Recent advances in the canonical LQG

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Motivation for quantum gravity

...Nevertheless, due to the inner-atomic movement of electrons, atoms would have to radiate not only electro-magnetic but also gravitational energy, if only in tiny amounts. As this is hardly true in Nature, it appears that quantum theory would have to modify not only Maxwellian electrodynamics, but also the new theory of gravitation.

Albert Einstein

Approaches to Quantum Gravity

- Asymptotically Safe Quantum Einstein Gravity,
- Lattice,
- Triangulations,
- Loop Quantum Gravity
- Non Commutative Geometry,
- String Theory.

Quantum Gravity within reach

- The recent advances in loop quantum gravity *Ashtekar - the book; Rovelli - the book; Thiemann - the book; Han, Huang, Ma Int. J. Mod. Phys. D16: 1397-1474 (2007); Ashtekar, L Class. Quant. Grav. 21: R53 (2004);* strongly suggest that the goal of constructing a candidate for quantum theory of gravity and the Standard Model is within reach. Remarkably, that goal can be addressed within the canonical formulation of the original Einstein's general relativity in four dimensional spacetime.

- Physical dynamics and spacetime emerge from general relativity where spacetime diffeomorphisms are treated as a gauge symmetry, due to the framework of the relational Dirac observables *Rovelli the book; Dittrich Class. Quantum Grav. 23, 6155, 2006, Thiemman the book*.

Deparametrization $+$ *LQG*

- The exact and most powerful example of the relational observables is the deparametrization technique *Kijowski, Smolski, Gornicka Phys. Rev. D41 (1990) - perfect fluid; Rovelli, Smolin Phys. Rev. Lett. 72 446 (1993) - scalar field; Brown, Kuchar Phys. Rev. D 51, 5600, (1995) dust*. This allows to map canonical General Relativity into a theory with a (true) non-vanishing Hamiltonian. All this can be achieved at the classical level.

- The framework of loop quantum gravity (LQG) itself, provides the quantum states, the Hilbert spaces, quantum operators of the geometry and fields, and well defined quantum operators for the constraints of General Relativity. This is exactly an ingredient missing in the deparametrization works.

New QFT needed

- Applying LQG techniques to perform the quantization step has the consequence that the quantum fields of the Standard Model have to be reintroduced within the scheme of LQG. The resulting quantum theory of gravity cannot be just coupled to the Standard Model in it's present form. The formulation of the full Standard Model within LQG will require some work. For this reason, we proceed step by step, increasing gradually the level of complexity.

The symmetry reduced models

- The first step was constructing various cosmological models by analogy with LQG by performing a symmetry reduction already at the classical level. They give rise to loop quantum cosmology (LQC) *Bojowald, Class. Quantum Grav. 17, 1489 (2000) - 18, 1071 (2001); Ashtekar, Pawlowski, Singh Phys. Rev. D 74, 084003 (2006); Ashtekar, Corichi, Singh Phys. Rev. D77: 024046 (2008)*. We have learned from them a lot about qualitative properties of quantum spacetime and its quantum dynamics *Kamiński, L, Pawłowski Class. Quantum Grav. 26, 035012 (2009)*.

- There are also more advanced symmetry reduced models includding quasi-local degrees of freedom *Martn-Benito, Mena Marugn, Wilson-Ewing arXiv: 1006.2369; Gambini, Pullin, Rastgoo Class. Quant. Grav. 26: 215011 (2009)*

Gravity quantized

- The knowledge we gained by considering the symmetry reduced models is very useful in performing the second step, that is introducing quantum models with the full set of the local gravitational degrees of freedom.

- The first quantum model of the full, four dimensional theory of gravity was obtained by applying the LQG techniques to the deparametrized model of gravity coupled to dust *Giesel, Thiemann Class. Quantum Grav. 27 175009 (2010)*.

- The second model is application of LQG to the deparametrized model of gravity coupled to a massless scalar field *Domagala, Giesel, L, Kaminski Phys.Rev.D82:104038,2010* . This is an exact generalization of the symmetry reduced models of LQC to the local degrees of freedom.

