How General is the Generalized Second Law?

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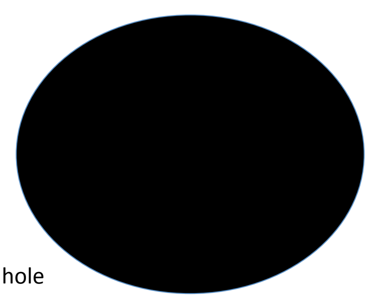
Wheeler to Bekenstein (1971):

"If I drop a teacup into a black hole, I conceal from all the world the increase of entropy."



Black hole

$$\Delta S_{\rm outside} < 0$$

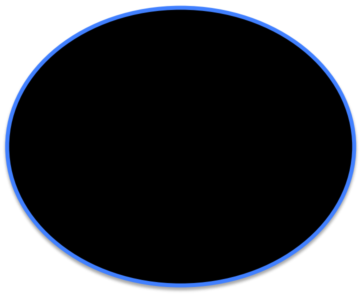
Black hole entropy

Bekenstein, 1972

$$S_{BH} = \alpha(\text{Horizon Area})$$

$$\alpha \sim L_{\text{Planck}}^{-2}$$

$$L_{\text{Planck}}^{2} = \hbar G/c^3 = (10^{-33}\,\text{cm})^2$$



Generalized second law:

$$\Delta(S_{\text{outside}} + S_{\text{BH}}) \geq 0$$

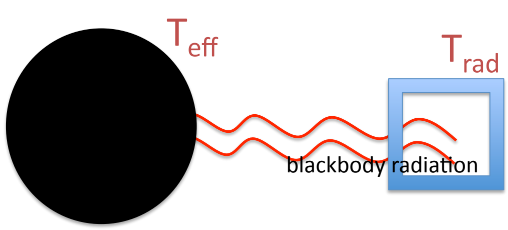
$$dS_{BH} = \frac{dE}{T_{\text{eff}}}$$

Black hole Clausius relation

$$T_{\text{eff}} = \frac{\hbar\kappa}{8\pi\eta}, \quad \kappa = \text{surface gravity} = \frac{1}{2R_{\text{horizon}}}$$

(non-rotating case)

Black hole "temperature"



GSL violated if $T_{rad} < T_{eff}$ …

…but this requires $\lambda_{rad} \gtrsim R_{horizon}$

Apparent GSL violation happens in fluctuation-dominated regime…

…the GSL is saved by the Hawking radiation, at temperature

$$T_H = \frac{\hbar\kappa}{2\pi}, \qquad \Longrightarrow \eta \equiv \frac{1}{4}, \text{ so } S_{BH} = \frac{A}{4L_P^2}$$

Hawking radiation

- is a QFT effect
- comes from *outside* the black hole
- is correlated to field fluctuations *inside* the black hole
- is associated with negative energy flux across the horizon
- leads to BH evaporation, which satisfies the GSL

Box lowering gedanken experiments

(Unruh & Wald, 1982)

Can lower a box full of radiation so close to the horizon that it has almost no Killing energy, then dump in its entropy …

… but the box will float when its energy density is equal to that of the ambient thermal atmosphere of the black hole. The dumped entropy is no more than that of the displaced atmosphere, while the increase of bh entropy is equal to that of the displaced atmosphere, hence the GSL stands.

"Mining" the atmosphere: lower an empty open box to near the horizon, close the box, and lift to infinity where it appears containing radiation!

The box radiates *negative* energy into the hole when pulled back up ---

--- the entropy of the hole goes down, the entropy of the outside goes up, and the atmosphere is restored. *Negative energy flux is associated with increase of entanglement between the inside and outside!*

WHY is the GSL true?? If truly fundamental, it should be tied to fundamental properties of QFT, and it should hold for arbitrary, rapid variations…

Could the GSL be a guiding light to aspects of spacetime in quantum gravity?

Important to understand the most general and precise formulation and proofs of the GSL.

