Symmetry constraints on perturbative $\mathcal{N} = 8$ supergravity

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Based on

arXiv:1009.1643 w/ Niklas Beisert, Dan Freedman, Michael Kiermaier, Alejandro Morales, Stephan Stieberger

arXiv:1007.4813 w/ Michael Kiermaier

arXiv:1003.5018 w/ Dan Freedman, Michael Kiermaier

Is $\mathcal{N} = 8$ supergravity UV finite in 4d?

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Results:

- SUSY prohibits $L = 1, 2$ divergences.

  [Grisaru (1977); van Nieuwenhuizen and Wu (1977)]

- Explicit calculations (unitary methods) demonstrate that the 4-graviton amplitude is finite at loop orders $L = 3, 4$.
  [Bern,Carrasco,Dixon,Roiban,Johansson (2007-2009)]

  → why?

- Cancelations beyond what is expected from SUSY
  → 'magic' or symmetries?

- Superfield arguments + string theory arguments.

  [Bossard, Drummond, Green, Howe, Russo, Stelle, Vanhove, ...]

Perturbative structure of $\mathcal{N} = 8$ supergravity in 4d

Questions:

- Why are the 3- and 4-loop 4-graviton amplitudes finite?
- What to expect from higher-loop orders?
- What about higher-point loop amplitudes?
- What can the symmetries of the $\mathcal{N} = 8$ theory teach us about the perturbative structure of the theory?

On-shell states and symmetries of $\mathcal{N} = 8$ supergravity

$2^8 = 256$ massless states

state	helicity		
1 graviton	$+2$	h^+	
$\vdots$			
70 scalars	0	φ^{abcd}	$(a, b, \ldots = 1, \ldots, 8)$
$\vdots$			
1 graviton	-2	h^-	

35 pairs of complex scalars are self-conjugate: $\overline{\varphi}_{abcd} = \frac{1}{4!}\epsilon_{abcdefgh}\varphi^{efgh}$.

Global $SU(8)$ R-symmetry:

$M_n^{\text{SUGRA}}(v^{12}, \varphi^{1245}, \ldots) = 0$ unless $SU(8)$-singlet.

Global continuous $E_{7(7)}$ symmetry spontaneously broken to $SU(8)$.

The $133 - 63 = 70$ scalars are the Goldstone bosons.

Outline

1. PART 1: $\mathcal{N} = 8$ SUSY and $SU(8)$.
2. PART 2: $E_{7(7)}$ constraints.
3. Current status.

Perturbative structure of $\mathcal{N} = 8$ supergravity in 4d

L-loop divergence ↔ counterterm

local operator of mass dimension $(2L + 2)$

for example: R^4 at 3-loop order

Perturbative structure of $\mathcal{N} = 8$ supergravity in 4d

L-loop divergence $\leftrightarrow$ counterterm
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for example: R^4 at 3-loop order

Our goal:
characterize candidate counterterm operators
to bound lowest possible order of a UV divergence

Chart of potential counterterms

Which operator is the first viable candidate counterterm?

L	$n = 4$	5	6			
3	R^4					
4	$D^2 R^4$	R^5				
5	$D^4 R^4$	$D^2 R^5$	R^6			
6	$D^6 R^4$	$D^4 R^5$	$D^2 R^6$	R^7		
7	$D^8 R^4$	$D^6 R^5$	$D^4 R^6$	$D^2 R^7$	R^8	
8	$D^{10} R^4$	$D^8 R^5$	$D^6 R^6$	$D^4 R^7$	$D^2 R^8$	R^9

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- Must require $\mathcal{N} = 8$ SUSY and $SU(8)$ R-symmetry.
- Role of $E_{7(7)}$?

Operators complicated

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Operators complicated, but their leading on-shell matrix elements are simple!

Notation

I will use 4d *spinor helicity* formalism to study on-shell matrix elements:

- If 4d momentum p_μ null, $p^2 = 0$, then

$$p_{\alpha\dot{\beta}} \; = \; p_\mu(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} \; = \; |p\rangle^{\dot{\alpha}}\,[p|^{\beta}$$

  with bra and kets being 2-component commuting spinors which are solutions to the massless Weyl eqn, $p_{\alpha\dot{\beta}}|p\rangle^{\dot{\beta}} = 0$.

