Extended Field Theories and Higher Gauge Theory

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Context: "Categorify" quantum mechanical description of states and processes.

We propose to represent:

- configuration spaces of physical systems by n-groupoids (or *n-stacks*), based on local symmetries
- process relating two systems through time by a **span** of groupoids, including a groupoid of "histories"
- higher spans for composition of systems

This can be represented in **Hilb** by "degroupoidification" (Baez/Dolan). We'll look for "higher" analogs.

Definition

A **groupoid G** is a category in which all arrows are invertible.

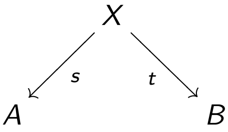
- Any group G is a groupoid with one object
- Given a set S with a group-action $G \times S \rightarrow S$ yields a transformation groupoid $S /\!\!/ G$ whose objects are elements of S; if $g(s) = s'$ then there is an arrow $g_s : s \rightarrow s'$
- "Physical" applications of groupoids arise mostly from $S /\!\!/ G$ associated to a G-action on S is a space of configurations.
- Morita equivalent groupoids are "physically indistinguishable". (E.g. full action groupoid; quotient with automorphisms)

Example

Moduli space for *gauge theory*, for (finite) gauge group G. Given M, the groupoid $\mathcal{A}_0(M, G) = hom(\pi_1(M), G) /\!\!/ G$ has:

- **Objects**: Flat connections on M
- **Arrows** Gauge transformations

A *span* of groupoids is a diagram:



whose arrows are groupoid homomorphisms (i.e. functors between groupoids).

In a span $A \leftarrow X \rightarrow B$, think of X as a space of *histories*; intuitively s and t pick the starting and terminating configuration in spaces A and B.
Fact: There's an induced map: $\mathcal{A}_0(-, G) : \mathbf{nCob} \rightarrow Span(\mathbf{Gpd})$, where the legs of the span are *restriction to the boundary*.

Definition

An n-dimensional Topological Quantum Field Theory is a monoidal functor

$$Z : \mathbf{nCob} \to \mathbf{Hilb}$$

where **nCob** has

- **Objects**: $(n-1)$-dimensional manifolds
- **Arrows**: n-dimensional cobordisms (manifolds with boundary, with ∂M a union of source and target objects)

So Z assigns Hilbert spaces to manifolds, linear maps to cobordisms (think of these as "spacetimes" connecting "space slices"). To a closed manifold, it assigns the *partition function* $Z(M)$.
We get a TQFT Z_G from $\mathcal{A}_0(-, G)$ using:

$$D : Span(\mathbf{Gpd}) \to \mathbf{Hilb}$$

(Baez/Dolan "degroupoidification")

For a groupoid **A**, assign the vector space of *equivariant functions* on the objects of **A** (or functions on *isomorphism classes* of **A**).
The standard inner product on $D(G)$ makes the $\delta_{[a]}$ orthogonal with length $\frac{1}{\# \mathrm{Aut}(a)}$. (For various good reasons.)
Then there is a pair of linear maps associated to a groupoid homomorphism $f : A \rightarrow B$:

- $f^* : \mathbb{C}^B \rightarrow \mathbb{C}^A$, with $f^*(g) = g \circ f$
- $f_* : \mathbb{C}^A \rightarrow \mathbb{C}^B$, adjoint to f^*

These adjoint maps "pull" and "push" functions.
Then for a span we get a "pull-push" map:

$$D(X, s, t)(g)([b]) = \sum_{[x] \in \underline{t^{-1}(b)}} \frac{\# \mathrm{Aut}(b)}{\# \mathrm{Aut}(x)} [g(s(x))]$$

(If a history x carries an action $S(x)$, we can modify this sum.)

Motivation: A TQFT assigns a number $Z(M) \in \mathbb{C}$ to a closed n-manifold, and a Hilbert space $Z(B) \in \mathbf{Hilb}$ to a codimension-1 boundary. What does it assign in codimension 2, 3... and to a point?

