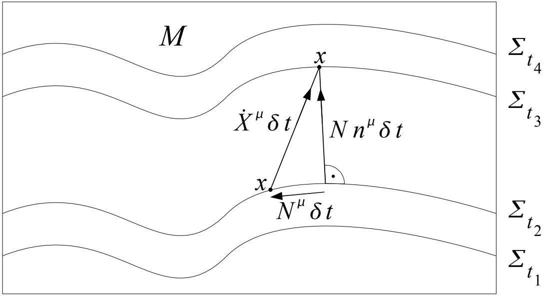
→ Compare results?

- Problem: Superstring / M-theory require
  - higher dimensions
  - supersymmetry
  - additional particles
- However: Higher dimensional Supergravities
  - are considered as the low-energy limits of Superstring / M-theory
  - have action of the type S_{GR} + more

→ develop loop quantisation methods for higher dim. Supergravities

Outline

ADM Phase Space: 3+1 Split



Foliation of M

M top. $\mathbb{R} \times \sigma$

$\Sigma_t := X_t(\sigma)$

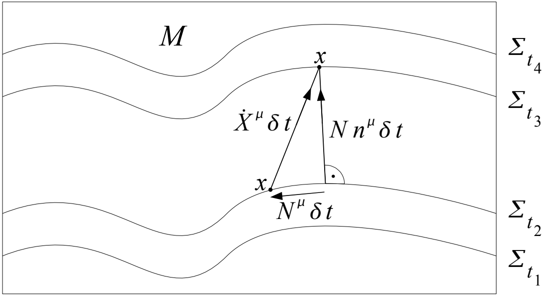
$X_t : \sigma \to M$

$\mu, \nu = 0, .., 3$: spacetime

$\downarrow$ pullback to σ

$a, b = 1, .., 3$: spatial

ADM Phase Space: 3+1 Split



Foliation of M

M top. $\mathbb{R} \times \sigma$

$\Sigma_t := X_t(\sigma)$

$X_t : \sigma \to M$

$\mu, \nu = 0, .., 3$: spacetime

$\downarrow$ pullback to σ

$a, b = 1, .., 3$: spatial

ADM phase space

[Arnowitt, Deser, Misner '62]

- **Canonical variables**
  spatial metric q_{ab}, conjugate momentum $P^{ab}(q_{cd}, K_{cd})$, K_{cd}: extrinsic curvature
- **Constraints**
  Totally constrained Hamiltonian: $H = \int_\sigma d^3x (N\mathcal{H} + N^a \mathcal{H}_a)$
  spatial diffeomorphism constraint $\mathcal{H}_a(q, P)$
  Hamiltonian constraint $\mathcal{H}(q, P) = -\frac{\pm}{\sqrt{\det(q)}}[q_{ac}q_{bd} - \frac{1}{2}q_{ab}q_{cd}]P^{ab}P^{cd} - \sqrt{\det(q)}R$
- **Observables**
  Dirac observables $\mathcal{O}$ obey $\{\mathcal{O}, \mathcal{H}\} = \{\mathcal{O}, \mathcal{H}_a\} = 0$

Classical theory in 3 + 1 dimensions

Phase space extension: Add $SO(3)$ gauge symmetry

Introduce densitised 3-bein E_j^a, extrinsic curvature K_a^i and Gauß law $G_{jk} = E^a_{[j} K_{a|k]}$

using $\{K_a^j, E_k^b\} = \delta_a^b \delta_k^j \quad \Rightarrow \quad \{q_{ab}(E,K), P^{cd}(E,K)\} = \delta^c_{(a}\delta^d_{b)}, \qquad \{P,P\} = \{q,q\} = 0$

spatial: $a, b = 1, .., 3$ so(3): $j, k = 1, .., 3$ $E_j^a = \sqrt{\det(q)} e_j^a$ $K_{ab} = K^j_{(a} e^j_{b)}$ $e_j^a e_a^k = \delta_j^k$ $q_{ab} = e_a^j e_b^k \delta_{jk}$

Classical theory in $3+1$ dimensions

Phase space extension: Add $SO(3)$ gauge symmetry

Introduce densitised 3-bein E^a_j, extrinsic curvature K^i_a and Gauß law $G_{jk} = E^a_{[j} K_{a|k]}$

using $\{K^j_a, E^b_k\} = \delta^b_a \delta^j_k \quad \Rightarrow \quad \{q_{ab}(E,K), P^{cd}(E,K)\} = \delta^c_{(a}\delta^d_{b)}, \qquad \{P,P\} = \{q,q\} = 0$

spatial: $a, b = 1, .., 3$ so(3): $j, k = 1, .., 3$ $E^a_j = \sqrt{\det(q)} e^a_j$ $K_{ab} = K^j_{(a} e^j_{b)}$ $e^a_j e^k_a = \delta^k_j$ $q_{ab} = e^j_a e^k_b \delta_{jk}$

