Spectral Actions

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- Uncanny Precision of Spectral Action
- Spectral Action For Robertson Walker metrics

Based on collaborative work with **Alain Connes** in publications:

- *The Spectral Action Principle, Comm. Math. Phys. 186, 731-750 (1997)*
- *Scale Invariance in the Spectral Action, J. Math. Phys. 47, 063504 (2006)*
- *Inner Fluctuations of the Spectral Action, J. Geom. Phys. 57, 1, (2006)*
- *Gravity and the Standard Model with Neutrino Mixing, Adv. Theor. Math. Phys. 11 991-1090 (2007).*
- *Boundary Terms in Quantum Gravity from Spectral Action of Noncommutative Space, Phys. Rev. Lett. 99 071302 (2007).*
- *Why the Standard Model Journ. Geom. Phys.* 58:38-47,2008.
- *Uncanny Precision of the Spectral Action, 2009, Spectral Action for Robertson Walker metrics, 2011.*

1 Introduction

- Taking GR as prototype for other forces where Geometry determines the dynamics, we will set to construct geometrical spaces and associate with these dynamical actions.
- Dirac operator is a basic ingredient in defining noncommutative spaces.
- Eigenvalues of Dirac operators define geometric invariants. The Spectral action is a function of these eigenvalues.
- The only restriction on the function is that it is a positive function.
- Principle although simple works in a large number of cases.

2 *A Brief Summary of AC NCG*

The basic idea is based on physics. The modern way of measuring distances is spectral. The units of distance is taken as the wavelength of atomic spectra. To adopt this geometrically we have to replace the notion of real variable which one takes as a function f on a set X, $f : X \to R$. It is now given by a self adjoint operator in a Hilbert space as in quantum mechanics. The space X is described by the algebra $\mathcal{A}$ of coordinates which is represented as operators in a fixed Hilbert space $\mathcal{H}$.The geometry of the noncommutative space is determined in terms of the spectral data $(\mathcal{A}, \mathcal{H}, \mathcal{D}, J, \gamma)$. A real, even spectral triple is defined by

- $\mathcal{A}$ an associative algebra with unit 1 and involution $*$.
- $\mathcal{H}$ is a complex Hilbert space carrying a faithful representation π of the algebra.
- $\mathcal{D}$ is a self-adjoint operator on $\mathcal{H}$ with the resolvent $(D-\lambda)^{-1}, \lambda \in \mathbf{R}$ of D compact.
- J is an anti–unitary operator on $\mathcal{H}$, a real structure (charge conjugation.)
- γ is a unitary operator on $\mathcal{H}$, the chirality.

We require the following axioms to hold:

- $J^2 = \epsilon$, ($\varepsilon = 1$ in zero dimensions and $\varepsilon = -1$ in 4 dimensions).
- $[a, b^o] = 0$ for all $a, b \in \mathcal{A}$, $b^o = Jb^*J^{-1}$. This is the zeroth order condition. This is needed to define the right action on elements of $\mathcal{H}$: $\zeta b = b^o \zeta$.
- $DJ = \varepsilon' JD, \quad J\gamma = \varepsilon'' \gamma J, \quad D\gamma = -\gamma D$ where $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$. The reality conditions resemble the conditions of existence of Majorana (real) fermions.
- $[[D, a], b^o] = 0$ for all $a, b \in \mathcal{A}$. This is the first order condition.
- $\gamma^2 = 1$ and $[\gamma, a] = 0$ for all $a \in \mathcal{A}$. These properties allow the decomposition
  $\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R$.
- $\mathcal{H}$ is endowed with $\mathcal{A}$ bimodule structure $a\zeta b = ab^o\zeta$.
- The notion of dimension is governed by growth of eigenvalues, and may be fractals or complex.
- $\mathcal{A}$ has a well defined unitary group

  $$\mathcal{U} = \{u \in \mathcal{A}; \quad u\,u^* = u^*u = 1\}$$

  The natural adjoint action of of $\mathcal{U}$ on $\mathcal{H}$ is given by $\zeta \to u\zeta u^* = u\,J\,u\,J^*\zeta \quad \forall \zeta \in \mathcal{H}$. Then

  $$\langle \zeta, D\zeta \rangle$$

  is not invariant under the above transformation:

  $$(u\,J\,u\,J^*)\,D\,(u\,J\,u\,J^*)^* = D + u\,[D, u^*] + J\,(u\,[D, u^*])\,J^*$$

- Then the action $\langle \zeta, D_A\zeta \rangle$ is invariant where

  $$D_A = D + A + \varepsilon' JAJ^{-1}, \quad A = \sum_i a^i\,[D, b^i]$$

and $A = A^*$ is self-adjoint. This is similar to the appearance of the interaction term for the photon with the electrons

$$i\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi \rightarrow i\overline{\psi}\gamma^{\mu}\left(\partial_{\mu} + ieA_{\mu}\right)\psi$$

to maintain invariance under the variations
$\psi \rightarrow e^{i\alpha(x)}\psi.$

- A real structure of KO-dimension $n \in \mathbb{Z}/8$ on a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is an antilinear isometry $J : \mathcal{H} \rightarrow \mathcal{H}$, with the property that

$$J^{\,2} = \varepsilon, \quad JD = \varepsilon' DJ, \quad \text{and} \quad J\gamma = \varepsilon''\gamma J \text{ (even case)}.$$

The numbers $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ are a function of n mod 8 given by

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

- The algebra $\mathcal{A}$ is a tensor product which geometrically corresponds to a product space. The spectral geometry of $\mathcal{A}$ is given by the product rule $\mathcal{A} = C^{\infty}(M) \otimes \mathcal{A}_F$ where the algebra $\mathcal{A}_F$ is finite dimensional, and

$$\mathcal{H} = L^2(M, S) \otimes \mathcal{H}_F, \quad D = D_M \otimes 1 + \gamma_5 \otimes D_F,$$

where $L^2(M, S)$ is the Hilbert space of L^2 spinors, and D_M is the Dirac operator of the Levi-Civita spin connection on M, $D_M = \gamma^{\mu}\left(\partial_{\mu} + \omega_{\mu}\right).$ The Hilbert space $\mathcal{H}_F$ is taken to include the physical fermions. The chirality operator is $\gamma = \gamma_5 \otimes \gamma_F$.

In order to avoid the fermion doubling problem $\zeta, \zeta^c, \zeta^*, \zeta^{c*}$ where $\zeta \in \mathcal{H}$, are not independent) it was shown that the finite dimensional space must be taken to be of K-theoretic dimension 6 where in this case $(\varepsilon, \varepsilon', \varepsilon") = (1, 1, -1)$ (so as to impose the condition $J\zeta = \zeta$). This makes the total K-theoretic dimension of the noncommutative space to be 10 and would allow to impose the reality (Majorana) condition and the Weyl condition simultaneously in the Minkowskian continued form, a situation very familiar in ten-dimensional supersymmetry. In the Euclidean version, the use of the J in the fermionic action, would give for the chiral fermions in the path integral, a Pfaffian instead of determinant, and will thus cut the fermionic degrees of freedom by 2. In other words, to have the fermionic sector free of the fermionic doubling problem we must make the choice

$$J_F^2 = 1, \qquad J_F D_F = D_F J_F, \qquad J_F \gamma_F = -\gamma_F J_F$$

In what follows we will restrict our attention to determination of the finite algebra, and will omit the subscript F.

3 Noncommutative Space of Standard Model

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- There are two main constraints on the algebra from the axioms of noncommutative geometry. We first look for involutive algebras $\mathcal{A}$ of operators in $\mathcal{H}$ such that,

$$[a, b^0] = 0\,, \quad \forall\, a, b \in \mathcal{A}\,.$$

 where for any operator a in $\mathcal{H}$, $a^0 = Ja^* J^{-1}$. This is called the order zero condition. We shall assume that the following two conditions to hold. We assume the representation of $\mathcal{A}$ and J in $\mathcal{H}$ is *irreducible.*

- Classify the irreducible triplets $(\mathcal{A}, \mathcal{H}, J)$.