Starting point:

Definition

An Extended (Topological) Field Theory is a monoidal 2-functor

$$Z : \mathbf{nCob_2} \to \mathbf{2Hilb}$$

where $\mathbf{nCob_2}$ has

- **Objects**: $(n-2)$-dimensional manifolds
- **Arrows**: $(n-1)$-cobordisms
- **2-Cells**: n-cobordisms with corners

We can say roughly:

Definition

A 2-Hilbert space (cf. Baez) is an abelian $H^\star$-category.

That is, 2-Hilbert spaces have:

- a "direct sum" $\oplus$
- $hom(x, y) \in \mathbf{Hilb}$ for objects x and y
- a "star structure":

$$\mathrm{hom}(x, y) \cong (\mathrm{hom}(y, x))^*$$

which we think of as finding the "adjoint of an arrow".

A **2-linear map** is a functor preserving all this structure.
There are **natural transformation** between 2-linear maps.
These form the 2-category **2Hilb**.

Conjecture (Baez/Baratin/Freidel/Wise)

Any 2-Hilbert space is of the following form: **Rep(A)***, the category of representations of a von Neumann algebra A on Hilbert spaces. The star structure takes the adjoint of a map.*

Example

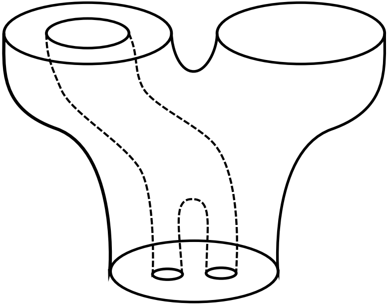
The 1-dimensional 2-Hilbert space is the category $\mathbf{Hilb} = \mathbf{Rep}(\mathbb{C})$.

Example

If $\mathbf{B}$ is a finite groupoid, the $\mathbf{Rep(B)}$ is a 2-Hilbert space, since $\mathbb{C}[\mathbf{B}]$ is a von Neumann algebra.

The "basis elements" (generators) of $\mathbf{Rep(B)}$ are labeled by $([b], V)$, where $[b]$ is an iso. class of objects in $\mathbf{B}$ and V an irreducible rep of $Aut(b)$.

To get an ETQFT, use the fact that cobordisms are actually **co**spans of manifolds (with corners):

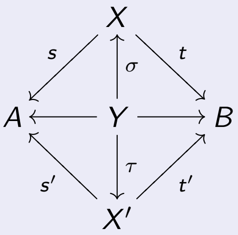
nCob$_2$	$Span^2(ManCorn)$
	$\begin{array}{ccccc} S^1 & \xrightarrow{i_A} & (A\coprod D) & \xleftarrow{i'_A\otimes i_D} & S^1\coprod S^1 \\ \downarrow i_1 & & \downarrow \iota_1 & & \downarrow i_2 \\ Y & \xrightarrow{\iota_3} & M & \xleftarrow{\iota_4} & Y \\ \uparrow i_2 & & \uparrow \iota_2 & & \uparrow i_1 \\ S^1\coprod S^1 & \xrightarrow[i_2]{} & Y & \xleftarrow[i_1]{} & S^1 \end{array}$

Applying $\mathcal{A}_0(-, G)$ to this gives spans of spans of groupoids.

The bicategory $Span_2(\mathbf{Gpd})$ has:

Definition (Part 1)

- **Objects**: Groupoids
- **Arrows**: Spans of groupoids
- Composition defined by "weak pullback" (a kind of gluing):
- tensor product from the product in **Gpd**
- **2-cells** (iso. classes of) spans of *span maps*:



Theorem

If $\mathbf{X}$ *and* $\mathbf{B}$ *are (reasonably nice) groupoids, a functor* $f : \mathbf{X} \to \mathbf{B}$ *gives a pair of 2-linear maps:*

$$f^* : \Lambda(\mathbf{B}) \to \Lambda(\mathbf{X})$$

with $f^* F = F \circ f$ *and (the restricted representation along* f*)*

$$f_* : \Lambda(\mathbf{X}) \to \Lambda(\mathbf{B})$$

the induced representation of F *along* f*.*

These are "adjoints" in the sense of maps between 2-Hilbert spaces. (The "inner product" is $\langle x, y \rangle = hom(x, y) \in \mathbf{Hilb}$, which takes values in the 1-dimensional 2-Hilbert space!)