- Spinor products $\langle 12\rangle \equiv \langle p_1|_{\dot{\alpha}}\,|p_2\rangle^{\dot{\alpha}}$ and $[12] = [p_1|^{\alpha}|p_2]_{\alpha}$ are just $\sqrt{|s_{12}|} = \sqrt{|2p_1\cdot p_2|}$ up to a complex phase.

- Note $[i\,j] = -[j\,i]$ and $\langle i\,j\rangle = -\langle j\,i\rangle$.

- Dimensional analysis: $\langle ij\rangle$ and $[ij]$ have mass dimension 1.

Analysis of potential counterterms

Instead of studying the operators, we analyze their leading matrix elements:

operator ↔ *matrix elements*

local ↔ polynomial in momenta and polarizations

↔ polynomial in $\langle ij \rangle$ and $[ij]$.

L-loop ↔ $\langle ij \rangle$, $[ij]$ polynomial has degree $2L + 2$.

$\mathcal{N} = 8$ SUSY ↔ SUSY Ward identities.

$SU(8)$-invariant ↔ $SU(8)$ Ward identities.

$E_{7(7)}$-compatible ↔ low-energy theorems

no such matrix elements ↔ no such operator ↔ no such counterterm.

If matrix elements do exist: determine multiplicities of such operators.

Tool box

- **"Little group scaling":**

For each external state $i = 1, \ldots, n$,

$$|i\rangle \to t_i|i\rangle \text{ and } |i] \to t_i^{-1}|i], \quad \Longrightarrow \quad A_n \to t_i^{-2h_i} A_n$$

where h_i is the helicity.

- **$\mathcal{N} = 4, 8$ maximal SUSY Ward identities:**

MHV: $\langle + + - - + + \ldots \rangle = \frac{\langle 34\rangle^{\mathcal{N}}}{\langle 12\rangle^{\mathcal{N}}} \langle - - + + + + \ldots \rangle$.

Example: n-gluon MHV amplitude (Parke-Taylor formula)

$$A_n(1^- 2^- 3^+ 4^+ \ldots n^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle \cdots \langle n1\rangle}$$

has mass dim. $4 - n$.

MHV = maximally helicity violating

Example of how we exclude operators as candidate counterterms.

4-loops: R^5 (mass dim. $2L + 2 = 10$)

10 derivatives in R^5 → leading 5-point interaction has 10 powers of momentum

→ 5-pt matrix element has mass dim. 10

and is polynomial of degree 10 in $\langle..\rangle$'s and $[..]$'s.

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$$\langle 34\rangle^4 [12]^2 [25]^2 [51]^2 = \frac{\langle 34\rangle^8}{\langle 12\rangle^8} \langle 12\rangle^4 [34]^2 [45]^2 [53]^2$$

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$\Longrightarrow$ No $\mathcal{N}=8$ SUSY matrix elements. So R^5 is not indep. supersymmetrizable.

Analysis

Carry out an analysis of matrix elements at MHV and NMHV level.
`[HE, Freedman, Kiermaier, 1003.5018]`

- Use superamplitudes.
- Use solution to SUSY Ward identities.
  `[HE, Freedman, Kiermaier, 0911.3169]`
- Use Gröbner basis: $\text{PolynomialRing}\big[\langle ij\rangle,[kl]\big]\Big/\text{Ideal}\big[\text{Shouten,mom.cons.}\big]$
  `[Beisert, HE, Freedman, Kiermaier, Morales, Stieberger, 1009.1643]`

RESULTS: Chart of potential counterterms

The matrix elements of a prospective counterterm must respect $\mathcal{N} = 8$ SUSY and $SU(8)$ Ward identities.

If *no*: excluded. If *yes*: we find multiplicities of such operators.

Explicit 4-pt calc. shows finite

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"None → ":
we proved no MHV and no NMHV, and conjectured no N^{k}MHV for $L < 7$ in `[HE, Freedman, Kiermaier, 1003.5018]` .
Conjecture proven by `[Howe, Heslop, Drummond, 1008.4939]`

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Let's now include $E_{7(7)}$

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Symmetries

- $\mathcal{N} = 8$ supergravity has a global continuous $E_{7(7)}$ symmetry which is spontaneously broken to $SU(8)$.

  The $133 - 63 = 70$ scalars are the Goldstone bosons.