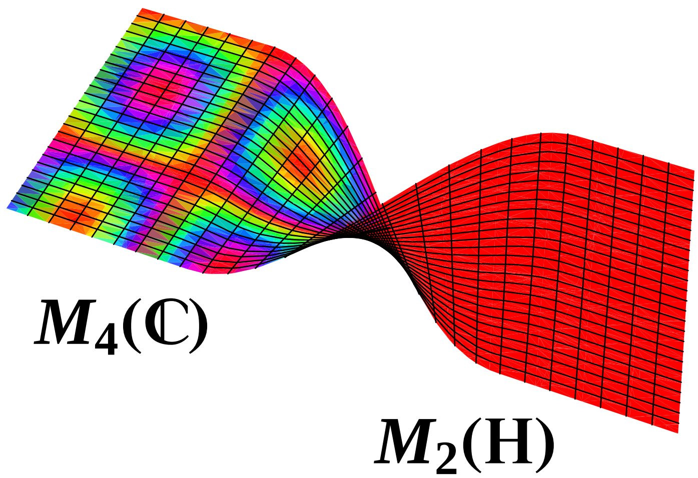
- In this case we can state the following theorem: *The center $Z(\mathcal{A}_{\mathbb{C}})$ is C or $C \oplus C$.*

- If the center $Z(\mathcal{A}_{\mathbb{C}})$ is $\mathbb{C}$ then $\mathcal{A}_{\mathbb{C}} = M_k(\mathbb{C})$ and $\mathcal{A} = M_k(\mathbb{C})$, $M_k(\mathbb{R})$ and $M_a(\mathbb{H})$ for even $k = 2a$, where $\mathbb{H}$ is the field of quaternions. These correspond respectively to the unitary, orthogonal and symplectic case. The dimension of $\mathcal{H}$ Hilbert spac is $n = k^2$ is a square and $J(x) = x^*, \quad \forall x \in M_k(\mathbb{C})$.

- If the center $Z(\mathcal{A}_{\mathbb{C}})$ is $\mathbb{C} \oplus \mathbb{C}$ then we can state the theorem: *Let H be a Hilbert space of dimension n. Then an irreducible solution with $Z(\mathcal{A}_{\mathbb{C}}) = C \oplus C$ exists iff $n = 2k^2$ is twice a square. It is given by $A_{\mathbb{C}} = M_k(\mathbb{C}) \oplus M_k(\mathbb{C})$ acting by left multiplication on itself and antilinear involution*

$$J(x, y) = (y^*, x^*)\,, \quad \forall x, y \in M_k(\mathbb{C})\,.$$

 With each of the $M_k(\mathbb{C})$ in $\mathcal{A}_{\mathbb{C}}$ we can have the three possibilities $M_k(\mathbb{C})$, $M_k(\mathbb{R})$, or $M_a(\mathbb{H})$, where $k = 2a$. At this point we make the *hypothesis* that we are in the "symplectic–unitary" case, thus restricting the algebra $\mathcal{A}$ to the form $\mathcal{A} = M_a(\mathbb{H}) \oplus M_k(\mathbb{C})$, $k = 2a$. The dimension of the Hilbert space $n = 2k^2$ then corresponds to k^2 fundamental fermions, where $k = 2a$ is an even number. The first possible value for k is 2 corresponding to a Hilbert space of four fermions and an algebra $\mathcal{A} = \mathbb{H} \oplus M_2(\mathbb{C})$. The existence of quarks rules out this possibility. The next possible value for k is 4 predicting the number of fermions to be 16.



Up to an automorphisms of A^{ev}, there exists a unique involutive subalgebra $A_F \subset A^{\mathrm{ev}}$ of maximal dimension admitting off-diagonal Dirac operators

$$\mathcal{A}_F = \{\lambda \oplus q, \lambda \oplus m \,|\, \lambda \in \mathbb{C}, q \in \mathbb{H}, m \in M_3(\mathbb{C})\}$$
$$\subset \mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C})$$

isomorphic to $C \oplus \mathbb{H} \oplus M_3(\mathbb{C})$.

We denote the spinors as follows

$$\begin{aligned}\psi_A = \psi_{\alpha I} &= (\psi_{\alpha 1}, \psi_{\alpha i}) \\ &= \left(\psi_{\dot{1}1}, \psi_{\dot{2}1}, \psi_{a1}, \psi_{\dot{1}i}, \psi_{\dot{2}i}, \psi_{ai}\right) \\ &\equiv (\nu_R, e_R, l_a, u_{Ri}, d_{Ri}, q_{ai})\end{aligned}$$

where $l_a = (\nu_L, e_L)$ and $q_{ai} = (u_{Li}, d_{Li})$. The component $\psi_{\dot{1}'1'} = \psi^c_{\dot{1}1}$ so that we get

$$\psi^*_A D^B_A \psi_B + \nu^{*c}_R k^{*\nu_R} \nu_R + cc$$

Needless to say the term $\psi^*_A D^B_A \psi_B$ contains all the fermionic interaction terms in the standard model.

Write the Dirac operator in the form

$$D = \begin{pmatrix} D^B_A & D^{B'}_A \\ D^B_{A'} & D^{B'}_{A'} \end{pmatrix},$$

where

$$A = \alpha I, \quad \alpha = 1, \cdots, 4, \quad I = 1, \cdots, 4$$
$$A' = \alpha' I', \quad \alpha' = 1', \cdots, 4', \quad I = 1', \cdots, 4'$$

Thus $D_A^B = D_{\alpha I}^{\beta J}$. We start with the algebra

$$\mathcal{A} = M_4(\mathbb{C}) \oplus M_4(\mathbb{C})$$

and write

$$a = \begin{pmatrix} X_\alpha^\beta \delta_I^J & 0 \\ 0 & \delta_{\alpha'}^{\beta'} Y_{I'}^{J'} \end{pmatrix}$$

In this form

$$a^o = Ja^* J^{-1} = \begin{pmatrix} \delta_\alpha^\beta Y_I^{tJ} & 0 \\ 0 & X_{\alpha'}^{*\beta'} \delta_{I',}^{J'} \end{pmatrix}$$

and clearly satisfy $[a, b^o] = 0$. The order one condition is

$$[[D, a], b^o] = 0$$

Write

$$b^o = \begin{pmatrix} \delta_\alpha^\beta W_I^J & 0 \\ 0 & Z_{\alpha'}^{\beta'} \delta_{I'}^{J'} \end{pmatrix}$$

then

$$[[D, a], b^o] = \begin{pmatrix} \left[\left[D_A^B, X\right], W\right] & \left(D_A^{B'} Y - X D_A^{B'}\right) Z - W \left(D_A^{B'} Y - X D_A^B \right. \\ \left(D_{A'}^B X - Y D_{A'}^B\right) W - Z \left(D_{A'}^B X - Y D_{A'}^B\right) & \left[\left[D_{A'}^{B'}, Y\right], Z\right] \end{pmatrix}$$

Explicitely the first two equations:

$$\left(D_{\alpha I}^{\gamma K} X_\gamma^\beta - X_\alpha^\gamma D_{\gamma I}^{\beta K}\right) W_K^J - W_I^K \left(D_{\alpha K}^{\gamma J} X_\gamma^\beta - X_\alpha^\gamma D_{\gamma K}^{\beta J}\right) = 0$$
$$\left(D_{\alpha I}^{\gamma' K'} Y_{K'}^{J'} - X_\alpha^\gamma D_{\gamma I}^{\gamma' K}\right) Z_{\gamma'}^{\beta'} - W_I^K \left(D_{\alpha K}^{\beta' K'} Y_{K'}^{J'} - X_\alpha^\gamma D_{\gamma K}^{\beta' J'}\right) = 0$$

We have shown that the only solution of the second equation is

$$D_{\alpha I}^{\beta' K'} = \delta_\alpha^{\dot{1}} \delta_{\dot{1}'}^{\beta'} \delta_I^1 \delta_{1'}^{K'} k^{*\nu_R}$$

and this implies that

$$D_{\alpha I}^{\beta J} = D_{\alpha(l)}^\beta \delta_I^1 \delta_1^J + D_{\alpha(q)}^\beta \delta_I^i \delta_j^J \delta_i^j$$
$$Y_{I'}^{J'} = \delta_{I'}^{1'} \delta_{1'}^{J'} Y_{1'}^{1'} + \delta_{I'}^{i'} \delta_{j'}^{J'} Y_{i'}^{j'}$$
$$X_{\dot{1}}^{\dot{1}} = Y_{1'}^{1'}, \; X_{\dot{1}}^\alpha = 0, \quad \alpha \neq \dot{1}$$