Connection formulation: Only $3+1$ dimensions [Sen, Ashtekar, Immirzi, Barbero]

Canonical transformation: $A^j_a = \epsilon^{jkl}\Gamma_{akl} + \gamma K^j_a$, where $\partial_a e^j_b + \Gamma^c_{ab} e^j_c + \Gamma_a{}^j{}_k e^k_b = 0$

$\Rightarrow \{A^j_a, E^b_k\} = \gamma \delta^b_a \delta^j_k \qquad \{A^j_a, A^k_b\} = \{E^a_j, E^b_k\} = 0 \qquad \gamma \in \mathbb{R}$ Immirzi parameter

Classical theory in $3+1$ dimensions

Phase space extension: Add $SO(3)$ gauge symmetry

Introduce densitised 3-bein E^a_j, extrinsic curvature K^i_a and Gauß law $G_{jk} = E^a_{[j} K_{a|k]}$

using $\{K^j_a, E^b_k\} = \delta^b_a \delta^j_k \quad \Rightarrow \quad \{q_{ab}(E,K), P^{cd}(E,K)\} = \delta^c_{(a}\delta^d_{b)}, \qquad \{P,P\} = \{q,q\} = 0$

spatial: $a, b = 1, .., 3$ so(3): $j, k = 1, .., 3$ $E^a_j = \sqrt{\det(q)} e^a_j$ $K_{ab} = K^j_{(a} e^j_{b)}$ $e^a_j e^k_a = \delta^k_j$ $q_{ab} = e^j_a e^k_b \delta_{jk}$

Connection formulation: Only $3+1$ dimensions [Sen, Ashtekar, Immirzi, Barbero]

Canonical transformation: $A^j_a = \epsilon^{jkl}\Gamma_{akl} + \gamma K^j_a$, where $\partial_a e^j_b + \Gamma^c_{ab} e^j_c + \Gamma_a{}^j{}_k e^k_b = 0$

$\Rightarrow \{A^j_a, E^b_k\} = \gamma \delta^b_a \delta^j_k \qquad \{A^j_a, A^k_b\} = \{E^a_j, E^b_k\} = 0 \qquad \gamma \in \mathbb{R}$ Immirzi parameter

Hamiltonian GR on a Yang-Mills phase space, such that

- Connection A^j_a and momentum E^a_j are real valued
- The Poisson algebra has the simple form $\{A^j_a, E^b_k\} = \delta^b_a \delta^j_k$, $\{A, A\} = \{E, E\} = 0$
- The Yang-Mills gauge group is compact

Quantum theory: Independence of dimension / c. group

Simple Poisson brackets: Holonomy-flux algebra

holonomies: $h_c(A) := \mathcal{P}\exp\left(\int_c A\right)$ fluxes: $E^n(S) := \int_S n_{IJ}(*E)^{IJ}$

$$\boxed{\{h(A), E(S)\} \propto h(A)}$$

$\rightarrow$ Preferred Poisson subalgebra

Quantum theory: Independence of dimension / c. group

Simple Poisson brackets: Holonomy-flux algebra

$$\text{holonomies:} \quad h_c(A) := \mathcal{P}\exp\left(\int_c A\right) \qquad \text{fluxes:} \quad E^n(S) := \int_S n_{IJ}(*E)^{IJ}$$

$$\boxed{\{h(A), E(S)\} \propto h(A)}$$

$\rightarrow$ Preferred Poisson subalgebra

Compact gauge group: Measure theory on projective limits

Ashtekar-Lewandowski measure $d\mu_{AL}$ on the space of distributional connections

$\rightarrow$ Quantum configuration space $L^2(\overline{\mathcal{A}}, d\mu_{AL})$

Quantum theory: Independence of dimension / c. group

Simple Poisson brackets: Holonomy-flux algebra

holonomies: $h_c(A) := \mathcal{P}\exp\left(\int_c A\right)$ fluxes: $E^n(S) := \int_S n_{IJ}(*E)^{IJ}$

$$\boxed{\{h(A), E(S)\} \propto h(A)}$$

$\rightarrow$ Preferred Poisson subalgebra

Compact gauge group: Measure theory on projective limits

Ashtekar-Lewandowski measure $d\mu_{AL}$ on the space of distributional connections

$\rightarrow$ Quantum configuration space $L^2(\overline{\mathcal{A}}, d\mu_{AL})$

Simple reality conditions: Scalar product

$$< \Psi, \Psi' > := \int d\mu_{AL}\, \Psi^* \Psi'$$

$\rightarrow$ Implementation of the reality conditions

[Rovelli, Smolin, Ashtekar, Isham, Lewandowski, Marolf, Mourao, ...]