  Low-energy theorems:

  In $\mathcal{N} = 8$ supergravity, single soft scalar limits vanish,

$$M_n(\varphi(p), \dots) \to 0 \quad \text{as} \quad p \to 0.$$

[Bianchi, HE, Freedman '0805; Arkani-Hamed, Cachazo, Kaplan '0808; Kallosh, Kugo '0811]

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  [Bossard, Hillmann, Nicolai (2010)]

- Counterterm operator $\mathcal{O}$: $E_{7(7)}$ compatible?

  Test if the single soft scalar limits of their matrix elements vanish.

  Soft scalar limits of the MHV 4-, 5- and 6-pt matrix elements trivially vanish

  We would like to calculate the *scalar-graviton* NMHV matrix element

  $$\lim_{p_1 \to 0} \langle \varphi \, \overline{\varphi} \; 3^- \, 4^- \, 5^+ \, 6^+ \rangle_{\mathcal{O}} = ?$$

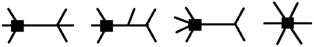
  to test if its single soft limit vanishes or not, when $\mathcal{O} = R^4,\ D^4R^4,\ D^6R^4$.

R^4 matrix elements

$$R^4$$

$$\left\langle \varphi \, \overline{\varphi} + + - - \right\rangle_{R^4}$$

Very hard to calculate from Feynman diagrams



We use a trick to extract the 6-point R^4 matrix elements from the closed string theory tree amplitude.

String effective action: $\alpha'^3 \sqrt{-g}\, e^{-6\phi} R^4$

(not quite what we want)

$e^{-6\phi} R^4$ versus R^4

- The α'^3-correction to the closed string tree amplitude are encoded in the supersymmetrization of

  $$\alpha'^3 \sqrt{-g}\, e^{-6\phi} R^4$$

  This preserves only $SU(4) \times SU(4)$.

- We *cannot* use the closed string tree amplitude directly to explore the 3-loop R^4 candidate counterterm of $\mathcal{N} = 8$ supergravity, because it has to be an $SU(8)$-invariant supersymmetrization.

Earlier work w/ $e^{-6\phi} R^4$ [Brödel & Dixon, 2009]

From $e^{-6\phi}R^4$ to R^4

How to obtain the matrix elements $\langle \varphi\, \overline{\varphi}\;\, 3^-\, 4^-\, 5^+\, 6^+ \rangle_{R^4}$ of the $SU(8)$-invariant supersymmetrization of R^4 from α'^3 of the string amplitude?

'Average' the α'^3 contributions of the string amplitude over $SU(8)$

$\Longrightarrow$

'Average' the matrix elements of $e^{-6\phi}R^4$ over $SU(8)$

$\Longrightarrow$

matrix elements of an $SU(8)$-invariant supersymmetric 8-derivative operator.

There is only ONE such operator, namely the desired R^4.

`[Freedman, Kiermaier, H.E. (March 2010)]`

Average of $SU(8)$

Product of two scalars ϕ^{abcd} contains one singlet: $(\varphi\,\overline{\varphi})_{\text{sing}} = \frac{1}{8!}\,\epsilon_{abcdefgh}\,\varphi^{abcd}\,\varphi^{efgh}$.

Thanks to $SU(4) \times SU(4)$, we get

$$\langle \varphi\,\overline{\varphi} + + - - \rangle_{R^4} \quad = \quad \frac{1}{35}\langle \varphi^{1234}\varphi^{5678} + + - - \rangle_{e^{-6\phi}R^4} - \frac{16}{35}\langle \varphi^{123|5}\varphi^{4|678} + + - - \rangle_{e^{-6\phi}R^4} + \frac{18}{35}\langle \varphi^{12|56}\varphi^{34|78} + + - - \rangle_{e^{-6\phi}R^4}\,.$$

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We calculate these 3 matrix elements from the α'-expansion of the closed string NMHV amplitudes, obtained via KLT

(α'-expansion of open string amplitude from Stieberger & Taylor)

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$$\lim_{p_1\to 0}\,\langle \varphi^{1234}\varphi^{5678} + + - - \rangle_{e^{-6\phi}R^4} \;=\; -12\,\zeta(3)\;\times [34]^4\langle 56\rangle^4\,,$$

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$$\lim_{p_1\to 0}\,\langle \varphi^{12|56}\varphi^{34|78} + + - - \rangle_{e^{-6\phi}R^4} \;=\; 0\,.$$

