Covariant and local deformations of quantum field theories

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Quantum Theory and Gravitation
June 20, 2011

Quantum theory and relativity

- Main topic of this conference:

> Combination of quantum theory and general relativity
> (quantum gravity)

- **This talk:** Let's take a step back and reconsider

> Combination of quantum theory and special relativity
> (quantum field theory)

- Also in quantum theory + special relativity, some fundamental problems are unsolved (construction of models)
- Some connections between the two settings by noncommutative geometry

Construction of models in quantum field theory

Models of QFT can be formulated in any approach. Known models:

- ✓ Free field theories (any dimension)
- ✓ perturbative quantum field theories (any dimension)
- ✓ Rigorous construction beyond perturbation theory ("constructive qft"): ... Fröhlich, Glimm, Hepp, Jaffe, Jost, Nelson, Osterwalder, Schrader, Segal, Wightman, ...
  $\rightarrow$ completely constructed models in $d = 1 + 1$ and $d = 2 + 1$ spacetime dimensions
- ✓ conformal/chiral field theories (see talk of Longo)
- ✓ Integrable QFT models in $d = 1 + 1$ dimensions

Construction of models in quantum field theory

However, in $d = 1 + 3$,

✗ No non-perturbative construction of interacting models yet!

In this sense, we don't yet have a proof for the compatibility of quantum theory and special relativity.

In this talk, I want to present a new approach to this problem, using operator-algebraic methods.

Algebraic formulation of QFT on Minkowski space

As usual in quantum theory,

- observables are modelled as operators A on a Hilbert space $\mathcal{H}$, constitute observable algebra
- states are modelled by unit vectors $\Psi \in \mathcal{H}$ and corresponding expectation value functionals

$$A \longmapsto \langle \Psi, A\Psi \rangle$$

(or mixtures thereof)

Particular state in QFT: Vacuum Ω. Consider a unitary representation U with positive energy of the Poincaré group $\mathcal{P}_+^\uparrow$ on $\mathcal{H}$, and demand $U(x, \Lambda)\Omega = \Omega$ for all $(x, \Lambda) \in \mathcal{P}_+^\uparrow$.

Algebraic formulation of QFT on Minkowski space

Physically realizable experiments are localized in space and time. To any region $O \subset \mathbb{R}^d$, there corresponds a localized observable algebra $\mathcal{A}(O) \subset \mathcal{B}(\mathcal{H})$.

Formally: $\phi(x) \in \mathcal{A}(O)$ for $x \in O$.

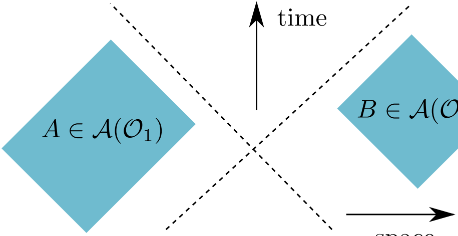
QFT properties of the correspondence $O \longmapsto \mathcal{A}(O)$:

- larger laboratories contain more observables: $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$ if $O_1 \subset O_2$
- Poincaré covariance:

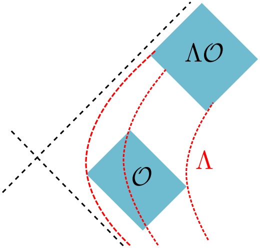
$$U(x,\Lambda)\mathcal{A}(O)U(x,\Lambda)^{-1} = \mathcal{A}(\Lambda O + x)$$

- Einstein causality: Spacelike separated observables commute,

$$\mathcal{A}(O_1) \subset \mathcal{A}(O_2)' \quad \text{if} \;\; O_1 \subset O_2' \,.$$



$$AB = BA$$



$$U(\Lambda)\mathcal{A}(\mathcal{O})U(\Lambda)^{-1} = \mathcal{A}(\Lambda\mathcal{O})$$

Given data $(\{\mathcal{A}(O)\}_{O\subset\mathbb{R}^d}, U, \Omega)$, one can investigate the physics of this model and compute

- Particle content, scattering states, S-matrix
- Short distance behaviour, associated quantum fields, OPE
- charge structure
- ...

However, it is difficult to construct examples (construction of interacting quantum field theories).