Outline

Hamiltonian Point of View

Phase space extension from the ADM-variables

New variables

- π^{aIJ}: densitized vielbein in the adjoint rep. of SO$(D+1)$, roughly $q^{ab} = \mathrm{Tr}(\pi^a \pi^b)$
- A_{aIJ}: so$(D+1)$ connection, roughly $A_{aIJ} = \underbrace{\Gamma^{\mathrm{hyb}}_{aIJ}(\pi)}_{\text{[Peldan '93]}} + \beta \underbrace{K_{ab}}_{\text{extr. curv.}} \pi^b{}_{IJ}$

spatial indices: $a, b = 1, .., D$, so$(D+1)$ indices $I, J = 0, .., D$,

Hamiltonian Point of View

Phase space extension from the ADM-variables

New variables

- π^{aIJ}: densitized vielbein in the adjoint rep. of SO$(D+1)$, roughly $q^{ab} = \mathrm{Tr}(\pi^a \pi^b)$
- A_{aIJ}: so$(D+1)$ connection, roughly $A_{aIJ} = \underbrace{\Gamma^{\mathrm{hyb}}_{aIJ}(\pi)}_{\text{[Peldan '93]}} + \beta \underbrace{K_{ab}}_{\text{extr. curv.}} \pi^b{}_{IJ}$

spatial indices: $a, b = 1, .., D$, so$(D+1)$ indices $I, J = 0, .., D$, $\partial_a e^J_b + \Gamma^c_{ab} e^J_c + \Gamma^{\mathrm{hyb}\,J}_{a}{}_K e^K_b = 0$

New constraints

- $G^{IJ} = D_a \pi^{aIJ} \approx 0$ SO$(D+1)$ Gauß constraint
- $S^{ab,IJKL} = \pi^{a[IJ} \pi^{b|KL]} \approx 0$ Simplicity constraint $\pi^{aIJ} = 2n^{[I} E^{aJ]}$, $n^I E^a_I = 0$

Hamiltonian Point of View

Phase space extension from the ADM-variables

New variables

- π^{aIJ}: densitized vielbein in the adjoint rep. of SO$(D+1)$, roughly $q^{ab} = \mathrm{Tr}(\pi^a \pi^b)$
- A_{aIJ}: so$(D+1)$ connection, roughly $A_{aIJ} = \underbrace{\Gamma^{\text{hyb}}_{aIJ}(\pi)}_{\text{[Peldan '93]}} + \beta \underbrace{K_{ab}}_{\text{extr. curv.}} \pi^b{}_{IJ}$

spatial indices: $a, b = 1, .., D$, so$(D+1)$ indices $I, J = 0, .., D$, $\partial_a e^J_b + \Gamma^c_{ab} e^J_c + \Gamma^{\text{hyb}\,J}_a{}_K e^K_b = 0$

New constraints

- $G^{IJ} = D_a \pi^{aIJ} \approx 0$ SO$(D+1)$ Gauß constraint
- $S^{ab,IJKL} = \pi^{a[IJ}\pi^{b|KL]} \approx 0$ Simplicity constraint $\pi^{aIJ} = 2n^{[I}E^{aJ]}$, $n^I E^a_I = 0$

Phase space extension

$q_{ab} = q_{ab}(\pi, A)$, $P^{ab} = P^{ab}(\pi, A)$, $\{A_{aIJ}, \pi^{bKL}\} = \beta \delta^b_a \delta^K_{[I}\delta^L_{J]}$

$\Rightarrow \{q_{ab}(A,\pi), P^{cd}(A,\pi)\} = \delta^c_{(a}\delta^d_{b)}$ (ADM brackets) $\beta \in \mathbb{R}$ free parameter