The first generalization: from black holes to arbitrary *causal horizons*!

"Causal Horizon"

The boundary of the causal past of an observer is their causal horizon.
If the observer lives infinitely long, this horizon has all the thermodynamic properties of a black hole.

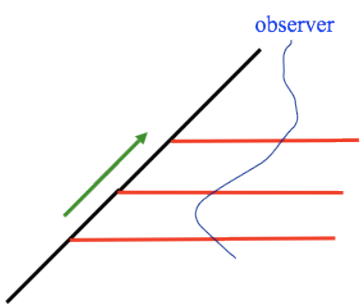
Examples:

- observer who remains outside a black hole
- inertial observer in an exponentially expanding cosmological spacetime
- uniformly accelerating observer in Minkowski spacetime

These *all* have entropy and temperature, and the first law and GSL hold for them, so "black holes" should play no special role.

Horizon Thermodynamics

The outside of a horizon is an OPEN system—
info can leave (but not enter).



But the generalized entropy

$$S_{\rm gen} = \frac{A}{4\hbar G} + S_{\rm out}$$

still increases. Area A is included as boundary term:

$$\frac{dS_{gen}}{dt} \geq 0$$

Generalized Second Law (GSL).

The rest of this talk will be a sketch of recent results by Aron C. Wall, which show that arbitrary, rapid semi-classical perturbations of a causal horizon satisfy the GSL with *any* horizon slicing.

Proof of the generalized second law for rapidly evolving Rindler horizons
PRD 82, 124019 (2010), arXiv:1007.1493

A proof of the generalized second law for rapidly changing fields and arbitrary horizon slices
arXiv: 1105.3445

See also Wall's review:
Ten proofs of the generalized second law
JHEP 06 (2009) 021, arXiV: 0901.3865

I use mostly the slides (slightly modified) from a talk given by Wall; this is *not* my work!

Rindler GSL proof summary

Basic ideas:

1. Relate generalized entropy to free (boost) energy in wedge.
2. Relate free energy to a quantity known as "relative entropy"
3. Apply theorem that says relative entropy can't increase.

free boost energy — relative entropy

$$S_{\text{gen}} = (TS_{\text{out}} - K)/T = -S(\rho\,|\,\sigma)$$

(up to additive constants)

Previous proofs of the GSL implicitly used the concept of relative entropy (Casini 08).

Semiclassical approximation

1. Pick a classical background and look at QFT state ρ .
2. Expand out metric in powers of $\hbar$:

$$g_{ab} = g^0_{ab} + g^{1/2}_{ab} + g^1_{ab} + \mathcal{O}(\hbar^{3/2})$$

classical background metric

quantized gravitons

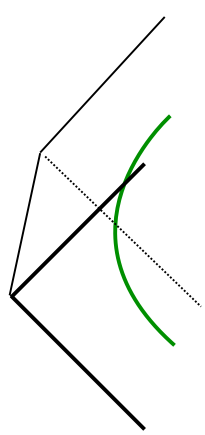
gravitational fields of matter/gravitons calculated using expectation of Einstein equation
$$\langle G_{ab} \rangle = 8\pi G \langle T_{ab} \rangle$$

3. See if the generalized entropy of the state ρ increases:

$$\frac{dS_{\text{gen}}}{dt} = \frac{d}{dt}\left[\frac{\langle A \rangle}{4\hbar G} - \text{tr}(\rho \ln \rho)\right] \geq 0$$

Entanglement entropy divergence must be renormalized!
This can be done by considering only entropy *differences*.

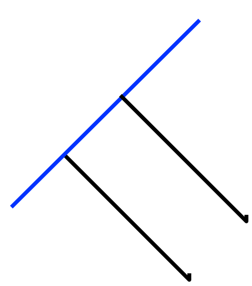
Rindler Wedges



perspective drawing of wedge
& accelerating observer

A Rindler wedge is the intersection of the past & future of uniformly accelerating worldline.

1-parameter family of Rindler wedges share same future horizon & fit inside each other.