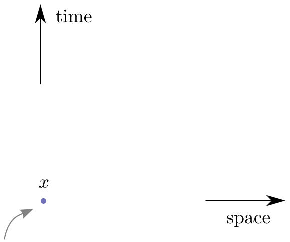
Rest of the talk: New constructive ideas within this setting

A strategy for constructing models

Two steps:

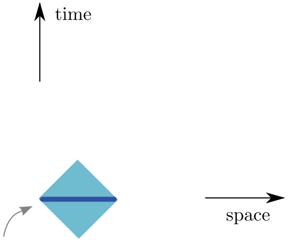
1. Define the particle content of the model
   - Fix a representation U of $\mathcal{P}_+^\uparrow$ on some Hilbert space $\mathcal{H}$.
   - Scattering theory: $\rightarrow$ may work with second quantized representation on a Fock space of scattering states without restriction.
2. Construct the observable algebras $\mathcal{A}(O)$
   - Uncertainty principle: The stricter the localization of the observables, the more involved are the momentum-space properties of these operators (particle production).
   - $\rightarrow$ Consider "weakly localized" observables first and obtain sharper localization by exploiting covariance and algebraic structure.

Localization regions



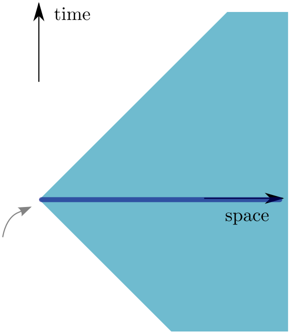
localization in spacetime point x

Localization regions



localization in bounded region – for constructive purposes essentially the same as points

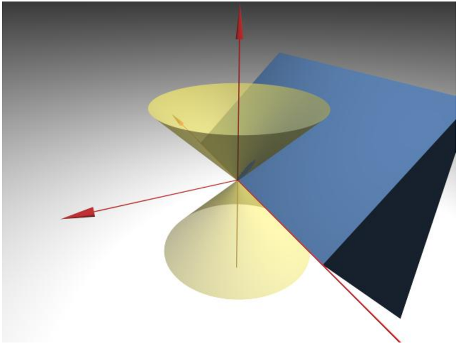
Localization regions



localization in unbounded region (wedge = causal completion of half space)

$$W_0 := \{x = (x_0, ..., x_{d-1}) \in \mathbb{R}^d \ : \ x_1 > |x_0|\}$$

Localization regions



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Wedge algebras

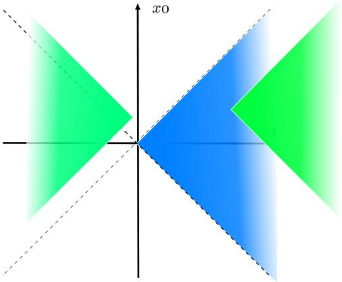
Conditions for a v. Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ to be localized in W_0:

- $U(x,\Lambda)\mathcal{M}U(x,\Lambda)^{-1} \subset \mathcal{M}$ for $(x,\Lambda) \in \mathcal{P}_+^\uparrow$ with $\Lambda W_0 + x \subset W_0$
- $U(x,\Lambda)\mathcal{M}U(x,\Lambda)^{-1} \subset \mathcal{M}'$ for $(x,\Lambda) \in \mathcal{P}_+^\uparrow$ with $\Lambda W_0 + x \subset W_0'$
- $\mathcal{M}\Omega \subset \mathcal{H}$ dense

$\rightarrow$ call then $(\mathcal{M}, U, \Omega)$ a causal triple.

Causal triples generate QFT models: Define

$$\mathcal{A}(W_0) := \mathcal{M}$$
$$\mathcal{A}(\Lambda W_0 + x) := U(x,\Lambda)\mathcal{M}U(x,\Lambda)^{-1}$$
$$\mathcal{A}(\bigcap_k W_k) := \bigcap_k \mathcal{A}(W_k)$$



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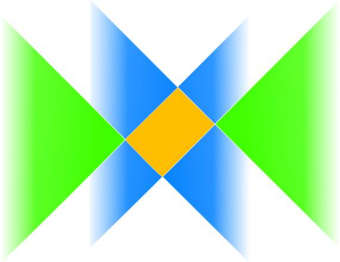

Theorem:

This definition yields a Poincaré covariant, local net on $\mathbb{R}^d$.