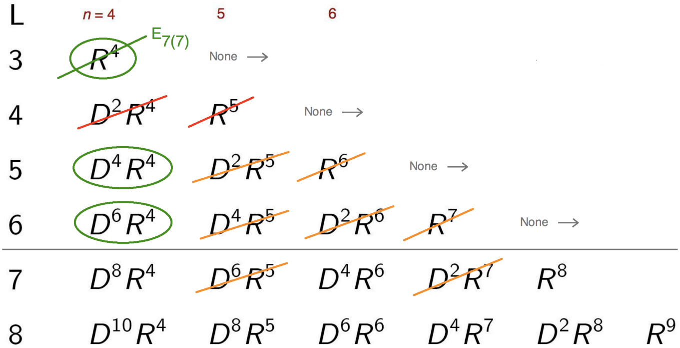
hence

$$\boxed{\lim_{p_1\to 0}\,\langle \varphi\,\overline{\varphi} + + - - \rangle_{R^4} \;=\; 2\zeta(3)\,\frac{6}{5}\,[34]^4\langle 56\rangle^4 \;\neq\; 0\,.}$$

Conclusion: the unique $SU(8)$-invariant supersymmetrization of R^4 is NOT $E_{7(7)}$-compatible.

Chart of potential counterterms in $\mathcal{N} = 8$ supergravity

Candidate counterterm operators must be $\mathcal{N} = 8$ SUSY and $SU(8)$-invariant and have $E_{7(7)}$ symmetry.



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Understand now why 3-loop 4-graviton amplitude is finite.

Observation 1

$$(\star) \qquad \text{Why} \quad \lim_{p_1\to 0} \left\langle \varphi^{12|56}\varphi^{34|78} ++--\right\rangle_{e^{-6\phi}R^4} \quad = \quad 0 \quad ?$$

- $\mathcal{N}=8$ supergravity:

  Global $E_{7(7)}$ symmetry spontaneously broken to $SU(8)$.

  The $133-63=70$ scalars are the Goldstone bosons, which transform in the **70**.

- For $\alpha'>0$:

  Global $SO(6,6)$ spontaneously broken to $SU(4)\times SU(4)$.

  There are $66-30=36$ Goldstone bosons. They transform in the $\mathbf{6}\otimes\mathbf{6}$.

- These are precisely scalars that decompose into products of two $\mathcal{N}=4$ SYM scalars:

  $\varphi_s = z\otimes z$ ex. $\varphi^{12|56}$

- Eq. $(\star)$ holds to all orders in α'. have checked explicit up to and incl. α'^7.

Observation 2: Duality and supersymmetry

Green, Miller, Russo, and Vanhove (GMRV) showed that duality and supersymmetry requires the SUSY operator R^4 to have a non-linear completion of $f_{R^4}\, R^4$, where f_{R^4} is a moduli-dependent automorphic function which satisfies

$$\Delta\, f_{R^4} = -42\, f_{R^4} \qquad \text{for} \quad D = 4$$

Here Δ is the Laplacian on the coset $E_{7(7)}/SU(8)$.

Compare:

Let's compare GMRV to our result:

$$\boxed{\lim_{p_1\to 0}\,\langle \varphi\,\overline{\varphi} + + - - \rangle_{R^4} \;=\; 2\zeta(3)\frac{6}{5}\,[34]^4\langle 56\rangle^4 \;\neq\; 0\,.}$$

Must come from local operator $(\varphi\overline{\varphi})_{\rm sing}R^4$, so that is part of the non-linear completion of R^4, i.e. $f_{R^4}R^4$ with

$$f_{R^4} \propto -2\zeta(3)\Big[1 - \frac{6}{5}\big(\varphi^{1234}\varphi^{5678} + 34\text{ others}\big) + \dots\Big]$$

The Laplacian on $E_{7(7)}/SU(8)$ is

$$\Delta = \Big(\frac{\partial}{\partial\varphi^{1234}}\frac{\partial}{\partial\varphi^{5678}} + 34\text{ inequivalent perms}\Big) \;+\; \dots$$

Indeed we find

$$\Delta\, f_{R^4} + 42\, f_{R^4} \;=\; -2\zeta(3)\Big(-\frac{6}{5}\times 35 + 42\Big) + O(\varphi\overline{\varphi}) \;=\; 0 + O(\varphi\overline{\varphi})$$

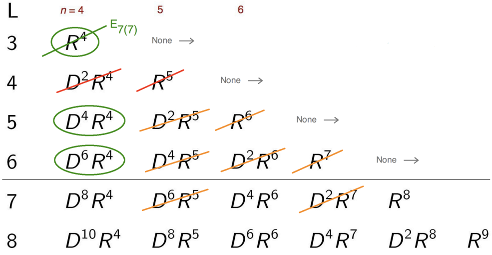
so our result matches GMRV!