We will be using the notation

$$\alpha = \dot{1}, \dot{2}, a \quad \text{where } a = 1, 2$$

From the property of commutation of the grading operator

$$g_\alpha^\beta = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}$$
$$[g, a] = 0 \quad a \in M_4(\mathbb{C})$$

the algebra $M_4(\mathbb{C})$ reduces to $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$. We further impose the condition of symplectic isometry on $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$

$$\sigma_2 \otimes 1_2 (\overline{a}) \sigma_2 \otimes 1_2 = a$$

reduces it to $\mathbb{H} \oplus \mathbb{H}$. Together with the above condition this implies that

$$X_\alpha^\beta = \delta_\alpha^{\dot{1}} \delta_{\dot{1}}^\beta X_{\dot{1}}^{\dot{1}} + \delta_\alpha^{\dot{2}} \delta_{\dot{2}}^{\beta'} \overline{X}_{\dot{1}}^{\dot{1}} + \delta_\alpha^a \delta_b^\beta X_a^b$$

and the algebra $\mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C})$ reduces to

$$\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$$

because $X_{\dot{1}}^{\dot{1}} = Y_{1'}^{1'}$.

With this we can form the Dirac operator of the product space of this discrete space times a four-dimensional Riemannian manifold

$$D = D_M \otimes 1 + \gamma_5 \otimes D_F$$

Since D_F is a 32×32 matrix tensored with the 3×3 matrices of generation space, D is 384×384 matrix.

Next we have to evaluate the operator

$$D_A = D + A + JAJ^{-1}$$

where

$$A = \sum a [D, b]$$

or in tensor notation

$$A_A^B = \sum a_A^C \left(D_C^D b_D^B - b_C^D D_D^B \right)$$

(there are no mixing terms like $D_{C}^{D'} b_{D'}^{B}$ because b is block diagonal).

Writing all components of the the full Dirac operator $D_{\alpha I}^{\beta J}$

$$(D)_{\dot{1}1}^{\dot{1}1} = \gamma^{\mu} \otimes D_{\mu} \otimes 1_3, \quad D_{\mu} = \partial_{\mu} + \frac{1}{4}\omega_{\mu}^{cd}(e)\,\gamma_{cd}, \quad 1_3 = \text{generations}$$

$$(D)_{\dot{1}1}^{a1} = \gamma_5 \otimes k^{*\nu} \otimes \epsilon^{ab} H_b \qquad k^{\nu} = 3 \times 3 \text{ neutrino mixing matrix}$$

$$(D)_{\dot{2}1}^{\dot{2}1} = \gamma^{\mu} \otimes (D_{\mu} + i g_1 B_{\mu}) \otimes 1_3$$

$$(D)_{\dot{2}1}^{a1} = \gamma_5 \otimes k^{*e} \otimes \overline{H}^{a}$$

$$(D)_{a1}^{\dot{1}1} = \gamma_5 \otimes k^{\nu} \otimes \epsilon_{ab} \overline{H}^{b}$$

$$(D)_{a1}^{\dot{2}1} = \gamma_5 \otimes k^{e} \otimes H_a$$

$$(D)_{a1}^{b1} = \gamma^{\mu} \otimes \left(\left(D_{\mu} + \frac{i}{2} g_1 B_{\mu} \right) \delta_a^b - \frac{i}{2} g_2 W_{\mu}^{\alpha} \left(\sigma^{\alpha}\right)_a^b \right) \otimes 1_3, \qquad \sigma^{\alpha} = \text{Pauli}$$

$$(D)_{\dot{1}i}^{\dot{1}j} = \gamma^{\mu} \otimes \left(\left(D_{\mu} - \frac{2i}{3} g_1 B_{\mu} \right) \delta_i^j - \frac{i}{2} g_3 V_{\mu}^{m} \left(\lambda^{m}\right)_i^j \right) \otimes 1_3, \qquad \lambda^{i} = \text{Gell-Mann}$$

$$(D)_{\dot{1}i}^{aj} = \gamma_5 \otimes k^{*u} \otimes \epsilon^{ab} H_b \delta_i^j$$

$$(D)_{\dot{2}i}^{\dot{2}j} = \gamma^{\mu} \otimes \left(\left(D_{\mu} + \frac{i}{3} g_1 B_{\mu} \right) \delta_i^j - \frac{i}{2} g_3 V_{\mu}^{m} \left(\lambda^{m}\right)_i^j \right) \otimes 1_3$$

$$(D)_{\dot{2}i}^{aj} = \gamma_5 \otimes k^{*d} \otimes \overline{H}^{a} \delta_i^j$$

$$(D)_{ai}^{bj} = \gamma^{\mu} \otimes \left(\left(D_{\mu} - \frac{i}{6} g_1 B_{\mu} \right) \delta_a^b \delta_i^j - \frac{i}{2} g_2 W_{\mu}^{\alpha} \left(\sigma^{\alpha}\right)_a^b \delta_i^j - \frac{i}{2} g_3 V_{\mu}^{m} \left(\lambda^{m}\right)_i^j \delta_a^b \right) \otimes 1_3$$

$$(D)_{ai}^{\dot{1}j} = \gamma_5 \otimes k^{u} \otimes \epsilon_{ab} \overline{H}^{b} \delta_i^j$$

$$(D)_{ai}^{\dot{2}j} = \gamma_5 \otimes k^{d} \otimes H_a \delta_i^j$$

$$(D)_{\dot{1}1}^{\dot{1}'1'} = \gamma_5 \otimes k^{*\nu_R} \sigma \qquad \text{generate scale } M_R \text{ by } \sigma \to M_R$$

$$(D)_{\dot{1}'1'}^{\dot{1}1} = \gamma_5 \otimes k^{\nu_R} \sigma$$

$$D_{A'}^{B'} = \overline{D}_{A}^{B}$$

where the matrix form would look like

$$\begin{pmatrix} \dot{1}1 & \dot{2}1 & a1 & \dot{1}i & \dot{2}i & ai \\ v_R & e_R & l_a & u_{iR} & d_{iR} & q_{iL} \end{pmatrix}$$

$$\begin{pmatrix} \dot{1}1 \\ \dot{2}1 \\ b1 \\ \dot{1}j \\ \dot{2}j \\ bj \end{pmatrix} \begin{pmatrix} (D)^{\dot{1}1}_{\dot{1}1} & 0 & (D)^{a1}_{\dot{1}1} & 0 & 0 & 0 \\ 0 & (D)^{\dot{2}1}_{\dot{2}1} & (D)^{a1}_{\dot{2}1} & 0 & 0 & 0 \\ (D)^{\dot{1}1}_{b1} & (D)^{\dot{2}1}_{b1} & (D)^{b1}_{a1} & 0 & 0 & 0 \\ 0 & 0 & 0 & (D)^{\dot{1}i}_{\dot{1}j} & 0 & (D)^{ai}_{\dot{1}j} \\ 0 & 0 & 0 & 0 & (D)^{\dot{2}i}_{\dot{2}j} & (D)^{ai}_{\dot{2}j} \\ 0 & 0 & 0 & (D)^{\dot{1}i}_{bj} & (D)^{\dot{2}i}_{bj} & (D)^{ai}_{bj} \end{pmatrix}$$

4 *The Spectral Action Principle (SAP)*

- There is a shift of point of view in NCG similar to Fourier transform, where the usual emphasis on the points on the points $x \in M$ of a geometric space is now replaced by the spectrum Σ of the operator D. The existence of Riemannian manifolds which are isospectral but not isometric shows that the following hypothesis is stronger than the usual diffeomorphism invariance of the action of general relativity

The physical action depends only on the Σ

This is the spectral action principle (SAP) . The spectrum is a geometric invariant and replaces diffeomorphism invariance.

