Toward the consistency of $\mathcal{N} = 8$ supergravity as a quantum field theory

Guillaume Bossard

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Quantum gravity

General relativity is non-renormalisable ($[G] = -2$)

$\hookrightarrow$ Infinity of undetermined parameters

$$e^{-1}\mathcal{L}^\flat = R + a_2(C^3 + \bar{C}^3) + a_3 C^2 \bar{C}^2 + b_3(C^4 + \bar{C}^4) + \cdots$$

associated to $\ln \frac{\Lambda}{\mu}$ divergences in perturbation theory.

Perturbation theory is not predictive!!

$\hookrightarrow$ Non-perturbative inputs required to fix ambiguities.

$\mathcal{N} = 8$ supergravity

A candidate model for quantum gravity

- ⋆ Large symmetry group: local supersymmetry $\times E_{7(7)}$
- ⋆ Amplitudes determined by unitarity cuts
- ⋆ Non-perturbative definition through string theory

$\mathcal{N} = 8$ supergravity

Direct implications of supersymmetry on divergences

$$e^{-1}\mathcal{L}^{\flat} = R + a_3 C^2 \bar{C}^2 + a_5 (\nabla C)^2 (\nabla \bar{C})^2 + \cdots$$

Implications of supersymmetry and $E_{7(7)}$ symmetry

$$e^{-1}\mathcal{L}^{\flat} = R + a_7 (\nabla^2 C)^2 (\nabla^2 \bar{C})^2 + \cdots$$

Can one determine all the ambiguities by symmetry?

$\mathcal{N} = 8$ supergravity

$O(6,6,\mathbb{Z})$ perturbative type II String theory on $\mathbb{R}^{1,3} \times T^6$

$\underset{\text{non-perturbative}}{\hookrightarrow}$ $E_{7(7)}(\mathbb{Z})$ *U*-duality

$\ell_s \to 0$

$R \to 0$

$$\kappa^2 = 8\pi {g_4}^2 {\ell_s}^2 \quad \left(= 8\pi \frac{{\ell_s}^6 {g_s}^2}{R^6} {\ell_s}^2 \right)$$

$E_{7(7)}$ invariant $\mathcal{N} = 8$ supergravity $\underset{\text{non-perturbative quantisation}}{\longrightarrow}$ $E_{7(7)}(\mathbb{Z})$

Outline

- Duality invariance: $E_{7(7)}$ invariant supergravity
- $\mathfrak{e}_{7(7)}$ current anomaly: Perturbative symmetry
- Supersymmetric counter-terms: Implications on log divergences
- Conclusion and outlook

[G. Bossard, C. Hillmann and H. Nicolai, 1007.5472]
[G. Bossard, P. S. Howe and K. S. Stelle, 1009.0743]
[G. Bossard and H. Nicolai, 1105.1273]
[G. Bossard, P. S. Howe, K. S. Stelle and P. Vanhove, 1105.6087]

$SU(8)$ invariant free theory

$\mathcal{N} = 8$ asymptotic states

h	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
$\mathfrak{su}(8)$	$\mathbf{1}$	$\overline{\mathbf{8}}$	$\overline{\mathbf{28}}$	$\overline{\mathbf{56}}$	$\mathbf{70}$	$\mathbf{56}$	$\mathbf{28}$	$\mathbf{8}$	$\mathbf{1}$

$$\bar{C}_{\dot\alpha\dot\beta\dot\gamma\dot\delta} \leftarrow \bar{\rho}_{\dot\alpha\dot\beta\dot\gamma\, i} \leftarrow \bar{F}_{\dot\alpha\dot\beta\, ij} \leftarrow \bar{\chi}_{\dot\alpha\, ijk} \leftarrow \phi_{ijkl} \rightarrow \chi_\alpha^{ijk} \rightarrow F_{\alpha\beta}^{ij} \rightarrow \rho^i_{\alpha\beta\gamma} \rightarrow C_{\alpha\beta\gamma\delta}$$