Plan of the talk

- Gauge symmetries of general relativistic theories
- The canonical GR + matter fields.
- The tools provided by LQG: the Hilbert spaces, quantum representations, infinity free regularizations of quantum operators of geometry, of the constraints, of the matter hamiltonians, the habitat.
- Background manifold versus purely combinatorial formulation
- LQG and massless scalar field
- Summary

The canonical gravity and matter fields

- The canonicaly conjugate coordinate and momentum variables defined on a 3-manifold Σ in local corditaes $x^a = x^1, x^2, x^3$:
  - $(\, A^i_a,\ E^a_i \,)\ \ a, b = 1, 2, 3, i = 1, 2, 3$ - the gravitational field, the Ashtekar-Barbero variables
  - $(\, \phi_I,\ \pi^I \,)\ I = 1, ...$ - the matter fields
  - $N(t, x),\ N^a(t, x)$ the laps and shift functions
- The constraints
  - $C^{\rm tot}(x) := C^{\rm gr} + C^{\phi} \ = \ 0$ scalar constraint
  - $C^{\rm tot}_a(x) \ = \ 0$ vector constraint - Diff(Σ) generators
  - ${C^{\rm tot}_{\rm Gauss}}^i(x) = 0$ the Gauss contraint - the frame rotation generators
  - $C^{\rm tot}_{\rm other}$ other matter constraints (for example the generators of the Maxwell/Yang-Mills field gauge transformations)
- Recent breakthrough: Generalization to $D > 3$ dimensional Σ *(Bodendorfer, Thiemann, Thurn)*: new simplicity constraints
- Dirac observables: $\mathcal{O}$ such that $\{\mathcal{O}, C\} = 0$, restricted to the solutions to the constraints. No natural dynamics. Extra input needed.

The relation with gravity

$\tau_1, \tau_2, \tau_3 \in$ su(2) are such that

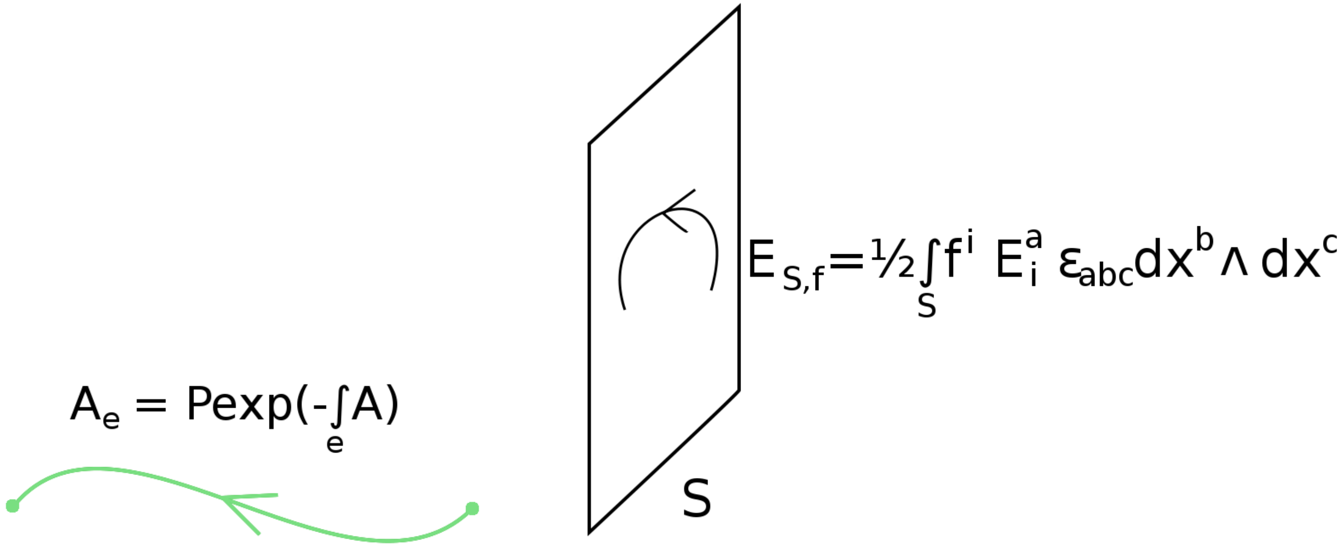
$$\eta(\tau_i, \tau_j) \ := \ -2\mathrm{Tr}(\tau_i \tau_j) = \delta_{ij}, \qquad (1)$$