In fact, the map f_* acts by:

$$f_*(F)(b) \cong \int_{f(x)\cong b}^{\oplus} \mathbb{C}[Aut(b)] \otimes_{\mathbb{C}[Aut(x)]} F(x)$$

(a direct sum/integral of induced representations), or also:

$$f_!(F)(b) \cong \int_{[x]|f(x)\cong b}^{\oplus} \hom_{\mathbb{C}[Aut(x)]}(\mathbb{C}[Aut(b)], F(x))$$

via the canonical *Nakayama isomorphism*:

$$N_{(f,F,b)} : f_!(F)(b) \rightarrow f_*(F)(b)$$

given by the *exterior trace map* (which uses a modified group average in each factor):

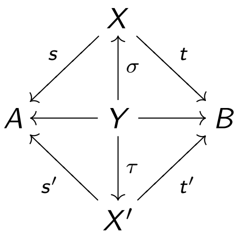
$$N : \int_{[x]|f(x)\cong b}^{\oplus} \phi_x \mapsto \int_{[x]|f(x)\cong b}^{\oplus} \frac{1}{vol(Aut(x))} \int_{g\in Aut(b)} g \otimes \phi_x(g^{-1})$$

The above can be summarized by saying f^* and f_* are "ambidextrous adjoints". There are maps between $F(x)$ and $f_* f^* F(x)$:

$$\eta_R(G)(x) : v \mapsto \int_{y|f(y)\cong x}^{\oplus} (g \mapsto g(v))$$

$$\epsilon_L(G)(x) : \int_{[y]|f(y)\cong x}^{\oplus} g_y \otimes v \mapsto \int_{[y]|f(y)\cong x} f(g_y)v$$

Use these to "pull" and "push" through the 2-cells:



Definition

Define the 2-functor Λ

$$\Lambda : Span_2(\mathbf{Gpd}) \rightarrow \mathbf{2Hilb}$$

as follows:

- Objects: $\Lambda(\mathbf{B}) = Rep(\mathbf{B}) := [\mathbf{B}, \mathbf{Vect}]$
- Arrows $\Lambda(X, s, t) = t_* \circ s^* : \Lambda(\mathbf{A}) \longrightarrow \Lambda(\mathbf{B})$
- 2-Cells: $\Lambda(Y, \sigma, \tau) = \epsilon_{L,\tau} \circ N \circ \eta_{R,\sigma} : (t)_* \circ (s)^* \rightarrow (t')_* \circ (s')^*$

Remark: The effect on arrows and 2-cells are both "pull-push" processes, of representations and intertwiners, respectively. When $\mathbf{A}$ and $\mathbf{B}$ are both 1 (so $Rep(\mathbf{A} = \mathbf{Hilb})$, this is *exactly* the Baez/Dolan degroupoidification (so gives the same TQFT).

- Physically, $A = \mathbb{C}[\mathbf{B}]$ is the algebras of **symmetries** of a system with configuration groupd **B**.
- The algebra of **observables** will be its commutant - (which depends on the choice of representation!)
- Basis elements are irreducible representations of the vN algebra - physically, these can be interpreted as **superselection sectors**. Any representation is a *direct sum/integral* of these.
- Then 2-linear maps are functors... given by tensoring with **Hilbert bimodules** between algebras. (When groupoids are trivial, this is a $\mathbb{C} - \mathbb{C}$ Hilbert bimodule: a Hilbert space.)
- The simple components of these bimodules are built from the matrix entries

$$\Lambda(X, s, t)_{([a],V),([b],W)} \simeq \int_{[x] \in \underline{(s,t)^{-1}([a],[b])}}^{\oplus} \hom(s^*(V), t^*(W)) \quad (1)$$

(by tensoring on left and right with V and W)

Example

Interesting case is $G = SU(2)$. The topology generates measurable sets to make $SU(2)$ a regular Borel space, with Haar measure μ.
The groupoid

$$\mathcal{G} = A_{SU(2)}(S^1) = SU(2) /\!\!/ SU(2)$$

gets a measure from Haar measure on $SU(2)$ (to define the groupoid von Neumann algebra).
We can get reps of $\mathcal{G}$ by integrating those indexed by $([g], V)$ for $g \in SU(2)$ and V an irrep of $Stab(g)$ ($SU(2)$ or $U(1)$).

