The Translation anomaly of Vacuum Einstein gravity

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Quantum Theory and Gravitation
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Context

$4d$ **Vacuum Classical General Relativity**

$$R_{\mu\nu} = 0.$$

Nor Loops nor Strings.

Main Claim

"Asymptotically flat spacetimes at spatial infinity

can be defined **without parity conditions**

when the Einstein-Hilbert action with Gibbons-Hawking term

is supplemented with an anomalous action

$$S = \int_{\mathcal{M}} \sqrt{-g} R + \int_{\rho=\Lambda} \sqrt{-\gamma}(K - \hat{K}) + \log \Lambda \ \mathcal{A}$$

which breaks asymptotic translation invariance."

Outline

- Motivation
- Framework : spatial infinity as an hyperboloid
- Origin of parity conditions
- Proposal to relax parity conditions
- Consequences of the construction
  - There is a translation anomaly
  - Supertranslations conserved charges distinguish phase spaces
  - Lorentz charges are non-linear functionals
- String theory analogue

Motivation

Why looking at vacuum spacetimes near spatial infinity ?

- We live in an asymptotically flat spacetime below the Λ scale
- Holography : can one use lessons from the

  anti-de Sitter spacetime / CFT correspondence
  [Maldacena, Witten, Gubser, Klebanov, Polyakov, . . .]
  [talk by Matthias Blau]

  in string theory to asymptotically flat spacetimes ?
- What is the most general definition of asymptotically flat spacetimes ? Unify Hamiltonian & Lagrangian approaches ? [Arnowitt, Deser, Misner, Geroch, Regge, Teitelboim, Ashtekar, . . .]
- Hidden symmetries / Electric-Magnetic duality [Cremmer, Julia, Pope, Lu, Henneaux, Teitelboim, . . .]

Hyperboloid representation of spatial infinity

A. Ashtekar & R. O. Hansen [J. Math. Phys. 19, 1542 (1978)]
R. Beig & B. Schmidt [Commun. Math. Phys. 87 (1982) 65]

Start with Minkowski spacetime in (t, r, θ, ϕ) coordinates. Perform the change of coordinates

$$t = \rho \sinh \tau, \qquad r = \rho \cosh \tau \,.$$

Minkowski spacetime takes the form

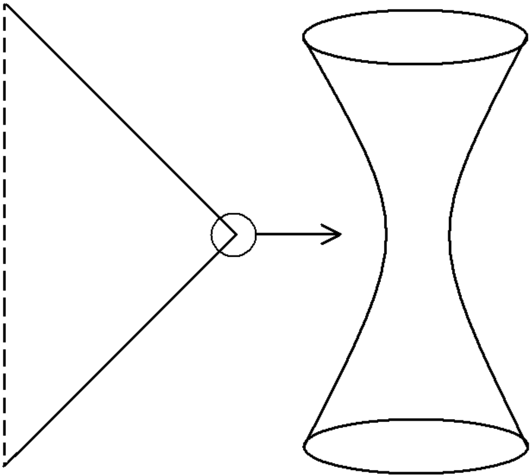
$$ds^2 = d\rho^2 + \rho^2 ds_H^2$$

where

$$ds_H^2 = h_{ab}^{(0)} dx^a dx^b = -d\tau^2 + \cosh^2 \tau (d\theta^2 + \sin^2 \theta d\phi^2) \,,$$

is the unit hyperboloid.

Hyperboloid representation of spatial infinity



Hyperboloid representation of spatial infinity

Asymptotically flat spacetimes are **defined** as spacetimes which admit a coordinate system $(\rho, \tau, \theta, \phi)$ such that the four-metric takes the form

$$\begin{aligned} ds^2 &= \Big(1 + \frac{2\sigma}{\rho} + o(\rho^{-1})\Big) d\rho^2 + O(\rho^0)\, d\rho\, dx^a \\ &\quad + \Big(\rho^2 h^{(0)}_{ab} + \rho(k_{ab} - 2\sigma h^{(0)}_{ab}) + o(\rho^1)\Big) dx^a dx^b\,, \end{aligned}$$

In Hamiltonian formalism, it gives the standard ADM boundary conditions

$$\begin{aligned} {}^3g_{ij} &= \delta_{ij} + \frac{1}{r}\,{}^3g^{(1)}_{ij} + o(r^{-1}), \\ \pi^{ij} &= \frac{1}{r^2}\pi^{ij}_{(2)} + o(r^{-2}) \end{aligned}$$

where ${}^3g^{(1)}_{ij}$, $\pi^{ij}_{(2)}$ are limited to be functions of σ and k_{ab}.