Hamiltonian Point of View

Phase space extension from the ADM-variables

New variables

- π^{aIJ}: densitized vielbein in the adjoint rep. of SO$(D+1)$, roughly $q^{ab} = \mathrm{Tr}(\pi^a \pi^b)$
- A_{aIJ}: so$(D+1)$ connection, roughly $A_{aIJ} = \underbrace{\Gamma^{\text{hyb}}_{aIJ}(\pi)}_{\text{[Peldan '93]}} + \beta \underbrace{K_{ab}}_{\text{extr. curv.}} \pi^{b}{}_{IJ}$

spatial indices: $a, b = 1, .., D$, so$(D+1)$ indices $I, J = 0, .., D$, $\partial_a e^J_b + \Gamma^c_{ab} e^J_c + \Gamma^{\text{hyb}}_a{}^J{}_K e^K_b = 0$

New constraints

- $G^{IJ} = D_a \pi^{aIJ} \approx 0$ SO$(D+1)$ Gauß constraint
- $S^{ab,IJKL} = \pi^{a[IJ}\pi^{b|KL]} \approx 0$ Simplicity constraint $\pi^{aIJ} = 2n^{[I}E^{aJ]}$, $n^I E^a_I = 0$

Phase space extension

$q_{ab} = q_{ab}(\pi, A)$, $P^{ab} = P^{ab}(\pi, A)$, $\{A_{aIJ}, \pi^{bKL}\} = \beta \delta^b_a \delta^K_{[I}\delta^L_{J]}$

$\Rightarrow \{q_{ab}(A,\pi), P^{cd}(A,\pi)\} = \delta^c_{(a}\delta^d_{b)}$ (ADM brackets) $\beta \in \mathbb{R}$ free parameter

Independence of the internal signature

Both Lorentzian and Euclidean GR can be formulated using either $SO(1,D)$ or $SO(D+1)$ as internal gauge group

Lagrangian Point of View

Canonical analysis of the Plebanski action [Plebanski '77]

Starting point

- GR $=$ BF-Theory $+$ Simplicity constraint (same as in spin foam models)

Lagrangian Point of View

Canonical analysis of the Plebanski action [Plebanski '77]

Starting point

- GR = BF-Theory + Simplicity constraint (same as in spin foam models)

Dirac stability algorithm

- Second class partner for simplicity constraint [Peldan '93]
- Solution of second class constraints $\Rightarrow$ ADM [Peldan '93]
- Dirac bracket is not of standard form, e.g. $\{A, A\} \neq 0$ [Alexandrov '00]

Lagrangian Point of View

Canonical analysis of the Plebanski action [Plebanski '77]

Starting point

- GR = BF-Theory + Simplicity constraint (same as in spin foam models)

Dirac stability algorithm

- Second class partner for simplicity constraint [Peldan '93]
- Solution of second class constraints $\Rightarrow$ ADM [Peldan '93]
- Dirac bracket is not of standard form, e.g. $\{A, A\} \neq 0$ [Alexandrov '00]

Gauge unfixing

- Well defined procedure to trade second class for first class constraints [Mitra, Rajaraman '89; Vytheesvaran '94]
- Second class partner can be removed to give same result as Hamiltonian derivation

Lagrangian Point of View

Canonical analysis of the Plebanski action [Plebanski '77]

Starting point

- GR = BF-Theory + Simplicity constraint (same as in spin foam models)

Dirac stability algorithm

- Second class partner for simplicity constraint [Peldan '93]
- Solution of second class constraints $\Rightarrow$ ADM [Peldan '93]
- Dirac bracket is not of standard form, e.g. $\{A, A\} \neq 0$ [Alexandrov '00]

Gauge unfixing

- Well defined procedure to trade second class for first class constraints [Mitra, Rajaraman '89; Vytheesvaran '94]
- Second class partner can be removed to give same result as Hamiltonian derivation

Restrictions compared to Hamiltonian derivation

- Only for matching internal and external signatures
- No free parameter

Quantum Theory

Conditions for loop quantisation

- A_{aIJ} and π^{aIJ} are real variables
- The Poisson algebra has the simple form $\{A_{aIJ}, \pi^{bKL}\} = \beta \delta_a^b \delta_{[I}^K \delta_{J]}^L$
- $SO(D+1)$ is compact

$\Rightarrow$ Loop quantisation methods apply!

Quantum Theory

Conditions for loop quantisation

- A_{aIJ} and π^{aIJ} are real variables
- The Poisson algebra has the simple form $\{A_{aIJ}, \pi^{bKL}\} = \beta \delta_a^b \delta_{[I}^K \delta_{J]}^L$
- $SO(D+1)$ is compact

$\Rightarrow$ Loop quantisation methods apply!