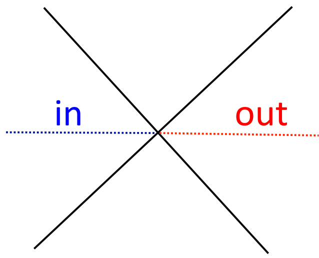
Go from bigger wedge to smaller wedge by restriction..

Rindler Wedges are Thermal

Bisognano-Wichmann
Davies, Unruh 75

QFT vacuum always KMS (i.e. thermal) in boost energy *K* when restricted to Rindler wedge, at temperature $T = \hbar/2\pi$.

Consequence of *boost & translation symmetry, & energy positivity*.



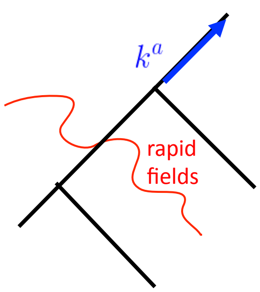
$$\mathrm{tr}_{\mathrm{in}}(|0\rangle\langle 0|) \propto e^{-2\pi K_{\mathrm{out}}/\hbar}$$

(formally) where the boost Killing energy on a slice Σ is:

$$K_{\mathrm{out}} = \int_{\mathrm{out}} T_{ab}\,\xi^a d\Sigma^b$$

So the vacuum is thermal. Next we will perturb it with quantum fields.

Area deficit ~ Boost Energy



Along each horizon generating lightray, the Raychaudhuri & Einstein equations hold:

$$\dot{\theta} = -\theta^2/2 - \sigma_{ab}\sigma^{ab} - 8\pi G T_{kk}$$

where $\theta = (1/A)(dA/d\lambda)$ is the expansion w.r.t. an affine parameter λ.

Linearize and integrate to get expression in terms of $K(\lambda)$, the boost energy of the wedge, up to constants.

$$A(\lambda) = A(\infty) - 8\pi G \int_{\lambda}^{\infty} T_{kk}(\lambda' - \lambda) d\lambda' = -8\pi G[K(\lambda) - K_{\mathrm{rad}}] + A(\infty)$$

constants

Horizon area canonically conjugate to boost time: generalizes Carlip & Teitelboim (95), Massar & Parentani (00) to dynamical situations.

So, up to a constant, the BH entropy is minus the boost energy outside!

$$S_{BH}(\lambda) = \frac{A(\lambda)}{4\hbar G} = -\frac{2\pi}{\hbar} K(\lambda) + const.$$

And, up to a constant, the generalized entropy is the free boost energy!

$$S_{gen}(\lambda) = -\frac{K(\lambda) - T S_{out}(\lambda)}{T}$$

Relative Entropy

Information theory property of two mixed states ρ and σ.

$$S(\rho\,|\,\sigma) = \mathrm{tr}(\rho\,\ln\,\rho) - \mathrm{tr}(\rho\,\ln\,\sigma)$$

(definition can be extended to arbitrary algebras of observables)

Properties:

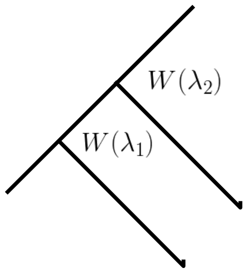
* Range is $[0, +\infty]$. Finite for nice enough states (no renormalization).
* $S(\rho\,|\,\rho) = 0$
* If σ is a KMS (thermal) state, proportional to free energy difference:

$$S(\rho\,|\,\sigma) = [T_\sigma^{-1}E - S]_\rho - [T_\sigma^{-1}E - S]_\sigma$$ (Araki & Sewell 77)

* Monotonicity: Always non-increasing under restriction to subsystems:

$$S(\rho\,|\,\sigma)_M \geq S(\rho\,|\,\sigma)_{M'}$$ when $M' \subset M$ (Araki 75)

Proof of the Rindler GSL



Let ρ be the state we are interested in proving the GSL for.