Unicity of the frame ?

"Given a spacetime metric $g_{\mu\nu}$, can one define uniquely the asymptotic expansion ? In particular σ and k_{ab} ?"

No !

Asymptotic Frame transformations

The class of asymptotically flat metrics is invariant under

- Four translations

$$\delta_{\xi[\zeta]}\sigma = 0, \qquad \delta_{\xi[\zeta]}k_{ab} = 0$$

- Six Lorentz transformations

$$\delta_{\xi(0)}\sigma = \mathcal{L}_{\xi(0)}\sigma, \qquad \delta_{\xi(0)}k_{ab} = \mathcal{L}_{\xi(0)}k_{ab}$$

- One arbitrary angle-dependent and τ-dependent supertranslation

$$\delta_{\xi[\omega]}\sigma = 0, \qquad \delta_{\xi[\omega]}k_{ab} = 2(\mathcal{D}_a\mathcal{D}_b\omega + \omega h^{(0)}_{ab})$$

- Four Logarithmic translations

$$\delta_{\xi[H]}\sigma = H, \qquad \delta_{\xi[H]}k_{ab} = 0, \quad \text{such that} \quad \mathcal{D}_a\mathcal{D}_b H + H h^{(0)}_{ab} = 0$$

- Diffeomorphisms in the interior

$$\delta_{\xi}\sigma = 0, \qquad \delta_{\xi}k_{ab} = 0$$

Parity Conditions

Role of surface integrals in the Hamiltonian formulation of general relativity

T. Regge & C. Teitelboim, [Ann. Phys. 88 (1974) 286]

Finiteness of rotations and boosts generators requires

$$\gamma^{(1)}_{ij}(-\mathbf{n}) = \gamma^{(1)}_{ij}(\mathbf{n}), \qquad \pi^{(2)\,ij}(-\mathbf{n}) = -\pi^{(2)\,ij}(\mathbf{n}),$$

where $\mathbf{n} = r^{-1}(x^1, x^2, x^3)$.

The Covariant Phase Space Of Asymptotically Flat Gravitational Fields

A. Ashtekar, L. Bombelli & O. Reula, [Analysis, Geometry and Mechanics : 200 Years After Lagrange, 1991]

Finiteness of the covariant phase space symplectic structure requires

$$\sigma(\tau, \theta, \phi) \;=\; \sigma(-\tau, \pi - \theta, \phi + \pi)$$

and, if one keeps k_{ab},

$$k_{ab}(\tau, \theta, \phi) \;=\; \pm k_{ab}(-\tau, \pi - \theta, \phi + \pi)\,.$$

Origin of the parity conditions

- The 3+1 canonical form admits a logarithmic divergence

$$\int_\Sigma d^3x \left(\delta_1\gamma_{ij}\delta_2\pi^{ij} - \delta_2\gamma_{ij}\delta_1\pi^{ij}\right) \sim \log\Lambda$$

- The covariant bulk symplectic structure

$$\Omega^{bulk}[\delta_1 g, \delta_2 g] = \frac{1}{32\pi G}(d^3x)_\mu \, P^{\mu\nu\alpha\beta\gamma\delta}\Big(\delta_1 g_{\alpha\beta} D_\nu \delta_2 g_{\gamma\delta} - (1 \leftrightarrow 2)\Big)$$

where

$$\begin{aligned} P^{\mu\nu\alpha\beta\gamma\delta} \quad &= \quad g^{\mu\nu} g^{\gamma(\alpha} g^{\beta)\delta} + g^{\mu(\gamma} g^{\delta)\nu} g^{\alpha\beta} + g^{\mu(\alpha} g^{\beta)\nu} g^{\gamma\delta} \\ &\quad - g^{\mu\nu} g^{\alpha\beta} g^{\gamma\delta} - g^{\mu(\gamma} g^{\delta)(\alpha} g^{\beta)\nu} - g^{\mu(\alpha} g^{\beta)(\gamma} g^{\delta)\nu}. \end{aligned}$$

admits a logarithmic divergence : $W \sim \log\Lambda$.