$\mathcal{N} = 8$ supergravity

The R^4 operator in $D = 4$:

- $\mathcal{N} = 8$ SUSY and $SU(8)$ invariant.
- NOT $E_{7(7)}$ invariant.
- Explains why R^4 is not a candidate counterterm...
- ...and why the 3-loop 4-point amplitude is finite.

[Bern, Carrasco, Dixon, Johansson, Kosower, Roiban '07]



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Next up: D^4R^4

Closed string effective action

$$\begin{aligned} S_{\text{eff}} \quad &= \quad S_{\text{SG}} - 2\,\alpha'^3\zeta(3)\,e^{-6\phi}\,R^4 - \alpha'^5\,\zeta(5)\,e^{-10\phi}\,D^4R^4 \\ &\quad + \tfrac{2}{3}\,\alpha'^6\,\zeta(3)^2\,e^{-12\phi}\,D^6R^4 - \tfrac{1}{2}\alpha'^7\zeta(7)\,e^{-14\phi}\,D^8R^4 + \dots . \end{aligned}$$

$SU(8)$ average procedure gives unique D^4R^4 matrix elements from α'^5 of closed string amplitude.

- NOT $E_{7(7)}$ invariant.
- Single soft limit shows SUSY operator is $f_{D^4R^4}\,D^4R^4$ with $f_{D^4R^4} \;\propto\; -\zeta(5)\Big[1 - \frac{12}{7}\big(\varphi^{1234}\varphi^{5678} + 34 \text{ others}\big) + \dots\Big]$
- Satisfies Green et al's $\Delta\, f_{D^4R^4} = -60\, f_{D^4R^4}$
- Conclude: D^4R^4 is *not* a candidate counterterm.
- $\mathcal{N} = 8$ SG finite at 5-loops in $D = 4$.

Next up: D^4R^4 and D^6R^4

Closed string effective action

$$\begin{aligned} S_{\text{eff}} \;&=\; S_{\text{SG}} - 2\,\alpha'^3\zeta(3)\,e^{-6\phi}R^4 - \alpha'^5\,\zeta(5)\,e^{-10\phi}D^4R^4 \\ &\quad + \tfrac{2}{3}\,\alpha'^6\,\zeta(3)^2\,e^{-12\phi}D^6R^4 - \tfrac{1}{2}\alpha'^7\zeta(7)\,e^{-14\phi}D^8R^4 + \ldots\,. \end{aligned}$$

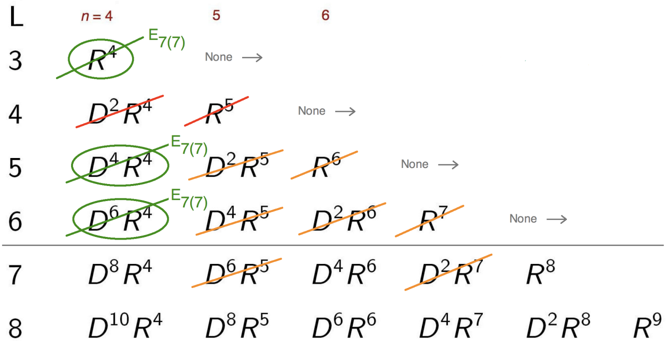
Matrix elements from α'^6 of closed string amplitude are polluted by pole terms R^4—R^4 from $\alpha'^3 \times \alpha'^3$.

- We calculate fully $\mathcal{N} = 8$ SUSY'ized R^4—R^4.
- Extract $\langle\varphi\,\overline{\varphi}++--\rangle_{R^4\text{—}R^4}$ and subtract it from $\langle\varphi\,\overline{\varphi}++--\rangle_{e^{-12\phi}D^6R^4}$.
- $SU(8)$ average then gives $\langle\varphi\,\overline{\varphi}++--\rangle_{D^6R^4}$, which has non-vanishing single soft scalar limit.
- Satisfies Green et al's $\Delta\, f_{D^6R^4} = -60\, f_{D^6R^4} - (f_{R^4})^2$.