- Apply this basic principle to the noncommutative geometry defined by the spectrum of the standard model to show that the dynamics of all the interactions, including gravity is given by the spectral action

$$\text{Trace } f\left(\frac{D_A}{\Lambda}\right) + \frac{1}{2}\langle J\psi, D_A\psi\rangle$$

where f is a test function, Λ a cutoff scale and ψ represents the fermions.

- The function f only plays a role through its momenta f_0, f_2, f_4 where

$$f_k = \int_0^\infty f(v)v^{k-1}dv, \quad \text{for } k > 0, \;\; f_0 = f(0).$$

These will serve as three free parameters in the model. $S_\Lambda[D_A]$ is the number of eigenvalues λ of D_A counted with their multiplicities such that $|\lambda| \leq \Lambda$.

To illustrate how this comes, expand the function f in terms of its Laplace transform

$$\mathrm{Trace} f\left(P\right) = \sum_s f_{s'} \mathrm{Trace}\left(P^{-s}\right)$$

$$\mathrm{Trace}\left(P^{-s}\right) = \frac{1}{\Gamma\left(s\right)} \int_0^\infty t^{s-1} \mathrm{Trace}\left(e^{-tP}\right) dt \qquad \mathrm{Re}\left(s\right) \geq 0$$

$$\mathrm{Trace}\left(e^{-tP}\right) \simeq \sum_{n\geq 0} t^{\frac{n-m}{d}} \int_M a_n\left(x,P\right) dv\left(x\right)$$

Gilkey gives generic formulas for the Seeley-deWitt coefficients $a_n\left(x,P\right)$ for a large class of differential operators P.

- The bosonic part gives an action that unifies gravity with $SU(2) \times U(1) \times SU(3)$ Yang-Mills gauge theory, with a Higgs doublet ϕ and spontaneous symmetry breaking. It is given by

$$S = \int \Big(\tfrac{1}{2\kappa_0^2} R + \alpha_0 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \gamma_0 + \tau_0 R^* R^* + \tfrac{1}{4} G^i_{\mu\nu} G^{\mu\nu i} + \tfrac{1}{4} F^\alpha_{\mu\nu} F^{\mu\nu\alpha} + \tfrac{1}{4} B_{\mu\nu} B^{\mu\nu} + \tfrac{1}{2} |D_\mu H|^2 - \mu_0^2 |H|^2 - \xi_0 R |H|^2 + \lambda_0 |H|^4 \Big) \sqrt{g}\, d^4x,$$

We can answer the following questions:

- Why the specific $U(1) \times SU(2) \times SU(3)$ gauge group.
- Why the particular representations.
- Why 16 fermions in one generation.
- Why the Higgs field and the spontaneous symmetry breaking.
- Why the Higgs mass, the fermion masses.
- A relation among the fermions and W mass

$$\sum_{\text{generations}} m_e^2 + m_\nu^2 + 3m_d^2 + 3m_u^2 = 8M_W^2.$$

a top quark mass of the order of $\frac{1}{\sqrt{2}} y_0 \, v \sim 173.683 \, y_0$ Gev.

The see-saw mechanism, however, suggests that the Yukawa coupling for the τ neutrino is of the same order as the top quark Yukawa coupling. Indeed, even if the tau neutrino mass has an upper bound of the order

$$m_{\nu_\tau} \leq 18.2 \text{ MeV},$$

the see-saw mechanism allows for a large Yukawa coupling term and in effect lowering the value of y_0 to $y_0 \sim 1.04$, which yields an acceptable value for the top quark mass of 179 Gev.

5 Spectral Action for NC Spaces with Boundary

In the Hamiltonian quantization of gravity it is essential to include boundary terms in the action as this allows to define consistently the momentum conjugate to the metric. This makes it necessary to modify the Einstein-Hilbert action by adding to it a surface integral term so that the variation of the action is well defined. The reason for this is that the curvature scalar R contains second derivatives of the metric, which are removed after integrating by parts to obtain an action which is quadratic in first derivatives of the metric. To see this note that the curvature $R\sim\partial\Gamma+\Gamma\Gamma$ where $\Gamma\sim g^{-1}\partial g$ has two parts, one part is of second order in derivatives of the form $g^{-1}\partial^{2}g$ and the second part is the square of derivative terms of the form $\partial g\partial g.$ To define the conjugate momenta in the Hamiltonian formalism, it is necessary to integrate by parts the term $g^{-1}\partial^{2}g$ and change it to the form $\partial g\partial g$. These surface terms, which turned out to be very important, are canceled by modifying the Euclidean action to

$$I=-\frac{1}{16\pi}\int\limits_{M}d^{4}x\sqrt{g}R-\frac{1}{8\pi}\int\limits_{\partial M}d^{3}x\sqrt{h}K,$$

where ∂M is the boundary of $M,$ h_{ab} is the induced metric on ∂M and K is the trace of the second fundamental form on ∂M. Notice that there is a relative factor of 2 between the two terms, and that the boundary term has to be completely fixed. This is a delicate fine tuning and is not determined by any symmetry, but only by the consistency requirement. There is no known symmetry that predicts this combination and it is always added by hand. In contrast we can compute the spectral action for manifolds with boundary. The hermiticity of the Dirac operator

$$(\psi|\,D\psi)=(D\psi|\,\psi)$$

is satisfied provided that $\pi_{-}\psi|_{\partial M}=0$ where $\pi_{-}=\frac{1}{2}\left(1-\chi\right)$ is a projection operator on ∂M with $\chi^{2}=1$. To compute the spectral action for manifolds with boundary we have to specify the condition $\pi_{-}D\psi|_{\partial M}=0.$ The result of the computation gives the remarkable result that the Gibbons-Hawking boundary term is generated without any fine tuning. Adding matter interactions, does not alter the relative sign and coefficients of these two terms, even when higher orders are included. The Dirac operator for a product space such as that of the standard model, must now be taken to be of the form

$$D = D_1 \otimes \gamma_F + 1 \otimes D_F$$

instead of

$$D = D_1 \otimes 1 + \gamma_5 \otimes D_F$$

because γ_5 does not anticommute with D_1 on ∂M.

We list the first relevant Seeley-deWitt coefficients for Laplacians which are square of Dirac operators

$$a_0 \left(P, \chi\right) = \frac{1}{16\pi^2} \int\limits_M d^4x \sqrt{g} \mathrm{Tr}\left(1\right),$$

$$a_1 \left(P, \chi\right) = 0,$$

$$a_2 \left(P, \chi\right) = \frac{1}{96\pi^2} \left(\int\limits_M d^4x \sqrt{g} \mathrm{Tr}\left(6E + R\right) + \int\limits_{\partial M} d^3x \sqrt{h} \mathrm{Tr}\left(2K + 12S\right) \right),$$

$$a_3 \left(P, \chi\right) = \frac{1}{384(4\pi)^{\frac{3}{2}}} \int\limits_{\partial M} d^3x \sqrt{h} \mathrm{Tr}\left(96\chi E + 3K^2 + 6K_{ab}K^{ab} + 96SK + 192S^2 - 12\nabla'_a \chi \nabla'^a \chi\right),$$

As a warm up, these results could be applied to the simple case of an ordinary Dirac operator

$$D = \gamma^\mu \left(\partial_\mu + \omega_\mu\right).$$

Therefore, in the above formulas we have

$$\omega'_\mu = \omega_\mu, \quad E = -\frac{1}{4}R, \quad \Phi = 0,$$

$$S = -\frac{1}{2}K\Pi_+, \quad \nabla'_a \chi = K_{ab}\chi\gamma^n\gamma^b$$

Substituting $\mathrm{Tr}(1) = 4$ and $\mathrm{Tr}(S) = -K$ we have for the first few terms

$$a_0(P,\chi) = \frac{1}{4\pi^2}\int_M d^4x\sqrt{g}$$

$$a_2(P,\chi) = -\frac{1}{24\pi^2}\left(\int_M d^4x\frac{1}{2}\sqrt{g}R + \int_{\partial M} d^3x\sqrt{h}K\right)$$

$$a_3(P,\chi) = \frac{1}{32(4\pi)^{\frac{3}{2}}}\int_{\partial M} d^3x\sqrt{h}\left(K^2 - 2K_{ab}K^{ab}\right)$$

The important point in the above result is the emergence of the combination

$$-\int_M d^4x\sqrt{g}R - 2\int_{\partial M} d^3x\sqrt{h}K$$

as the lowest term of the gravitational action which is known to be the required correction to the Einstein action involving the surface term so as to make the Hamiltonian formalism consistent. This is remarkable because both the sign and the coefficient are correct. The only assumption made is that normal boundary conditions are taken such that they enforce the hermiticity of the Dirac operator. This is yet another miracle concerning correct signs obtained in the spectral action of the Dirac operator. We also notice that the relative coefficient between R and K depends, in general, on the nature of the Laplacian. The desired answer is true for the square of the Dirac operator, but *not* for a general Laplacian. We note that there other boundary conditons may lead to different results.