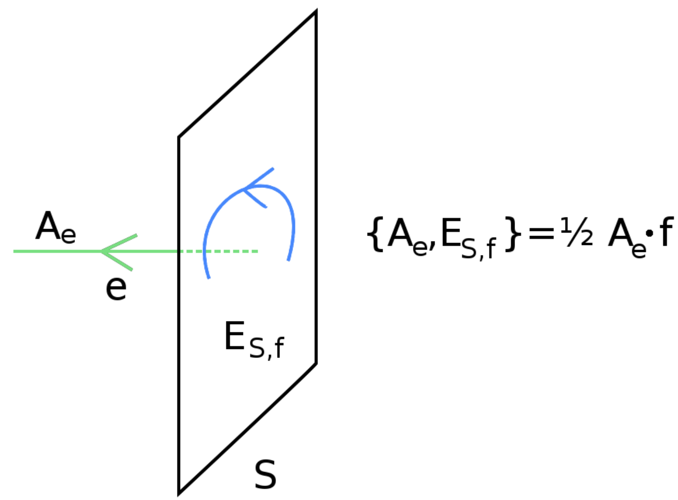
and τ^1, τ^2, τ^3 is the dual basis. The relation with the intrinsic/extrinsic geometry e^i_a / K^i_a of Σ is

$$A^i_a \ = \ \Gamma^i_a + \gamma K^i_a, \qquad E^a_i \ = \ \frac{1}{16\pi G\gamma} e^j_b e^k_c \epsilon^{abc} \epsilon_{ijk}, \qquad (2)$$

where $\epsilon^{123} = 1 = \epsilon_{123}$ and ϵ^{abc}, ϵ_{ijk} are completely antisymmetric.

The holonomy-flux variables





The holonomy and flux variables generate a Lie algebra **HF**.

*The Quantum HF *-algebra*

The abstract Quantum Holonomy-Flux algebra **QHF** *(Sahlmann 2000)* is the *-algebra with unity obtained from the complexification of the classical holonomy-flux algebra and by the replacement

$$\{\cdot,\cdot\} \mapsto \frac{1}{i\hbar}[\cdot,\cdot].$$

There is a unique Diff(M) invariant state *(L, Okolow, Sahlmann, Thiemann 2005)*

$$\omega \ : \ \mathbf{QHF} \ \to \ \mathbb{C}.$$

Through the GNS construction it defines a quantum representation of the parallel transport functionals and the flux functionals as operators $\hat{A}_e{}^A_B$, $\hat{E}_{S,f}$, in the resulting Hilbert space $\mathcal{H}_{\rm gr}$, and $|0\rangle \in \mathcal{H}_{\rm gr}$ such that:

$$\langle 0|\hat{E}_{S,f}...\hat{A}_e^M{}_N|0\rangle \ = \ \omega(\hat{E}_{S,f}...\hat{A}_e^M{}_N)$$

Notable analogies: the von Neumann's uniqueness theorem for Quantum Mechanics, the unique Poincare invariant vacuum in QFT.

More about the state

$$\langle 0| \ \cdot \ \hat{E}_{S,f}|0\rangle \ = \ 0$$

$$\langle 0|\psi(\hat{A}_e)|0\rangle \ = \ \int_{\bar{\mathcal{A}}} d\mu(\bar{A})\psi(\bar{A})$$