Let σ be the Minkowski vacuum state.

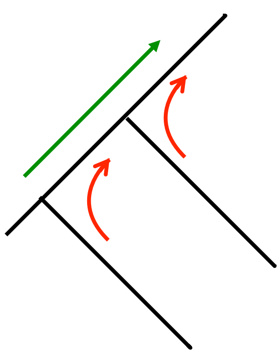
Since σ is thermal in each wedge, $S(\rho\,|\,\sigma)$ is the free boost energy up to terms constant in each wedge:

$$S(\rho\,|\,\sigma) = [\tfrac{2\pi}{\hbar}K - S_{\text{out}}]_\rho = -\frac{A}{4\hbar G} - S_{\text{out}} = -S_{\text{gen}}$$

$S_{\text{out}} = S_\rho - S_\sigma$ is the *renormalized* entropy.

Relative entropy is monotonic under restriction, so the GSL holds!

Rindler Symmetry

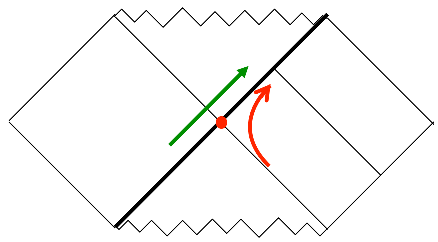


Argument just given requires each wedge to have a boost symmetry so that the vacuum state σ is thermal.

Commutator of two boosts is a null translation symmetry. Vacuum state σ invariant under this too.

Rindler horizon invariant under 2d Lie group. That's why it works.

Black Holes have less symmetry



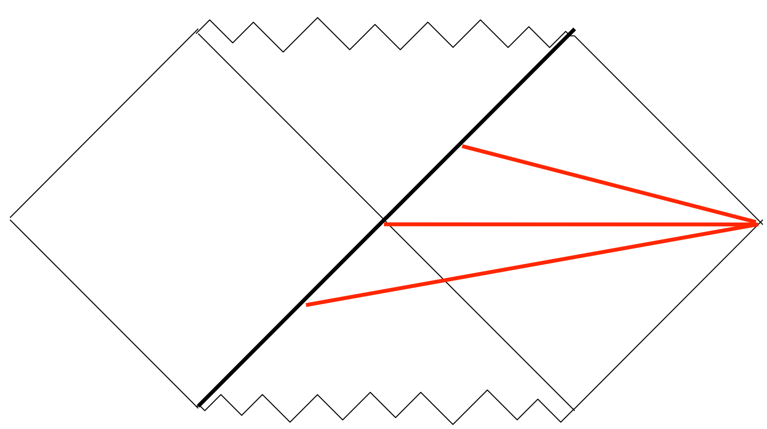
Spacetime has a Killing boost symmetry only about the bifurcation surface.

No null translation Killing symmetry.

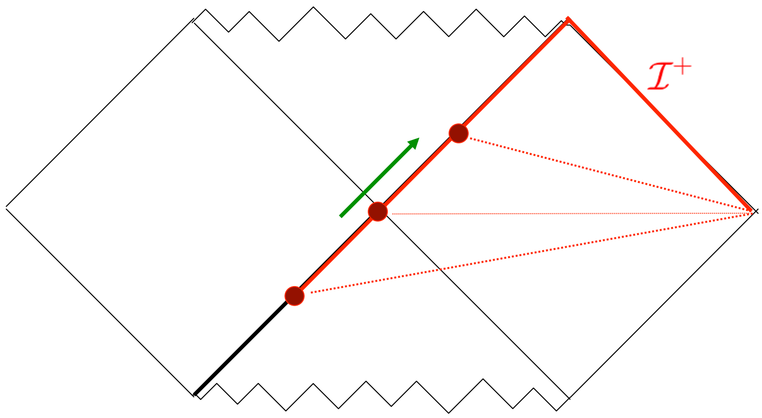
Kerr is even worse because no thermal Hartle-Hawking state exists at all (angular momentum is unbounded below).