5.1 Dilaton as the Dynamical Scale

Replacing the cutoff scale Λ in the spectral action, replacing $f(\frac{D^2}{\Lambda^2})$ by $f(P)$ where $P = e^{-\phi}D^2e^{-\phi}$ modifies the spectral action with dilaton dependence to the form

$$\mathrm{Tr}\ F(P) \simeq \sum_{n=0}^{6} f_{4-n}\int d^4x\sqrt{g}e^{(4-n)\phi}a_n\left(x, D^2\right)$$

One can then show that the dilaton dependence almost disappears from the action if one rescales the fields according to

$$G_{\mu\nu} = e^{2\phi} g_{\mu\nu}$$
$$H' = e^{-\phi} H$$
$$\psi' = e^{-\frac{3}{2}\phi}\psi$$

With this rescaling one finds the result that the spectral action is

$$I\left(g_{\mu\nu} \to G_{\mu\nu}, H \to H', \psi \to \psi'\right) + \frac{24 f_2}{\pi 2}\int d^4x\sqrt{G}G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

scale invariant (independent of the dilaton field) except for the kinetic energy of the dilaton field ϕ. The dilaton field has no potential at the classical level. It acquires a Coleman-Weinberg potential through quantum corrections, and thus a vev. The dilaton acquires a very small mass. The Higgs sector in this case becomes identical with the Randall-Sundrum model. In that model there are two branes in a five dimensional space, one located at $x_5 = 0$ representing the invisible sector, and another located at $x_5 = \pi r_c$, the visible sector. The physical masses are set by the symmetry breaking scale $v = v_0 e^{-kr_c\pi}$ so that $m = m_0 e^{-kr_c\pi}$. If the bare symmetry breaking scale is taken at $m_0 \sim 10^{19}$ Gev, then by taking $kr_c\pi = 10$ one gets the low-energy mass scale $m \sim 10^2$ Gev. It is not surprising that the Randall-Sundrum scenario is naturally incorporated in the noncommutative geometric model, because intuitively one can think of the discrete space as providing the different brane sectors.

6 SA for Superstring

We start with the $N = 1$ non-linear sigma model with background fields $G_{\mu\nu}[\Phi]$ and $B_{\mu\nu}[\Phi]$ which are symmetric and antisymmetric in $\mu\nu$ respectively. Here $\Phi(\xi, \theta_+, \theta_-)$ is a superfield with ξ the coordinates on the two-dimensional world sheet. The two-dimensional action is

$$I = \frac{T}{2}\int d^2\xi d\theta_+ d\theta_- \left(G_{\mu\nu}[\Phi] + B_{\mu\nu}[\Phi]\right)\left(D_-\Phi^\mu D_+\Phi^\nu\right)$$

whrere T is the string tension and the component form of the superfield Φ^μ is

$$\Phi^\mu = X^\mu + i\theta_+\psi^{\mu+} - i\theta_-\psi^{\mu-} + i\theta_+\theta_- F^\mu$$

The supercharges $j_\pm$ are then

$$\frac{i}{T}(\pm j_\pm) = \pm i\psi^{\mu\pm}G_{\mu\nu}\partial_\pm X^\nu - \frac{1}{6}\psi^{\mu\pm}\psi^{\nu\pm}\psi^{\rho\pm}H_{\mu\nu\rho}.$$

The conserved supersymmetry charges are

$$Q = \int_0^{2\pi} d\sigma\, j(\sigma)$$

$$\overline{Q} = \int_0^{2\pi} d\sigma\, \overline{j}(\sigma)$$

It is a tedious exercise to show that after quantization, the charges Q and $\overline{Q}$ form a supersymmetry algebra with the properties

$$Q^2 = \frac{1}{2}P = \overline{Q}^2$$

$$\{Q, \overline{Q}\} = \frac{1}{2}H$$

where P is the momentum generating reparametrizations on the circle and H is the Hamiltonian.

The spectral action is identified with the partition function

$$I = \int \frac{d\tau d\overline{\tau}}{\tau_2^2}\mathrm{Tr}\left|\sum_{NS\oplus R}\left(e^{2\pi i(\tau Q^2}(-1)^\epsilon(1-\Gamma)\right)\right|^2$$

where $\epsilon = 0$ for the NS sector, and $\epsilon = 1$ for the Ramond sector over the states in the trace. This action, has space-time supersymmetry as can be verified by counting the number of fermionic and bosonic states (massive as well as massless) and showing they are the same. The parameter $\tau = \tau_1 + i\tau_2$ is the modular parameter (although τ was used up to now as the two-dimensional time). The total partition function, including the ghosts is

$$\int \frac{d\tau d\overline{\tau}}{\tau_2^2}\int \frac{d^8p}{(2\pi)^8}e^{-2\pi p^2\tau_2}\left|\frac{1}{2\eta(\tau)^4}\left(\theta_3^4(0|\tau) - \theta_4^4(0|\tau)^4 - \theta_2^4(0|\tau)^4 - \theta_1^4(0|\tau)\right)\right|^2$$

and as expected, because of supersymmetry, the partition function vanishes. The ghost contributions cancel the contributions of two bosonic and two fermionic coordinates. Since the superghost part is independent of the background, these contributions would be the same even in a curved background. The difficulty is to compute the spectral action in an arbitrary background including the dilaton, and the space-time supersymmetric vacuum so that a space-time gravitino background, as well as a two and three forms would be included. This is not an easy problem to solve since this will make space-time supersymmetry explicit without invoking the Green-Schwarz superstring and κ symmetry . For here we shall limit our considerations to the background we started with (which is not the most general, and can be perturbed to more general backgrounds by transformations which are automorphisms of the algebra $\Omega(M)$). To compute the spectral action in an arbitrary background is a very complicated. We shall only determine the lowest order terms in a perturbative expansion. One starts by splitting the dependence of the fields in the partition function in terms of zero modes and oscillators. In the NS-sector there are no fermionic zero modes and the coordinates $X^\mu(\sigma)$ have a constant part X_0^μ . The Hamiltonian of the zero modes is

$$H_{\rm NS}^0 = -[\nabla_0^a \nabla_{0a} + \omega_{0b}{}^{ab} \nabla_{0a}]$$

In the R-sector, there are fermionic zero modes χ_0^a and the zero modes Hamiltonian is

$$\begin{aligned} H_{\rm R}^0 = [& -\nabla_0^a \nabla_{0a} + \omega_{0b}{}^{ab} \nabla_{0a} + 4 \chi_0^a \bar\chi_0^b \chi_0^c \bar\chi_0^d R_{abcd}^0 \\ & + \frac{2}{3} (\chi_0^a \bar\chi_0^b \bar\chi_0^c \bar\chi_0^d + \bar\chi_0^a \chi_0^b \chi_0^c \chi_0^d + \bar\chi_0^a \chi_0^b \chi_0^c \bar\chi_0^d) \nabla_{0a} H_{0bcd} \\ & + 2 (\chi_0^b \chi_0^c + \bar\chi_0^b \bar\chi_0^c) H_{0bca} \nabla_{0a} \\ & + \frac{1}{3} (\chi_0^a \chi_0^b \bar\chi_0^c \bar\chi_0^d + \chi_0^a \bar\chi_0^b \bar\chi_0^c \chi_0^d + \chi_0^a \bar\chi_0^b \bar\chi_0^c \bar\chi_0^d + \bar\chi_0^a \bar\chi_0^b \chi_0^c \chi_0^d) H_{0ab}{}^{e} H_{0ecd}] \end{aligned}$$