Higher gauge theory: for a 2-group $\mathcal{G}$, define a 3-functor $Z_{\mathcal{G}} : \mathbf{nCob_3} \to \mathbf{3Hilb}$.

Definition

A 2-group is a 2-category with one object, and all arrows and 2-cells invertible.

But concretely, they're realized by *crossed modules*, which have:

- Groups G, H
- A map $\partial : H \to G$
- An action $G \rhd H$

Satisfying some relations.

Example

The **Poincaré 2-Group** has $G = SO(3,1)$, $H = R^{3,1}$, $partial = 1$ (the constant map), and $G \rhd H$ in the canonical way.

Think of H as the group of *automorphisms of* $1 \in G$.

Definition

Fixing a 2-group $\mathcal{G}$, the contravariant 2-functor

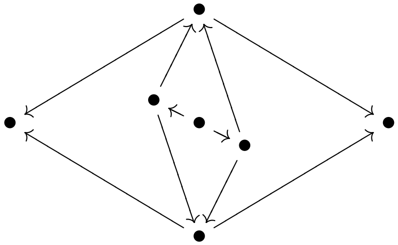
$$\mathcal{A}_0^{(2)} = 2Fun[\Pi_2(-), \mathcal{G}]$$

assigns to a manifold M the 2-groupoid $\mathcal{A}_0^{(2)}(M)$ with:

- Objects: 2-functors ("2-connections")
- Arrows: natural transformations ("gauge transformations")
- 2-Cells: modifications (...)

A 2-connection defines holonomies along paths *and surfaces*, valued in parts of the 2-group.

There's an induced map $Span_3(\mathbf{ManCorn}) \to Span_3(\mathbf{2Gpd})$, where $Span_3(-)$ has, as 3-cells, equivalence classes of diagrams shaped like:



(2)

Composition is again by weak pullback. (Note that 2-cells and 3-cells of **2Gpd** can appear in $Span_3(\mathbf{2Gpd})$ by weakening the assumption that this commutes.)

As before, $nCob_3$ lives in $Span^3(\mathbf{ManCorn})$.

We would like to define an extended TQFT via a 3-functor:

$$\Lambda^{(2)} : Span_3(\mathbf{2Gpd}) \to \mathbf{3Hilb}$$

using an extended version of the "pull-push" construction.

- **Objects**: $\Lambda^{(2)}(\mathcal{X}) = Rep(\mathcal{X})$
- **Arrows**: Pull-push 2-group representations (where push is "induced 2-group representation along $\mathcal{F}$")
- **2-Cells**: Pull-push 1-intertwiners
- **3-Cells**: Pull-push of "2-intertwiners"

(Though note the definition of **3Hilb** is still somewhat unclear. But $Rep(\mathcal{X})$ should certainly be an example.)

Irreducible representations of 2-groupoid $\mathcal{G}$ should be labelled by:

- A class $[y]$ of object in $\mathcal{G}$
- An irreducible representation of the 2-group $Aut(y)$

Theorem (BBFW)

An irreducible representation of a 2-group given by $(G, H, \triangleleft, \partial)$ is described by:

- *An space X, with action $X \triangleleft G$ of the group of objects*
- *A G-equivariant field of H-characters on X (supported on an orbit of $X \triangleleft G$*

Eventually: One hopes this pattern will repeat with representations of n-groupoids for all n.
Then we can say what the field theory "assigns to a point".

Note: For 2-groups, we have irreducible *representations*, but also irreducible *intertwiners*.

Puzzle: If an irreducible group(oid) representation is a superselection sector, what is an irreducible 2-group(oid) representation?
(Guess: a sector for a theory on the boundary of the codimension-3 surfaces. Irreducible intertwiners should define sectors for the codimension-2 surfaces.)