  The inhom. term is from R^4—R^4.
- NOT $E_{7(7)}$ invariant.
- Conclude: D^6R^4 is *not* a candidate counterterm.
- $\mathcal{N} = 8$ SG finite at 6-loops in $D = 4$.

Landscape of potential counterterms

$\mathcal{N} = 8$ SUSY and $SU(8)$-invariant candidate counterterm operators.



L	$n = 4$	5	6			
3	$\cancel{R^4}$ $E_{7(7)}$	None →				
4	$\cancel{D^2 R^4}$	$\cancel{R^5}$	None →			
5	$\cancel{D^4 R^4}$ $E_{7(7)}$	$\cancel{D^2 R^5}$	$\cancel{R^6}$	None →		
6	$\cancel{D^6 R^4}$ $E_{7(7)}$	$\cancel{D^4 R^5}$	$\cancel{D^2 R^6}$	$\cancel{R^7}$	None →	
7	$D^8 R^4$	$\cancel{D^6 R^5}$	$D^4 R^6$	$\cancel{D^2 R^7}$	R^8	
8	$D^{10} R^4$	$D^8 R^5$	$D^6 R^6$	$D^4 R^7$	$D^2 R^8$	R^9

[HE, Kiermaier, 1007.4813]

[Beisert, HE, Freedman, Kiermaier, Morales, Stieberger, 1009.1643]

What do we know about $L \geq 7$ loops?

$\mathcal{N} = 8$ SUSY and $SU(8)$-singlet candidate counterterm operators and $SU(8)$ **70** operators for their single soft scalar limits.

7-loop	4-pt	5-pt	6-pt	7-pt	8-pt	9-pt	10-pt	11-pt	12-pt	13-pt	14-pt	15-pt	16-pt
singlet	D^8R^4 $1\times\mathrm{MHV}$	~~D^6R^5~~	D^4R^6 $2\times\mathrm{NMHV}$	~~D^2R^7~~	R^8 $3\times\mathrm{N}^2\mathrm{MHV}$	~~$\varphi^2D^2R^7$~~	φ^2R^8 $4\times\mathrm{N}^3\mathrm{MHV}$	~~$\varphi^4D^2R^7$~~	φ^4R^8 $6\times\mathrm{N}^4\mathrm{MHV}$	~~$\varphi^6D^2R^7$~~	φ^6R^8 $8\times\mathrm{N}^5\mathrm{MHV}$	~~$\varphi^8D^2R^7$~~	φ^8R^8 $10\times\mathrm{N}^6\mathrm{MHV}$
			↙ soft		↙ soft		↙ soft		↙ soft		↙ soft		↙ soft
70		φD^8R^4 $2\times$		φD^4R^6 $4\times$		φR^8 $6\times$		φ^3R^8 $9\times$		φ^5R^8 $14\times$		φ^7R^8 $19\times$	

8-loop	4-pt	5-pt	6-pt	7-pt	8-pt	9-pt	10-pt	11-pt	12-pt	13-pt	14-pt
singlet	$D^{10}R^4$ $1\times\mathrm{MHV}$	D^8R^5 $1\times\mathrm{MHV}$	D^6R^6 $3\times\mathrm{NMHV}$	D^4R^7 $3\times\mathrm{NMHV}$	D^2R^8 $8\times\mathrm{N}^2\mathrm{MHV}$	R^9 $8\times\mathrm{N}^2\mathrm{MHV}$	$\varphi^2D^2R^8$ $25\times\mathrm{N}^3\mathrm{MHV}$	φ^2R^9 $22\times\mathrm{N}^3\mathrm{MHV}$	$\varphi^4D^2R^8$ $66\times\mathrm{N}^4\mathrm{MHV}$	φ^4R^9 $51\times\mathrm{N}^4\mathrm{MHV}$	$\varphi^6D^2R^8$ $153\times\mathrm{N}^5\mathrm{MHV}$
		↙	↙	↙	↙	↙	↙	↙	↙	↙	↙
70		$\varphi D^{10}R^4$ $3\times$	φD^8R^5 $4\times$	φD^6R^6 $17\times$	φD^4R^7 $16\times$	φD^2R^8 $81\times$	φR^9 $63\times$	$\varphi^3D^2R^8$ $232\times$	φ^3R^9 $211\times$	$\varphi^5D^2R^8$ $1033\times$	

Multiplicities found using $SU(2,2|8)$.