With these operators it is possible to use the heat kernel expansion to evaluate the trace of the exponential in the form

$$\mathrm{Tr}(e^{-\tau_2 \mathcal{P}}) = \sum_{n=0}^{\infty} a_n(\mathcal{P})\, \tau_2^{\frac{n-D}{2}}$$

where $a_n(\mathcal{P})$ are the Seeley-de Wit coefficients corresponding to the operator $\mathcal{P}$ and $D = 10$ is the dimension of the target manifold. Using the results of heat kernel expansion for a general second order operator, one finds the following results

$$\mathrm{Tr}(e^{-\tau_2 H^0_{\mathrm{NS}}}) = \frac{a_0(H^0_{\mathrm{NS}})}{\tau_2^5} + \frac{a_2(H^0_{\mathrm{NS}})}{\tau_2^4} + \cdots$$

where

$$a_0(H^0_{\mathrm{NS}}) = \frac{1}{(2\pi)^5}\int d^{10}X_0\sqrt{G[X_0]}$$

and the center of mass coordinates X_0^μ become coordinates on the manifold. The next term in the expansion is

$$a_2(H^0_{\mathrm{NS}}) = \frac{1}{(2\pi)^5}\int d^{10}X_0\sqrt{G[X_0]}(\frac{1}{6}R[X_0])$$

Similarly, for the R-sector, we have $a_0(H^0_{\mathrm{R}}) = a_0(H^0_{\mathrm{NS}})$ while for the next term a_2 we have

$$a_2(H^0_{\mathrm{R}}) = \frac{1}{(2\pi)^5}\int d^{10}X_0\sqrt{G}(-\frac{1}{12}R[X_0] - \frac{1}{24}H_{0\mu\nu\rho}H_0^{\mu\nu\rho})$$

Higher orders in the expansion would involve higher curvature terms, and will receive contributions from the oscillator parts. This can be done in a perturbative expansion using normal coordinates. To lowest orders, and for the a_0 terms, this is given by an expansion of the terms appearing. This implies that the coefficient of the a_0 term vanishes which is the cosmological constant. For the a_2 terms we have to expand , to lowest order in τ, $(\theta_3^4 - \theta_4^4)$ multiplying the NS-sector and $-(\theta_2^4 + \theta_1^4)$ multiplying the R-sector. The net contributions to lowest order is proportional to

$$\int d^{10}X_0\sqrt{G[X_0]}(\frac{1}{4}R[X_0] + \frac{1}{24}H_{0\mu\nu\rho}H_0^{\mu\nu\rho})$$

Comparing this with the superstring effective action at low energies we find that they are identical to this order. This is extremely encouraging, as we had no free parameters to adjust. The challenging problem that remains is to find a closed expression for the spectral action as a function of the background geometry in analogy with the calculation of the elliptic genus in where modular invariance plays an important role. Of course the effective superstring action has more terms depending on the dilaton, three-form, vector and gravitinos. It is possible to include the dilaton by adding to the non-linear sigma model the Weyl breaking term.

7 Uncanny Precision of Spectral Action

We investigate the accuracy of the approximation of the spectral action by the first terms of its asymptotic expansion. We consider the concrete example given by the four-dimensional geometry

$$S_a^3 \times S_\beta^1$$

where S_a^3 is the round sphere of radius a as a model of space, while S_β^1 is a circle of radius β viewed as a model of imaginary periodic time at inverse temperature β. We compute directly the spectral action and compare it with the sum of the first terms of the asymptotic expansion. In section two we start with the round sphere S_a^3 and use the known spectrum of the Dirac operator together with the Poisson summation formula, to estimate the remainder when using a smooth test function. Thus for instance an inner diameter of 10 in cutoff units yields the accuracy of the first hundred decimal places, while an inner diameter of 10^{31} corresponding to the visible universe at inverse temperature of 3 Kelvin and a cutoff at Planck scale, yields an astronomical precision of 10^{62} accurate decimal places.

7.1 The product $S^3 \times S^1$

We now want to move to the 4-dimensional Euclidean case obtained by taking the product $M = S^3 \times S^1$ of S^3 by a small circle. We take the product geometry of a three dimensional geometry with Dirac operator D_3 by the one dimensional circle geometry with Dirac

$$D_1 = \frac{1}{\beta} i \nabla_\theta$$

so that the spectrum of D_1 is $\frac{1}{\beta}(\mathbb{Z} + \frac{1}{2})$.

In fact, in order to estimate the size of the remainder in the asymptotic expansion of the spectral action for the product $M = S^3 \times S^1$, we shall now use the two dimensional form of ,

$$\sum_{\mathbb{Z}^2} g(n + \frac{1}{2}, m + \frac{1}{2}) = \sum_{\mathbb{Z}^2} (-1)^{n+m} \hat{g}(n, m)$$

where the Fourier transform is given by

$$\hat{g}(x, y) = \int_{\mathbb{R}^2} g(u, v) e^{-2\pi i (xu + yv)} du dv$$

For the operator D , and taking for D_3 the Dirac operator of the 3-sphere S_a^3 of radius a, the eigenvalues of D^2/Λ^2 are obtained by collecting the following

$$(\frac{1}{2}+n)^2\,(\Lambda a)^{-2}+(\frac{1}{2}+m)^2\,(\Lambda\beta)^{-2}\;,\quad n,m\in\mathbb{Z}$$

with the multiplicity $2n(n+1)$ for each $n, m \in \mathbb{Z}$. Thus, more precisely

$$\mathrm{Tr}(h(D^2/\Lambda^2))=\sum_{\mathbb{Z}^2}2n(n+1)h((\frac{1}{2}+n)^2\,(\Lambda a)^{-2}+(\frac{1}{2}+m)^2\,(\Lambda\beta)^{-2})$$

which is of the form:

$$\mathrm{Tr}(h(D^2/\Lambda^2))=\sum_{\mathbb{Z}^2}g(n+\frac{1}{2},m+\frac{1}{2})$$

where

$$g(u,v)=2(u^2-\frac{1}{4})h(u^2\,(\Lambda a)^{-2}+v^2\,(\Lambda\beta)^{-2})$$

Thus we get:

$$\hat{g}(0,0)=2\pi\,(\Lambda\beta)\,(\Lambda a)^3\int_0^\infty h(\rho^2)\rho^3d\rho-\pi\,(\Lambda\beta)\,(\Lambda a)\int_0^\infty h(\rho^2)\rho d\rho$$

We estimate of the remainder, given by the sum

$$\sum_{(n,m)\neq(0,0)}(-1)^{n+m}\hat{g}(n,m)$$

and show that the accuracy of the asymptotic expansion is at least of the order of $e^{-\frac{\pi}{2}(\mu\Lambda)^2}$. Indeed the term $\Lambda^4\beta a^3$ is the dominant volume term in the spectral action and the other terms in the formula for C are of order one. Thus for instance for a size $\mu\Lambda\sim 100$ one gets that the asymptotic expansion accurately delivers the first 6820 decimal places of the spectral action.

In our simplified physical model we test the approximation of the spectral action by its asymptotic expansion for the Euclidean model

$$E(t)=S^3_{a(t)}\times S^1_{\beta(t)}$$

where space at a given time t is given by a sphere with radius $a(t)$ and $\beta(t)$ is a uniform value of inverse temperature. One can then easily see that the above approximation to the spectral action is fantastically accurate, going backwards in time all the way up to one order lower than the Planck energy. In doing so the radius $a(t)$ varies between at least $\sim 10^{61}$ Planck units and 10 Planck units (i.e. 10^{-34} m), while the temperature varies between $2.7^{\circ}K$ and $(10^{31})^{\circ}K$. It is for an inner size less than 10 in Planck units that the approximation does break down.

This implies that all the Seeley coefficients a_{2n} vanish for $n \geq 2$, and we shall check this directly for a_4 and a_6.