where the commutative subalgebra

$$\mathrm{Cyl}$$

of the Quantum HF algebra generated by the connection variables

$$\hat{A}_e{}^M{}_N$$

defines a C* algebra *(Ashtekar-Isham,Baez)*, whose Gelfand-Naimark spectrum,

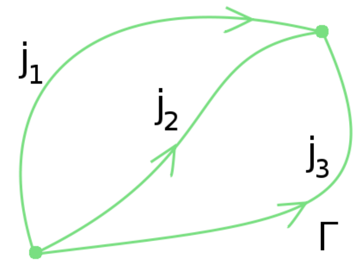
$$\bar{\mathcal{A}}$$

can be endowed with a Diff(Σ) invariant measure *(Ashtekar,L)*

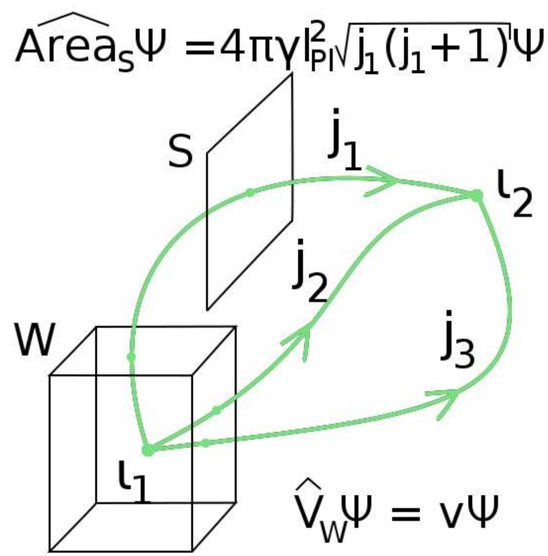
$$\mu_0$$

Quantum Geometry *(Rovelli,Smolin,Ashtekar,L)*

$$\mathcal{H}_{\rm gr} = \bigoplus_{\Gamma} \mathcal{H}_{{\rm gr},\Gamma} = \bigoplus_{\Gamma,j} \mathcal{H}_{{\rm gr},\Gamma,j}$$



Γ - graph in M, j- labelling with irreducible representations of SU(2)



No divergences in the construction of the geometry operators.

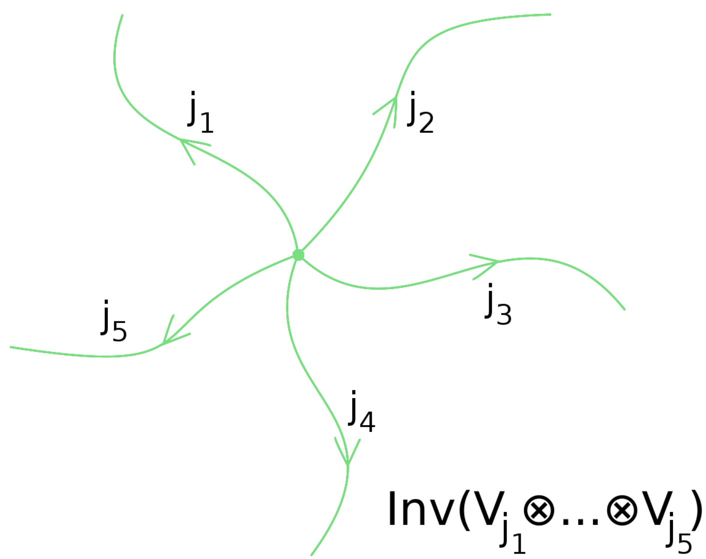
The YM gauge transformations

The Yang-Mills gauge transformations:

$$\hat{A}_e \mapsto \; g(\text{end of } e)^{-1} \hat{A}_e g(\text{begining of } e),$$

$$\hat{E}_{S,f} \;\mapsto\; \hat{E}_{S,g^{-1}fg}$$

The invariant elements in $\mathcal{H}_{\text{gr}}$ are defined by the invariants of the product of the representations $\rho_{j_1} \otimes ... \otimes \rho_{j_m}$: the spin networks *(Penrose, Rovelli, Smolin, Baez)*.



The Diff(Σ) gauge transformations

- Diffeomorphisms of Σ act naturally in $\mathcal{H}_{\rm gr}$,

$$u : {\rm Diff}(\Sigma) \ \to \ U(\mathcal{H}_{\rm gr}).$$

- A diffeomorphism averaging map $\eta_{\rm diff}$ maps elements of $\mathcal{H}_{\rm gr}$ into diffeomorphism invariant distributions - solutions to the diffeomorphism constraint *(Ashtekar, L, Marolf, Mourao, Thiemann)*. It is defined in a domain ${\rm Cyl} \subset \mathcal{H}_{\rm gr}$,

$$\eta_{\rm diff} : \mathcal{H}_{\rm gr} \supset {\rm Cyl} \ \to {\rm Cyl}^*$$

$$\eta(|\Psi\rangle) \ := \ \text{“}\int_{\rm Diff} d\mu(\varphi)\text{''} \langle\Psi| u(\varphi)^*.$$

- The resulting distributions are normalizable in a new scalar product

$$(\eta(\Psi)|\eta(\Psi''))_{\rm diff} \ := \ \langle\eta(\Psi), \Psi'\rangle,$$

and set a Hilbert space $\mathcal{H}_{\rm gr,diff}$ of solutions.