- In principle, every QFT can be described in this way.
- Controlling the size of the intersections? Model-independent effective criteria presently only known in $d = 1 + 1$ (von Neumann type, Reeh-Schlieder property) ▸ Buchholz/GL 2004 ▸ GL 2008

Constructive algebraic quantum field theory

Problem of constructive algebraic QFT

Find von Neumann algebras $\mathcal{M}$ such that $(\mathcal{M}, U, \Omega)$ is a causal triple.

- Modular (Tomita-Takesaki) construction of interaction-free nets
  Brunetti/Guido/Longo 2002
- Algebraic construction of integrable models on $\mathbb{R}^2$ Schroer 1997-2001
  GL 2003-2006 Buchholz/GL 2004
- Constructions of QFT on non-commutative Minkowski space
  Grosse/GL 2007, Grosse/GL 2008
- Deformations of quantum field theories Buchholz/Summers 2008
  Buchholz/GL/Summers 2010 Dappiaggi/GL/Morfa-Morales 2010 Longo/Witten 2010
  Dybalski/Tanimoto 2010 GL 2011 Morfa-Morales 2011 Bostelmann/GL/Morsella 2011

A concrete example of a QFT deformation

Recall Rieffel's deformations of C^*-dynamical systems ▸ Rieffel 1992 :

- C^*-algebra $\mathcal{A}$ with strongly continuous $\mathbb{R}^d$-action α by automorphisms of $\mathcal{A}$.
- For given antisymmetric $(d \times d)$-matrix Q, introduce new product

$$A \times_Q B := (2\pi)^{-d} \int dp\, dx\, e^{-ipx}\, \alpha_{Qp}(A)\alpha_x(B)$$

- Integral defined in an oscillatory sense for smooth elements A, B
- Product is associative, compatible with unit and *-involution of $\mathcal{A}$
- $A \times_0 B = AB$, deformation of noncommutative dynamical systems
- Used in deformation quantization, noncommutative geometry (NC tori), and models of noncommutative spacetimes (generalizes Moyal product)

A concrete example of a QFT deformation

Analogous deformation of Hilbert space operators ("warped convolution")

GL/Grosse 2007 · Buchholz/Summers 2008 · Buchholz/GL/Summers Comm.Math.Phys. 304, 95-123, 2011

- Take $\alpha_x(A) = U(x)AU(x)^{-1}$ and $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ suitable smooth subalgebra (i.e. $x \mapsto U(x)AU(x)^{-1}$ is smooth in norm)
- Deform operators $A \in \mathcal{A}$ according to

$$A_Q := (2\pi)^{-d} \int dp\, dx\, e^{-i(p,x)}\, U(Qp)AU(-Qp)U(x)$$

  Convergence of integral can be controlled on a dense domain, then extension to full Hilbert space
- Q: antisymmetric $(d \times d)$-matrix, deformation parameter.
- Analytic control over oscillatory integral $\sim$ summation of perturbation series

Properties of the deformation map $A \mapsto A_Q$

$$A_Q := (2\pi)^{-d} \int dp\, dx\, e^{-i(p,x)}\, U(Qp)AU(-Qp)U(x)$$

- $A_0 = A$ (deformation)
- $A \mapsto A_Q$ is linear
- $(A_Q)^* = (A^*)_Q$ observables stay observables
- Representation of deformed product $(\mathcal{A}, \times_Q)$:

$$A_Q B_Q = (A \times_Q B)_Q$$

Theorem

The map $A \mapsto A_Q$ extends to a faithful representation of Rieffel's deformed C^*-algebra $(\overline{\mathcal{A}}, \times_Q)$.

Interplay of deformation with covariance and locality

- **Covariance:** $U(x,\Lambda)$ Poincaré transformation, A observable. Then

$$U(x,\Lambda)A_Q U(x,\Lambda)^{-1} = \big(U(x,\Lambda)AU(x,\Lambda)^{-1}\big)_{\Lambda Q \Lambda^{-1}}$$

  $\rightarrow$ in a Lorentz covariant theory, cannot restrict to a single deformation parameter Q ▸ Doplicher/Fredenhagen/Roberts 1995 ▸ GL/Grosse 2007

- **Vacuum:** $A_Q\Omega = A\Omega$.
- **Locality:** Assume $A, B \in \mathcal{A}$ satisfy

$$[U(Qp)AU(Qp)^{-1},\, U(-Qq)BU(-Qq)^{-1}] = 0$$

  for all $p, q \in \mathrm{Sp}U$. Then $[A_Q,\, B_{-Q}] = 0$. ▸ Buchholz/Summers 2008

Application to causal triples

Assume a causal triple $(\mathcal{M}, U, \Omega)$ is given. Define

$$\mathcal{M}_Q := \{A_Q \,:\, A \in \mathcal{M} \text{ smooth}\}''$$

and keep $U, \Omega, \mathcal{H}$ unchanged.