This vanishing of the Seeley coefficients does not hold for the 4 sphere and it is worth understanding the difficulties in using the Poisson summation in the same way for the 4 sphere. The problem when one tries to use the Poisson formula as above is that, *e.g.* for the heat kernel, one is dealing with a function like $|x|e^{-tx^2}$ which is not smooth and whose Fourier transform does not have rapid decay at ∞.

8 Dirac operator for Robertson–Walker metrics

The starting point in the evaluation of a spectral action is the Dirac operator of the relevant geometry. The (Euclidean) Robertson-Walker metric of dimension 4 with the symmetry of the round sphere S^3 is read from the line element

$$ds^2 = \ dt^2 + a^2\,(t)\left(d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right).$$

the Dirac operator

$$\begin{aligned} D &= \gamma^a e^\mu_a \frac{\partial}{\partial x^\mu} + \frac{1}{4}\gamma^c\omega_{cab}\gamma^{ab} \\ &= \gamma^0\left(\frac{\partial}{\partial t} + \frac{3a'}{2a}\right) + \frac{1}{a}D_3 \end{aligned}$$

where the gamma matrices γ^a are antihermitian satisfying $(\gamma^a)^2 = -1$, and

$$D_3 = \gamma^1\left(\frac{\partial}{\partial\chi} + \cot\chi\right) + \gamma^2\frac{1}{\sin\chi}\left(\frac{\partial}{\partial\theta} + \frac{1}{2}\cot\theta\right) + \gamma^3\frac{1}{\sin\chi\sin\theta}\frac{\partial}{\partial\varphi}.$$

D_3 is directly related to the Dirac operator on S^3 which is given by

$$\mathrm{Dirac}_{S^3} = i\sigma^1\left(\frac{\partial}{\partial\chi} + \cot\chi\right) + i\sigma^2\frac{1}{\sin\chi}\left(\frac{\partial}{\partial\theta} + \frac{1}{2}\cot\theta\right) + i\sigma^3\frac{1}{\sin\chi\sin\theta}\frac{\partial}{\partial\varphi}.$$

In fact more precisely one has

$$\gamma^0 D_3 \sim \mathrm{Dirac}_{S^3} \oplus -\mathrm{Dirac}_{S^3}$$

Thus the square of the Dirac operator is

$$D^2 = -\left(\frac{\partial}{\partial t} + \frac{3a'}{2a}\right)^2 + \frac{1}{a^2}(\gamma^0 D_3)^2 - \frac{a'}{a^2}\gamma^0 D_3$$

One can view the vectors in the Hilbert space of spinors as functions $\xi(t)$ of t with values in

$$L^2(S^3, \Sigma) \oplus L^2(S^3, \Sigma)$$

where Σ is the spinor bundle on S^3 and the square norm of ξ is given by

$$\int ||\xi(t||^2 a(t)^3 dt$$

corresponding to the volume form for the Robertson–Walker metric

$$\sqrt{g} = a^3 \sin^2\chi \sin\theta$$

The spectrum of the Dirac operator for the round sphere S^d of unit radius is given by

$$\mathrm{Spec}(D) = \{\pm(\frac{d}{2} + k) \mid k \in \mathbb{Z}, k \geq 0\}$$

where the multiplicity of $(\frac{d}{2} + k)$ is equal to $2^{[\frac{d}{2}]}\binom{k+d-1}{k}$. For $d = 3$ we get the multiplicity $(k+1)(k+2)$ for the eigenvalues $(\frac{3}{2} + k)$, which means the multiplicity $x^2 - \frac{1}{4}$ for all odd half-integers. Thus decomposing the vector valued function $\xi(t)$ in terms of the eigenfunctions of the operators $\pm\mathrm{Dirac}_{S^3}$ for the eigenvalue λ, one reduces the problem to the direct sum of the one dimensional problems corresponding to the scalar operators in one dimension given by

$$-\left(\frac{\partial}{\partial t} + \frac{3a'}{2a}\right)^2 + \frac{1}{a^2}\lambda^2 - \frac{a'}{a^2}\lambda, \quad \lambda = \pm(\frac{3}{2} + n)$$

One can move back to the standard measure dt in the variable t by the transformation

$$v\,(t) = a^{-\frac{3}{2}} u\,(t)$$

which gives the direct sum of the operators $H_n^{\pm}$ where

$$H_n^{\pm} = -\left(\frac{d^2}{dt^2} - \frac{\left(n + \frac{3}{2}\right)^2}{a^2} \pm \frac{\left(n + \frac{3}{2}\right) a'}{a^2}\right)$$

which occurs with multiplicity $2(n+1)(n+2)$

9 Feynman Kac formula up to a_6

We resolve the above puzzle by showing that the operator used above, namely the direct sum of the operators $H_n = H_n^+$ with multiplicity $\mu(n)$, admits the same even terms in its spectral expansion as the Dirac operator and the natural symmetry of the latter entails the vanishing of the local formulas for the odd terms while justifying the computation of the even ones. We also show that this natural symmetry allows one to use the Poisson summation formula instead of the Euler–Maclaurin formula, and this simplification allows us to compute the local formula up to a_{10} while giving an algorithm to compute terms of arbitrary order.

We have seen that the square of the Dirac operator is

$$D^2 = \bigoplus \frac{1}{2}\mu(n)\left(H_n^+ \oplus H_n^-\right)$$

Now observe that the operator H_n^- can be viewed as the "time reversal" of $H_n = H_n^+$

$$H_n^T = -\frac{d^2}{dt^2} + V_n^T(t)$$
$$V_n^T(t) = \frac{(n+\frac{3}{2})}{a^2}\left(\left(n+\frac{3}{2}\right)+a'\right)$$

and this implies that the spectral asymptotics for the direct sum of the H_n^- with multiplicity $\mu(n)$ are obtained from the above ones simply by replacing the derivatives $a^{(k)}$ by $(-1)^k a^{(k)}$. Thus no new computation is needed for the first terms and one checks that, for the Dirac operator, the non vanishing odd terms such as a_1, a_3 etc. cancel out, while the even terms remain unchanged. We shall in fact go further and show how to use this added symmetry to simplify the above computations trading the Euler–Maclaurin formula for the Poisson summation formula. We consider the function

$$f_s(x) = (x^2 - \frac{1}{4})e^{u(b-x)x}$$

and the relevant sum is now

$$\sum_{-\infty}^{\infty} f_s(n+\frac{1}{2})$$

Using the Poisson summation formula it is very well approximated by

$$\int_{-\infty}^{\infty} f_s(x+\frac{1}{2})dx = \frac{e^{\frac{b^2 u}{4}}\sqrt{\pi}\,(2+(-1+b^2)\,u)}{4u^{3/2}}$$

There is an overall factor of 2 coming from spinors and one multiplies by $\frac{1}{2\sqrt{\pi s}}$ and then integrates in $D[\alpha]$ as above with

$$u = s\int_0^1 a^{-2}(t+\sqrt{2s}\,\alpha(v))\,dv$$

and

$$ub = s\int_0^1 a'^{-2}(t+\sqrt{2s}\,\alpha(v))\,dv$$

so that

$$b = \int_0^1 a'^{-2}(t+\sqrt{2s}\,\alpha(v))\,dv/\left(\int_0^1 a^{-2}(t+\sqrt{2s}\,\alpha(v))\,dv\right)$$

Thus the relevant expression is

$$\frac{1}{4}\int \frac{e^{\frac{b^2 u}{4}}\,(2+(-1+b^2)\,u)}{\sqrt{s}\,u^{3/2}} D[\alpha]$$

In this section we use the Feynman Kac formula to compute the spectral action. We use the following formula for the local expression of the trace of e^{-sH_n},

$$e^{-2s(-\frac{1}{2}\partial_t^2+V)}(t,t) = \frac{1}{2\sqrt{\pi s}}\int \exp\left(-2s\int_0^1 V(t+\sqrt{2s}\,\alpha(u))\,du\right)D[\alpha]$$

where α is the Brownian bridge which is a Gaussian random variable with covariance given by

$$E(\alpha(u)\alpha(v)) = u(1-v)$$

for $u \le v$. In our case the potential V is given by $-\frac{1}{2}\partial_t^2 + V = \frac{1}{2}H_n$

$$V(t) = \frac{1}{2}V_n(t)$$

and the above formula becomes

$$e^{-sH_n}(t,t) = \frac{1}{2\sqrt{\pi s}}\int \exp\left(-s\int_0^1 V_n(t+\sqrt{2s}\,\alpha(v))\,dv\right)D[\alpha]$$