The habitat space

The span of the states $\langle\Psi|$ averaged with an extra weight factor f which depends smoothly on location of the vertices of Ψ in Σ *(L,Marolf)* ,

$$\eta(f, |\Psi\rangle) \ := \ \text{``}\int_{\mathrm{Diff}} d\mu(\varphi)\text{''} f \langle\Psi| u(\varphi)^* \ \in \ \mathrm{Cyl}^*$$

Useful for the off-shell properties.

What diffeomorphisms?

- ‘Entire’ diffeomorphisms versus diffeos ‘modulo’ a set of points:
  - Entire (“piecewise analytic”) diffeomorphisms, typical for the canonical LQG: the graphs endowed with the differential structure at the nodes, and the linking/knoting characteristics, $\mathcal{H}_{\mathrm{gr,diff}}$ unseparable.
  - Diffeomorphisms analytic, modulo a finite set of points *(Rovelli)*: graphs endowed with the linking/knoting characteristics, no diff structure, the Hilbert space $\mathcal{H}_{\mathrm{gr,diff}}$ separable *(Rovelli)*.
- Replacing ‘analytic’ by ‘smooth’ *(Baez-Sawin, L-Thiemann, Fleischhack)*:
  - Webs, that is generalized graphs infinitely many times self overlaping
  - The LQG framework is extendable includding the measure, the Hilbert spaces, the operators.
  - Infinities caused by the infinite number of isolated intersections.
- Occasionally, the diffeomorphisms are restricted to be ‘preserving’ a finite set of points.

‘Combinatorial’ versus ‘combinatorial and more’ LQG

- Combinatorial: just graphs, colored by the representations and Hilbert spaces, the analog of $\mathcal{H}_{\rm gr,diff}$, can be defined in a purely combinatoric way *(Rovelli)*, the combinatorial volume operator $\hat{V}_{\rm RS}$ and other quantum operators, problems with the consistency of the volume *(Giesel, Thiemann)*, a classical differential manifold structure emerges from the quantum theory together with geometry, the topology is that of the graphs.
- Combinatorial and more: the graphs endowed with differential structure at the vertices and the linking/knoting characteristics, $\mathcal{H}_{\rm gr,diff}$ unseparable, can be defined purely from the graphs and their additional structure, the differentiable structure sensitive volume operator $\hat{V}_{\rm ALGT}$, consistent with the identities following from the classical theory, a classical geometry emerges from the quantum geometry, topology and the differential structure is that of the graphs.

Matter fields compatible with LQG: example

Scalar field defined on Σ: $\{\phi(x), \pi(x)\} \;=\; \delta(x,y)$
The polymer variables compatible with LQG:

$$N_{\lambda,x}(\phi) \;=\; \exp(i\lambda\phi(x)), \quad \pi(f) \;=\; \int_\Sigma \pi f$$
$$\{N_{\lambda,x}, \pi(f)\} \;=\; i\lambda f(x) N_{\lambda,x}.$$

The quantum *-algebra:

$$[\hat{N}_{\lambda,x}, \hat{\pi}(f)] \;=\; -\lambda f(x)\hat{N}_{\lambda,x}$$

A unique Diff(Σ) invariant state:

$$\omega(\hat{a}) \;=\; 0 \;\; \text{unless} \;\; \hat{a} = 1$$

The resulting Hilbert space:

$$\mathcal{H}_{\mathrm{Pol}} \;=\; \mathrm{L}^2(\mathbb{R}_{\mathrm{Bohr}}{}^{\Sigma},\; \mu_{\mathrm{Haar}})$$

.
Now, the Diff(Σ) averaging is applied to $\mathcal{H}_{\mathrm{gr}} \otimes \mathcal{H}_{\mathrm{Pol}}$.

The tools provided by LQG

- the kinematical Hilbert space for the geometric degrees of freedom $\mathcal{H}_{\rm gr}$, analogous Hilbert space for other gauge fields, the polymer Hilbert space for non-gauge fields;
- the quantum operators of the geometry of Σ and of the matter fields (no divergences);
- the quantum operators of the Gauss constraint and the Hilbert space $\mathcal{H}_{\rm gr,Gauss}$ of the invariants (the same with matter);
- the unitary action of the diffeomorphism groups Diff(Σ) and the Hilbert spaces $\mathcal{H}_{\rm gr,diff}$ of the invariant distributions (the same with matter);
- the quantum scalar constraint operators $\hat{C}^{\rm gr}$ defined in $\mathcal{H}_{\rm gr,diff}$, the quantum scalar constraint operators for matter fields, (no divergences - the same with matter) *(Thiemann)*.