The parity conditions are imposed so that the logarithmic divergences cancel.

The action principle

Holographic renormalization of asymptotically flat spacetimes

R. B. Mann and D.Marolf, [hep-th/0511096]

- The variation of the action

$$S = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}\,R + \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\,(K-\hat{K})$$

gives only non-vanishing terms at past and future boundaries once

$$k^a_a = 0.$$

If one neglect past/future boundaries, the variational principle holds independently of parity conditions.

Origin of parity conditions in the action principle

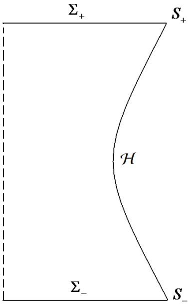


Figure: The variational principle is defined in the spacetime delimited by initial and final time slices $\Sigma_\pm$ and the hyperbolic cut-off $\mathcal{H}$.

$$\delta S = \mp \frac{\log \Lambda}{16\pi G} \int_{S_\pm} d^2 S n_a \left(4\sigma \mathcal{D}^a \delta\sigma + \frac{1}{2} k^{bc} \mathcal{D}^a \delta k_{bc} \right) + \delta\left(\dots\right) + O(\Lambda^0)$$

Regularization of the action / symplectic form

The action contains an additional boundary term which influences the dynamics.

- The regularized action is

$$S = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}\,R + \frac{1}{8\pi G}\int_{\rho=\Lambda} d^3x\sqrt{-h}\,(K-\hat{K}) + \frac{\log\Lambda}{4\pi G}\mathcal{A} + S_{(0)},$$

  where the spacetime radial cutoff is $\rho = \sqrt{r^2-t^2} = \Lambda$ and

$$\mathcal{A} = \frac{1}{8\pi G}\int_{H} d^3x\sqrt{-h^{(0)}}\left(\sigma(\Box+3)\sigma + \frac{1}{8}k^{ab}(\Box-3)k_{ab}\right)$$

- The main property of this action is that the symplectic structure and charges are finite without parity conditions.

Properties of the anomalous action

- The counterterm breaks (super/log)-translation invariance
- Lorentz invariance is preserved
- The anomalous action is identically zero when parity conditions hold.

  $\Rightarrow$ Our result is consistent with the analysis of Regge & Teitelboim, Ashtekar, Bombelli & Reula.
- The anomaly is invariant under translations, supertranslations and logarithmic translations

  $\Rightarrow$ The anomaly obeys the Wess-Zumino consistency conditions.

Properties of the phase space without parity conditions

- Translations are canonically associated with the ADM four-momenta

$$\mathcal{Q}_{(\mu)}[g;\bar{g}] \quad = \quad \mathcal{Q}^{ADM}_{(\mu)}[g;\bar{g}]$$

- For each Lorentz transformation, one can define the Lorentz charge

$$\mathcal{Q}_{\xi_{(0)}}[g;\bar{g}] \quad = \quad \frac{1}{8\pi G}\int_S d^2S\sqrt{-h_{(0)}}T^{(2)}_{ab}\xi^a_{(0)}n^b \neq \mathcal{Q}^{ADM}_{\xi_{(0)}}[g;\bar{g}].$$

where $T^{(2)}_{ab}$ contains linear terms **and quadratic** terms in σ and k_{ab}.

- One can **also** define two additional boundary Noether charges

$$T^{(\sigma)}_{ab} \quad \equiv \quad -\frac{2}{\sqrt{-h_{(0)}}}\frac{\delta L^{(\sigma)}}{\delta h^{(0)\,ab}}, \qquad T^{(k)}_{ab} \equiv -\frac{2}{\sqrt{-h_{(0)}}}\frac{\delta L^{(k)}}{\delta h^{(0)\,ab}}$$

which could shift Lorentz charges.