- Which Q? Dictated by symmetries of localization wedge W_0.
- Take

$$Q := \begin{pmatrix} 0 & \kappa_{\mathsf{e}} & 0 & 0 \\ \kappa_{\mathsf{e}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa_{\mathsf{m}} \\ 0 & 0 & -\kappa_{\mathsf{m}} & 0 \end{pmatrix}, \qquad \kappa_{\mathsf{m}} \in \mathbb{R},\ \kappa_{\mathsf{e}} \geq 0\,.$$

  Then

- $\Lambda W_0 \subset W_0 \Longleftrightarrow \Lambda Q \Lambda^{-1} = Q$
- $\Lambda W_0 \subset W_0' \Longleftrightarrow \Lambda Q \Lambda^{-1} = -Q$
- $Q \mathrm{Sp} U \subset W_0$

Theorem:

Let $(\mathcal{M}, U, \Omega)$ be a causal triple and Q a matrix of the specified form. Then also $(\mathcal{M}_Q, U, \Omega)$ is a causal triple. $({\mathcal{M}_Q}' = {\mathcal{M}'}_{-Q})$

Deformed causal triple generates a deformed quantum field theory. Application to triple of free field theory yields new models.

Properties of the deformed theories:

- Hamiltonian unchanged under deformation; interaction encoded in position of $\mathcal{M}_Q$ in $\mathcal{B}(\mathcal{H})$
- Deformed theory fully Poincaré covariant
- Localization in wedges: explicit observables A_Q. Localization in sub-wedge regions: Indirect characterization of observables via intersections of algebras (but Reeh-Schlieder locally violated)
- Deformation induces interaction: Two-particle scattering matrix depends on Q via phase factors e^{ipQq}.
- Effect of the deformation on thermal aspects under investigation ▸ Huber

Generalized deformation procedures ‣ GL 2011, arXiv: 1104.1948

- Warped convolution can be generalized to a large family of deformation procedures
  (by making use of the Borchers-Uhlmann tensor algebra)
- $\rightarrow$ deformed quantum fields on Fock space

$$\phi_K(x) := \int dp\, dy\, e^{i(p,y)}\, U(y)\phi(x)U(y)^{-1}K(p)$$

with specific operator-valued kernels K,

$$(K(p)\Psi)_n(q_1, ..., q_n) = k_n(p; q_1, ..., q_n) \cdot \Psi_n(q_1, ..., q_n)$$

analyticity properties of the k_n imply wedge-localization of ϕ_K.

- produces an infinite family of causal triples $(\mathcal{M}_K, U, \Omega)$, $K \in \mathcal{K}$.
- Examples of interacting covariant quantum field theories in $d = 1 + 3$ which are at least wedge-local.

Generalized deformation procedures

- Two-particle scattering can be computed; two particle S-matrix involves k_2^2.
- Resulting S-matrix "too simple" for a local interacting theory in $d > 1+1$ (no particle production) ▸ Åks 1965
- $\to$ Need to replace multiplication operator kernels k_n by integral operators

$$(K(p)\Psi)_n(q_1,..,q_n) = \sum_m \int dq_1' \cdots dq_m' \, k_{nm}(p;q_1,..,q_n;q_1',...,q_m') \, \Psi_m(q_1',..,q_m')$$

Situation under investigation (joint project with J. Schlemmer).

- In $d = 1+1$, deformations reproduce known integrable models (Sinh-Gordon,...). ▸ Schroer 1997 ▸ GL 2006

Conclusions

- Unsolved problem in QFT: Non-perturbative construction of interacting models in $d = 1 + 3$
- Algebraic approach gives new perspective on the construction problem
- complementary to other approaches
- First models obtained by operator-algebraic deformations
- General theory of deformations of nets of von Neumann algebras wanted ("landscape of all QFTs with given particle content")