Without including bothe sectors of the Dirac operator, we would have to evaluate the sum with multiplicity included,

$$\sum_{n=0}^{\infty} \mu(n) e^{-sH_n}(t,t)$$

by using the Euler-Maclaurin formula, which means replacing the discrete index n by the continuous variable $x = n + \frac{3}{2}$ and the sum over n by

$$\int_{\frac{3}{2}}^{\infty} k_s(x)dx + \frac{1}{2}k_s\left(\frac{3}{2}\right) - \frac{k_s'\left(\frac{3}{2}\right)}{12} + \frac{k_s''\left(\frac{3}{2}\right)}{720} - \frac{k_s^{(4)}\left(\frac{3}{2}\right)}{30240} + \dots$$

where the function k_s is given by

$$k_s(x) = (4x^2 - 1)\frac{1}{2\sqrt{\pi s}} \int e^{u(b-x)x} D[\alpha]$$

where

$$u = s \int_0^1 a^{-2}(t + \sqrt{2s}\,\alpha(v))\, dv$$

and

$$ub = s \int_0^1 a'^{-2}(t + \sqrt{2s}\,\alpha(v))\, dv$$

so that

$$b = \int_0^1 a'^{-2}(t + \sqrt{2s}\,\alpha(v))\, dv / \left(\int_0^1 a^{-2}(t + \sqrt{2s}\,\alpha(v))\, dv \right)$$

To obtain these terms we have used $x = n + \frac{3}{2}$ and

$$-s \int_0^1 V_n(t + \sqrt{2s}\,\alpha(v))\, dv = u(b - x)x$$

We use the Taylor expansion of a^{-2} and of a'^{-2} for the expressions of u and b to get an asymptotic expansion when $s \to 0$. The terms of the expansion are coming from the general formula

$$\int_0^1 F(t + \sqrt{2s}\,\alpha(v))\, dv = F(t) + \sum \frac{F^{(k)}(t)}{k!}(\sqrt{2s})^k x_k(\alpha)$$

where

$$x_k(\alpha) = \int_0^1 \alpha(v)^k \, dv$$

which gives the wrong answer where the odd terms in the expansion do not vanish, and the even terms differ by a factor of two. When summing both sectors, we can use instead the Poisson summation formula. Consider the function

$$f_s(x) = (x^2 - \frac{1}{4})e^{u(b-x)x}$$

and the relevant sum is now

$$\sum_{-\infty}^{\infty} f_s(n + \frac{1}{2})$$

Using the Poisson summation formula it is very well approximated by

$$\int_{-\infty}^{\infty} f_s(x + \frac{1}{2})dx = \frac{e^{\frac{b^2 u}{4}} \sqrt{\pi} \, (2 + (-1 + b^2) \, u)}{4 u^{3/2}}$$

There is an overall factor of 2 coming from spinors and one multiplies by $\frac{1}{2\sqrt{\pi s}}$ and then integrates in $D[\alpha]$ as above with

$$u = s \int_0^1 a^{-2}(t + \sqrt{2s}\, \alpha(v)) \, dv$$

and

$$ub = s \int_0^1 a'^{-2}(t + \sqrt{2s}\, \alpha(v)) \, dv$$

so that

$$b = \int_0^1 a'^{-2}(t + \sqrt{2s}\, \alpha(v)) \, dv / \left(\int_0^1 a^{-2}(t + \sqrt{2s}\, \alpha(v)) \, dv \right)$$

Thus the relevant expression is

$$\frac{1}{4} \int \frac{e^{\frac{b^2 u}{4}} \, (2 + (-1 + b^2) \, u)}{\sqrt{s} \, u^{3/2}} D[\alpha]$$

One then repeats the same computation as above and finds that all the terms a_j for j odd vanish, while they are unchanged for even j. For instance when we compute a_4 we find the following coefficients as a polynomial expression in the $x(j)$ This agrees with the spectral action which gives for the $\frac{1}{s}$ term

$$a_2 = \frac{1}{4\pi^2}\int \sqrt{g} d^4x$$

while

$$R = 6\left(\frac{a''}{a} + \frac{a'^2}{a^2} - \frac{1}{a^2}\right)$$

which is negative for the sphere and gives (using $|S_a^3| = 2\pi^2 a^3$)

$$a_2 = \frac{1}{4}\int dt a^3 \left(\frac{a''}{a} + \frac{a'^2}{a^2} - \frac{1}{a^2}\right)$$

We now look at the a_3 term, including the corrections it gives

$$a_3 = \frac{1}{6}\left(a'(t)\left(-3 + 4a(t)a''(t)\right) + 2a(t)^2 a^{(3)}(t)\right)$$

and one finds that it is the total derivative of the following expression which vanishes at both ends of the time interval

$$-\frac{a(t)}{2} + \frac{1}{3}a(t)^2 a''(t)$$

The next term is the a_4 term, one performs the computation in the same way as above and obtains

$$a_4 = \frac{1}{120}\left(-\left(5 + 4a'^2\right)a''(t) + 3a(t)a''^2 + 3a(t)\left(3a'^{(3)}(t) + a(t)a^{(4)}(t)\right)\right)$$

This term agrees with what one gets from the Gilkey formula and is very interesting because it does not vanish for the sphere case where it gives

$$a_4(\text{sphere}) = \frac{11 \sin^3 t}{120}, \quad \int_0^\pi a_4(\text{sphere})dt = \frac{4}{3} \times \frac{11}{120}$$

but it is a "topological" term since it is the derivative of the following expression

$$-\frac{1}{24}a'(t) - \frac{7a'^3}{360} + \frac{1}{40}a(t)a'(t)a''(t) + \frac{1}{40}a(t)^2 a^{(3)}(t)$$

In this expression the last two terms vanish at the end points of the time interval, but not the first two. In fact the variation of the first two across the interval will not be zero in general except when $a'(t)$ vanishes at the boundary which is the case of $S^1 \times S^3$. In the sphere case the derivative at the end points is ± 1 and this corresponds to smoothing out the conical singularity at the boundary.

The computation of a_6 is more complicated. To do the computation one needs to compute the integrals of polynomials in the $x_j(\alpha)$ under the Gaussian measure $D[\alpha]$ in order to obtain the coefficients. We list in the appendix the table of the integrals which are needed to compute up to a_{10}. It gives

$$a_6 = -\frac{a'^2 a''(t)}{240 a(t)^2} - \frac{a'^4 a''(t)}{84 a(t)^2} + \frac{a''^2}{120 a(t)} + \frac{a'^2 a''^2}{21 a(t)} - \frac{1}{90} a''^3 + \frac{a'^{(3)}(t)}{240 a(t)}$$

$$+\frac{a'^3 a^{(3)}(t)}{84 a(t)} - \frac{1}{20} a'(t) a''^{(3)}(t) - \frac{a(t) a^{(3)}(t)^2}{1680} - \frac{1}{240} a^{(4)}(t) - \frac{1}{120} a'^2 a^{(4)}(t)$$

$$+\frac{a(t) a''^{(4)}(t)}{840} + \frac{1}{140} a(t) a'^{(5)}(t) + \frac{a(t)^2 a^{(6)}(t)}{560}$$

One checks that it agrees with the computation using the Gilkey universal formula. This procedure gives as well expressions for a_8, a_{10}.

10 Other works

- Spectral action for Moyal planes (Gayral, Iochum)
- Spectral action for loop quantum gravity (Aastrup, Grimstrup and Nest).
- Renomralization for Yang-Mills spectral action (Suijlekom)
- SA for manifolds with torsion (Pfaeffle, Stephan)
- Noncommutative Cosmology for SA (Sakellariadou, Nelson and Marcolli, Pierpaoli)
- SA and induced Gravity (Barrett)
- SA and anomalies (Lizzi, Andrianov)

11 Conclusions

The spectral action principle is a very simple idea that has produced incredible results. It satsfies the Moto

Minimal Input, Maximal Outcome