Constraint operator representation

- Gauß constraint: as in LQG
- Spatial diffeomorphism constraint: as in LQG
- Hamiltonian constraint: as in LQG (generalised Thiemann's trick)
- Simplicity constraints: several proposals in the literature

$\Rightarrow$ Constraint operators can be defined!

Generalisations

Standard and exotic matter

- Dirac, Weyl, Majorana fermions
- Gauge fields with compact gauge group (standard model)
- Scalar fields
- Rarita-Schwinger fields (gravitinos)
- Abelian p-form fields

Generalisations

Standard and exotic matter

- Dirac, Weyl, Majorana fermions
- Gauge fields with compact gauge group (standard model)
- Scalar fields
- Rarita-Schwinger fields (gravitinos)
- Abelian p-form fields

So far not treatable

- Non-Abelian p-form fields ($\to$ higher gauge theory)
- Non-compact gauge groups

Generalisations

Standard and exotic matter

- Dirac, Weyl, Majorana fermions
- Gauge fields with compact gauge group (standard model)
- Scalar fields
- Rarita-Schwinger fields (gravitinos)
- Abelian p-form fields

So far not treatable

- Non-Abelian p-form fields ($\rightarrow$ higher gauge theory)
- Non-compact gauge groups

Canonically quantisable[1] Supergravities include

- $d = 4, \;\; N = 8$ (perturbatively quantisable?)
- $d = 10, N = 1$ (Superstring theory)
- $d = 11, N = 1$ (M-theory)

1: There exists a representation of a Poisson subalgebra $\mathfrak{P}$ (which splits points in phase space) on a Hilbert space $\mathcal{H}$ whose scalar product implements the classical reality conditions and supports a representation of the constraint operators (of the master constraint algebra).

Outline

Black Hole Entropy

Calculation of black hole entropy

- Thermodynamic analogy: Bekenstein '73; QFTCS: Hawking '74
- String theory: Strominger, Vafa, Maldacena '96
- Loop quantum gravity: Krasnov '96; Rovelli 96'; Ashtekar et. al. '97

$\Rightarrow$ Calculation possible in different theories!

Black Hole Entropy

Calculation of black hole entropy

- Thermodynamic analogy: Bekenstein '73; QFTCS: Hawking '74
- String theory: Strominger, Vafa, Maldacena '96
- Loop quantum gravity: Krasnov '96; Rovelli 96'; Ashtekar et. al. '97

$\Rightarrow$ Calculation possible in different theories!

Special interest

- Calculate entropy of a supersymmetric extremal black hole in higher dimensions
- Compare to results coming from string theory

Cosmology

Cosmology from different points of view

- Classical cosmology: Einstein, de Sitter, Friedmann, ...
- String cosmology: Veneziano '91
- Loop quantum cosmology: Bojowald '01

$\Rightarrow$ Calculation possible in different theories!

Cosmology

Cosmology from different points of view

- Classical cosmology: Einstein, de Sitter, Friedmann, ...
- String cosmology: Veneziano '91
- Loop quantum cosmology: Bojowald '01

$\Rightarrow$ Calculation possible in different theories!

Special interest

- Investigate SLQC in higher dimensions
- Compare to results coming from string cosmology and possibly from experiments

Outline

Conclusion

- $D + 1$ dim. GR on a $SO(D + 1)$ Yang-Mills phase space

Conclusion

- $D+1$ dim. GR on a $SO(D+1)$ Yang-Mills phase space
- LQG methods apply $\rightarrow$ rigorous quantisation exists

Conclusion

- $D + 1$ dim. GR on a $SO(D + 1)$ Yang-Mills phase space
- LQG methods apply $\rightarrow$ rigorous quantisation exists
- Extensions to interesting Supergravities exist

Conclusion

- $D + 1$ dim. GR on a $SO(D + 1)$ Yang-Mills phase space
- LQG methods apply $\rightarrow$ rigorous quantisation exists
- Extensions to interesting Supergravities exist
- LQG-based calculations of
  - Higher dimensional (supersymmetric) black hole entropy
  - Higher dimensional (supersymmetric) quantum cosmology

  are in reach.

Conclusion

- $D+1$ dim. GR on a $SO(D+1)$ Yang-Mills phase space
- LQG methods apply $\rightarrow$ rigorous quantisation exists
- Extensions to interesting Supergravities exist
- LQG-based calculations of
  - Higher dimensional (supersymmetric) black hole entropy
  - Higher dimensional (supersymmetric) quantum cosmology

  are in reach.

Thank you for your attention!