GR+massless scalar field

Fields A^i_a, E^a_i, ϕ, π defined on a 3-manifold Σ.

$$\{A^i_a(x), E^b_j(y)\} \;=\; \delta(x,y)\delta^b_a\delta^i_j, \quad \{\phi(x),\pi(y)\} \;=\; \delta(x,y). \tag{3}$$

The constraints:

$$C(x) \quad = \quad C^{\rm gr}(x) \;+\; \frac{1}{2}\frac{\pi^2(x)}{\sqrt{q(x)}} + \frac{1}{2}q^{ab}(x)\phi_{,a}(x)\phi_{,b}(x)\sqrt{q(x)} \tag{4}$$

$$C_a(x) \quad = \quad C^{\rm gr}_a(x) \;+\; \pi(x)\phi_{,a}(x). \tag{5}$$

Replace $C(x)$ by $C'(x)$ (deparametrised scalar constraint):

$$C'(x) \quad = \quad \pi(x) - h(x), \tag{6}$$

$$h_{\pm,\pm} \quad := \quad \pm\sqrt{-\sqrt{q}C^{\rm gr} + / - \sqrt{q}\sqrt{(C^{\rm gr})^2 - q^{ab}C^{\rm gr}_a C^{\rm gr}_b}}. \tag{7}$$

$$\pi > 0, \;\; C^{\rm gr} \; < \; 0. \tag{8}$$

$$\{C'(x), C'(y)\} \;=\; 0, \;\; \{h(x), h(y)\} \;=\; 0. \tag{9}$$

Quantum solutions and the Dirac observables

$$\Psi \in (\mathcal{H}_{\rm gr} \otimes \mathcal{H}_{\rm Pol})_{\rm diff} \quad \text{or} \quad \Psi \in \left(\text{Cyl}_{\rm gr,Pol}\right)^*_{\rm diff} \tag{10}$$

$$\Psi(\phi) \;=\; e^{i\int d^3x\phi(x)\hat{h}(x)}\psi, \tag{11}$$

where $\left(\text{Cyl}_{\rm gr,Pol}\right)_{\rm diff} \subset (\mathcal{H}_{\rm gr} \otimes \mathcal{H}_{\rm Pol})_{\rm diff}$

$$\psi \in \mathcal{H}_{\rm gr,diff} \tag{12}$$

$$\hat{h}_{\rm phys} \;=\; \int d^3x \sqrt{-2\sqrt{\hat{q}}(x)\hat{C}^{\rm gr}}, \tag{13}$$

and in fact ψ are constructed from elements of the subspace of $\mathcal{H}_{\rm gr,diff,\mathit{x}}$ corresponding to the nonpositive part of the spectrum of $\sqrt{\hat{q}}(x)\hat{C}^{\rm gr}(x)$.
The “physical” Hilbert product in the space of solutions:

$$\left(e^{i\int\hat{\phi}\hat{h}}\psi \,|\, e^{i\int\hat{\phi}\hat{h}}\psi'\right)_{\rm phys} \;:=\; (\psi|\psi')_{\rm gr,diff}. \tag{14}$$

The Dirac observables

The action of a general Dirac observable can be written with a help of an operator $\hat{L}$ defined in $\mathcal{H}_{\mathrm{gr,diff}}$,

$$\mathcal{O}(\hat{L})e^{i\int d^3x\hat{\phi}(x)\hat{h}(x)}\psi \;=\; e^{i\int d^3x\hat{\phi}(x)\hat{h}(x)}\hat{L}\psi \tag{15}$$

Those observables form an algebra

$$\mathcal{O}(\hat{L})\mathcal{O}(\hat{L}') \;=\; \mathcal{O}(\hat{L}\hat{L}') \tag{16}$$

(suppose the operators $\hat{L}$ are bounded).
A 1-dim group of automorphisms labelled by a parameter ϕ_0:

$$\mathcal{O}(\hat{L}) \;\mapsto\; \mathcal{O}_{\phi_0}(\hat{L}) \;=\; \mathcal{O}(e^{-i\int d^3x\phi_0\hat{h}(x)}\hat{L}e^{i\int d^3x\phi_0\hat{h}(x)}) \tag{17}$$