Properties of the phase space without parity conditions

- Supertranslations are associated with finite and conserved charges

$$Q_{(\omega)}[g;\bar{g}] = \frac{1}{4\pi G}\int_S d^2S\sqrt{-h_{(0)}}n_a\left(\sigma\mathcal{D}^a\omega - \mathcal{D}^a\sigma\omega\right).$$

  Here supertranslations **need to obey**

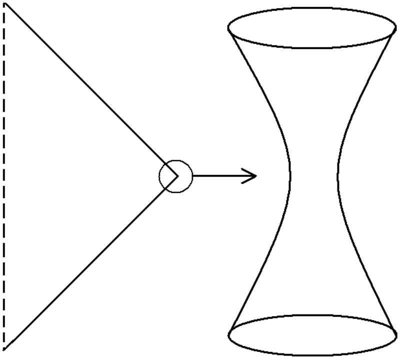
$$(\Box + 3)\omega = 0$$

  It solves a discrepancy between Hamiltonian and Lagrangian definitions of supertranslations.

  Supertranslations distinguish phase spaces.

- Logarithmic translations are degenerate direction of the symplectic structure - they are associated with zero charges.

What does the translation anomaly tell us ?



$$\mathcal{A} = \frac{1}{8\pi G}\int_H d^3x\sqrt{-h^{(0)}}\left(\sigma(\Box+3)\sigma + \frac{1}{8}k^{ab}(\Box-3)k_{ab}\right)$$

The Weyl Anomaly in Anti-de Sitter spacetimes

- Asymptotically anti-de Sitter spacetimes in $d = 5$ dimensions can be defined as

$$ds^2 = \frac{dr^2}{r^2} + \left(r^2 g^{(0)}_{ab} + g^{(2)}_{ab} + \frac{\log r}{r^2} i_{ab} + \frac{1}{r^2} g^{(4)}_{ab} + \dots \right) dx^a dx^b$$

- The action can be regularized as

$$S = \int_{\mathcal{M}} \sqrt{-g}(R + 2\Lambda) + \int_{\partial\mathcal{M}} \sqrt{-\gamma}\,(K + 1 + R[\gamma]) + \log\Lambda\ \mathcal{A}$$

where

$$\mathcal{A} = -\frac{l^3}{16\pi G_5}\sqrt{-g_{(0)}}\left(R^{(0)}_{ab} R^{ab}_{(0)} - \frac{1}{3} R^2_{(0)} \right)$$

The Weyl Anomaly in Anti-de Sitter spacetimes

The Holographic Weyl anomaly

M. Henningson & K. Skenderis, [hep-th/9806087]

$$\mathcal{A} = -\frac{l^3}{16\pi G_5}\sqrt{-g_{(0)}}\left(R^{(0)}_{ab}R^{ab}_{(0)} - \frac{1}{3}R^2_{(0)}\right)$$

- The anomaly **matches** with the anomaly of $\mathcal{N}=4$ super-Yang-Mills theory with $SU(N)$ gauge group, when $N \to \infty$,

$$\mathcal{A} = -\frac{N^2}{\pi^2}\left(E_{(4)} - \frac{1}{64}W^{(0)}_{abcd}W^{abcd}_{(0)}\right)$$

  upon using the AdS/CFT dictionary

$$G_5 = \frac{G_{10}}{l^5\pi^3}, \qquad l = (4\pi g_{str}N)^{1/4}, \qquad G_{10} = 8\pi^6 g_{str}^2$$

- The Weyl anomaly is a clue / check of the AdS/CFT correspondence in string theory.

Conclusion

- Asymptotically flat spacetimes at spatial infinity **without parity conditions** on the asymptotic fields define a consistent phase space when the Einstein-Hilbert action is supplemented with an anomalous action

$$S = \int_{\mathcal{M}} \sqrt{-g} R + \int_{\partial \mathcal{M}} \sqrt{-\gamma}(K - \hat{K}) + \log \Lambda \ \mathcal{A}$$

  which breaks global asymptotic translation invariance.
- Lorentz charges are non-linear functionals of the asymptotic fields. They admit a shift ambiguity by the Noether charges of the anomaly action.
- Supertranslations conserved charges distinguish phase spaces. Logarithmic translations are pure gauge.
- Interpretation of the anomaly unclear

Thank you !

For the future :

- Find an exact solution of vacuum Einstein's equations which violate parity conditions (an analytical solution might exist).
- Understand the translation anomaly in terms of an holographically dual QFT (hard)
- Understand supertranslations better in between Hamiltonian and covariant phase space formalisms
- Connect the definitions between spatial and null infinity (supertranslations are also present in the BMS group)