Proof does not work—take near-horizon limit?

Instead of using spacelike slices:



Push forward to the horizon itself



The horizon has translation symmetry (in fact, independently on each generator!), even though the full spacetime does not.

Horizon Algebra $\mathcal{A}(H)$

Wall's generalization of the proof to arbitrary cuts of arbitrary horizons
Assumes the QFT algebra can be restricted to the horizon and obeys:

- **Determinism** – taken together with the algebra at future null infinity it determines the entire bulk algebra
- **Ultralocality** – operators on different horizon generators commute
- **Local Lorentz Symmetry** – invariance under independent translations and boosts of each generator
- **Stability** – generator of translations along horizon has non-negative spectrum (quantum ANEC holds on each horizon generator)

Wall *proves* these hold for free fields of any spin, 1+1 CFT's, and for Interacting fields in tree-level perturbation theory.

Restrict fields to horizon algebra

Possible to restrict free fields operators to the event horizon itself.
Tricky since fields must be smeared only across the horizon.
One finds that:

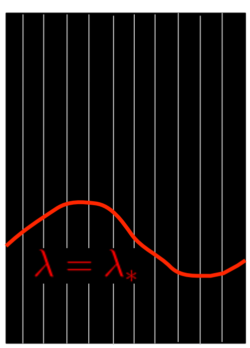
* Φ can't be restricted, but $\nabla_k \Phi$ can be. Normally derivatives hurt but in the field is already smeared in the k direction, and null mass shell tells us that $\nabla_+ \Phi \sim p_+ \Phi \sim \frac{1}{p_-}\Phi$ so it actually helps.

* The horizon algebra is ultralocal; each horizon generator is independent.

* There is an infinite dimensional symmetry group: translations and dilations of each horizon generator *independently*. (boosts = dilations on the horizon)

* Can accommodate arbitrary (nonderivative) potentials $V(\Phi)$ at the level of naïve Fock space perturbation theory. (No effect on horizon algebra.)

Arbitrary horizons, arbitrary slices:



a piece of a horizon, with grey generators

Because each horizon generator can be independently translated, can translate to wiggly slices.

Can define a canonical vacuum state w.r.t. all null translation symmetries (Sewell 81).

σ is KMS above *any* slice w.r.t. dilations about that slice. Formally,

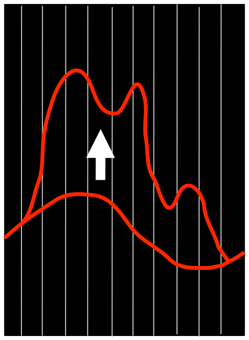
$$\sigma_{\lambda > \lambda_*} = e^{-2\pi K(\lambda_*)/\hbar}$$

where

$$K(\lambda_*) = \int_{\lambda_*}^{\infty} T_{kk}(\lambda - \lambda_*)\, d\lambda$$

This works on any background with a stationary horizon even when no Hartle-Hawking state can be defined on the bulk spacetime (e.g. Kerr).

Adapting the Proof of the GSL



a piece of a horizon, with grey generators

GSL can now be proven analogously to Rindler case for semi-classical perturbations to any stationary horizon:

Let ρ be the bulk state we are interested in, Let σ be a product of the vacuum state on the horizon with an arbitrary faithful state at Scri+. For each slice:

* σ is thermal with respect to $K(\lambda_*)$, which is proportional to the area A of the slice λ_*.

* $S(\rho\,|\,\sigma) = -S_{\rm gen}$ up to additive constants.

* Thus the GSL holds by monotonicity of relative entropy.

Conclusion

The concepts and methods that enter Wall's proof are deeply tied to the foundations of QFT.

The approach is very promising for gaining improved understanding of the role of renormalization in horizon entropy (including higher curvature corrections), and possibly the extension beyond semi-classical gravity.

Moreover, if any step in the argument cannot be proved, it should perhaps be adopted as a postulate of quantum gravity!