Encodes the dynamics whose generator is the physical hamiltonian

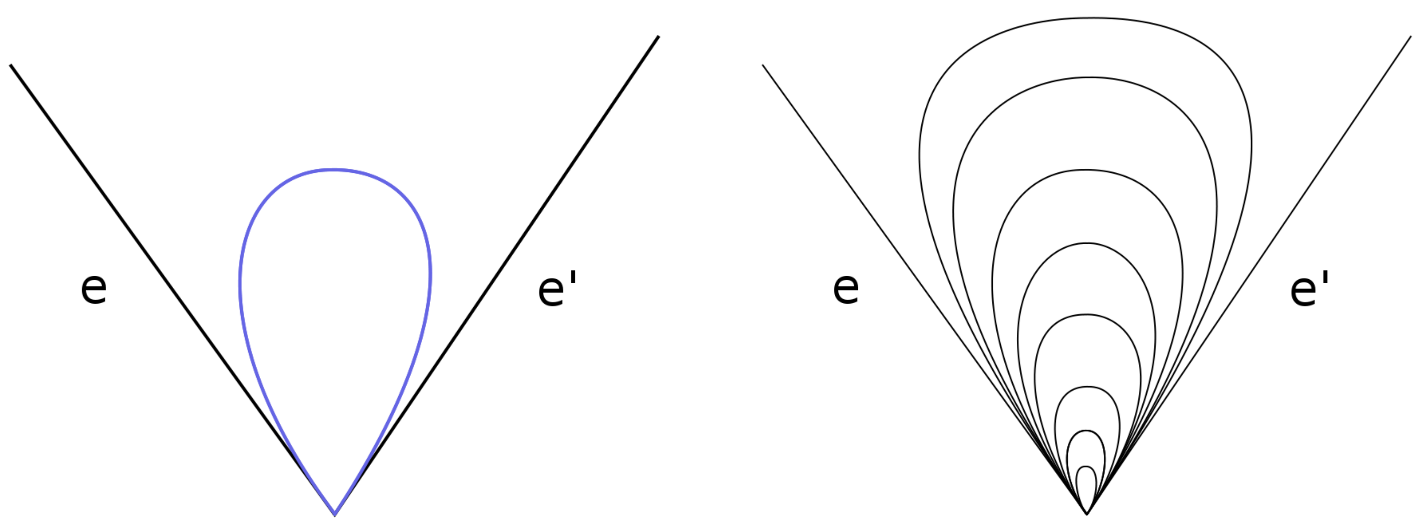
$$\frac{d}{d\phi_0}\mathcal{O}_{\phi_0}(\hat{L}) \;=\; i[\hat{h}_{\mathrm{phys}}, \mathcal{O}_{\phi_0}(\hat{L})] \tag{18}$$

The Hamiltonian

For the simplicity consider first the Euclidean case

$$-2\sqrt{q}C^{\mathrm{gr}} = -2c_k^{ij}F_{ab}^k E_i^a E_j^b \tag{19}$$

No volume operator!
The corresponding symmetric operator creates loops and anihilates. For the commutators to vanish, we are alowed to use only loops



The Lorentzian case

In the Lorentzian case *(Thiemann - the book)*:

$$-2\sqrt{q}C^{\rm gr} \;=\; 2c_k^{ij}F_{ab}^kE_i^aE_j^b + 4qR \tag{20}$$

where only the quantum operator R (the 3-dim Ricci scalar) required a new idea *(Domagala, Kaminski)*. The new operator has the form:

$$\widehat{qR} \;=\; f(\widehat{\rm Area}_{S_I}, \hat{V}_{U_J}). \tag{21}$$

Summary

- ..."but there is no quantum gravity"... Wrong!
- The canonical LQG provides more and more soluble models of quantum gravity with all the local degrees of freedom. The first model was LQG coupled to dust (Giesel-Thiemann). The current second model describes gravity coupled to massless scalar field. It is an exact generalization of the cosmological models of Loop Quantum Cosmology to the full theory with the local gravitational degrees of freedom.
- With this new model we can address the issues of general relativity which were analysed with the symmetry reduced LQC, namely:
  - The Big-Bang
  - The gravitational collaps
- Thank You and do not miss Simone Speziale's talk!