`[Beisert, HE, Freedman, Kiermaier, Morales, Stieberger, 1009.1643]`

For $n > 4$ none of the $L = 7$ operators are $E_{7(7)}$ compatible.
This means that the 4-graviton amplitude determines whether the theory is finite or not at $L = 7$.

Standings

SUSY, $SU(8)$, $E_{7(7)}$ $\Longrightarrow$ $\mathcal{N} = 8$ supergravity in 4d finite up to 7-loop order.

First divergence at $L = 7$?

Candidate full superspace integral — but it vanishes!!

But there is a new 7/8th superspace integral counterterm available.

[Bossard, Howe, Stelle, Vanhove (2011)]

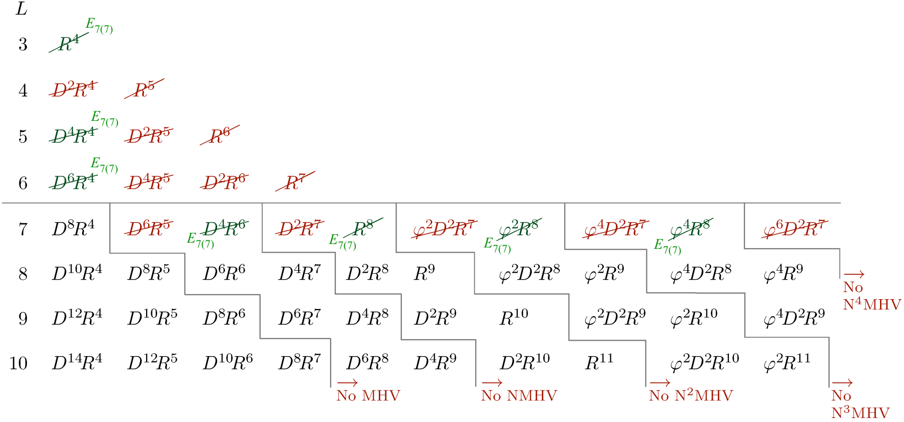
First divergence at $L = 8$?

Candidate full superspace integral available [Kallosh (1981), Howe & Lindstrom (1981)]

→ Looks like more than SUSY and $E_{7(7)}$ is needed for finiteness.

Landscape of potential counterterms

$\mathcal{N}=8$ SUSY and $SU(8)$-invariant candidate counterterm operators.



L										
3	$\cancel{R^4}$ $E_{7(7)}$									
4	$\cancel{D^2R^4}$	$\cancel{R^5}$								
5	$\cancel{D^4R^4}$ $E_{7(7)}$	$\cancel{D^2R^5}$	$\cancel{R^6}$							
6	$\cancel{D^6R^4}$ $E_{7(7)}$	$\cancel{D^4R^5}$	$\cancel{D^2R^6}$	$\cancel{R^7}$						
7	D^8R^4	$\cancel{D^6R^5}$	$E_{7(7)}$ $\cancel{D^4R^6}$	$\cancel{D^2R^7}$	$E_{7(7)}$ $\cancel{R^8}$	$\cancel{\varphi^2D^2R^7}$	$E_{7(7)}$ $\cancel{\varphi^2R^8}$	$\cancel{\varphi^4D^2R^7}$	$E_{7(7)}$ $\cancel{\varphi^4R^8}$	$\cancel{\varphi^6D^2R^7}$
8	$D^{10}R^4$	D^8R^5	D^6R^6	D^4R^7	D^2R^8	R^9	$\varphi^2D^2R^8$	φ^2R^9	$\varphi^4D^2R^8$	φ^4R^9
9	$D^{12}R^4$	$D^{10}R^5$	D^8R^6	D^6R^7	D^4R^8	D^2R^9	R^{10}	$\varphi^2D^2R^9$	φ^2R^{10}	$\varphi^4D^2R^9$
10	$D^{14}R^4$	$D^{12}R^5$	$D^{10}R^6$	D^8R^7	D^6R^8	D^4R^9	D^2R^{10}	R^{11}	$\varphi^2D^2R^{10}$	φ^2R^{11}

$\longrightarrow$ No N^4MHV

$\longrightarrow$ No MHV

$\longrightarrow$ No NMHV

$\longrightarrow$ No N^2MHV

$\longrightarrow$ No N^3MHV

[HE, Kiermaier, 1007.4813]

[Beisert, HE, Freedman, Kiermaier, Morales, Stieberger, 1009.1643]