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56 polarisation vectors e_i^m of the free quantum field

$$A_i^m(x) \equiv \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2|p|}} \sum_\sigma \Big(e^{-ix\cdot p} e^{*\,m}_{\ i}(\sigma,p) a(\sigma,p) + e^{ix\cdot p} e_i^m(\sigma,p) a^\dagger(\sigma,p) \Big) \ ,$$

split into $\mathbf{28} \oplus \overline{\mathbf{28}}$

$$J^m{}_n e_i^n(\sigma,p) \equiv \Omega^{mp} G_{pn} e_i^n(\sigma,p) = ih(\sigma) e_i^m(\sigma,p)$$

$SU(8)$ invariant free theory

e_i^m satisfy

$$\sum_\sigma e_i^m(\sigma,p) e^{*n}_j(\sigma,p) = -\Omega^{mn}\varepsilon_{ijk}\hat{p}^k + G^{mn}(\delta_{ij} - \hat{p}_i\hat{p}_j) \ .$$

Feynman rules are not manifestly Lorentz invariant

$$\left\langle 0 \middle| T\left\{ A_i^m(x)\, A_j^n(y) \right\} \middle| 0 \right\rangle = -i \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot(x-y)}}{{p_0}^2 - p^2 + i\varepsilon} \left(\Omega^{mn}\varepsilon_{ijk}\frac{p_0}{|p|}\hat{p}^k - G^{mn}(\delta_{ij} - \hat{p}_i\hat{p}_j) \right)$$

with the local Lagrangian [M. Henneaux and C. Teitelboim]

$$\mathcal{L}_0 = \frac{1}{2}\Omega_{mn}\varepsilon^{ijk}\partial_0 A_i^m \partial_j A_k^n - G_{mn}\partial_{[i}A_{j]}^m \partial_{[i}A_{j]}^n + b_m \partial_i A_i^m$$

Maxwell equations and Dirac duality

The Bianchi identity implies the existence of the potential

$$\varepsilon^{\mu\nu\sigma\rho}\partial_\nu F_{\sigma\rho} = 0 \rightsquigarrow F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

The vacuum Maxwell equation, of a dual potential

$$\nabla^\nu F_{\mu\nu} = 0 \rightsquigarrow \frac{1}{2}\varepsilon_{\mu\nu}{}^{\sigma\rho}F_{\sigma\rho} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

These equations are $U(1)$ invariant

$$\begin{pmatrix} A'_\mu \\ B'_\mu \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} A_\mu \\ B_\mu \end{pmatrix}$$

Maxwell equations and Dirac duality

The $U(1)$ invariant Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}\varepsilon^{ijk}\left(\partial_0 A_i \partial_j B_k - \partial_0 B_i \partial_j A_k\right) - \partial_{[i}A_{j]}\partial^{[i}A^{j]} - \partial_{[i}B_{j]}\partial^{[i}B^{j]}$$

defines the second order Euler–Lagrange equations

$$\begin{aligned} \varepsilon^{ijk}\partial_j\left(\partial_0 B_k + \varepsilon_k{}^{lh}\partial_l A_h\right) &= 0 \\ \varepsilon^{ijk}\partial_j\left(\partial_0 A_k - \varepsilon_k{}^{lh}\partial_l B_h\right) &= 0 \end{aligned}$$

According to the Poincaré Lemma

$$\begin{aligned} \partial_0 B_k - \partial_k B_0 &= -\varepsilon_k{}^{ij}\partial_i A_j \\ \partial_0 A_k - \partial_k A_0 &= \varepsilon_k{}^{ij}\partial_i B_j \end{aligned}$$

Maxwell equations and Dirac duality

The $U(1)$ invariant Lagrangian

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According to the Poincaré Lemma

$$\partial_\mu B_\nu - \partial_\nu B_\mu = \varepsilon_{\mu\nu}{}^{\sigma\rho}\partial_\sigma A_\rho$$

Maxwell–Einstein and Dirac duality

ADM decomposition of the metric

$$g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \,,$$

The $U(1)$ invariant Lagrangian

$$\mathcal{L} = \frac{1}{2}\varepsilon^{ijk}\big(\partial_0 A_i \partial_j B_k - \partial_0 B_i \partial_j A_k + 4N^\ell \partial_{[i} A_{\ell]} \partial_j B_k\big) \\ - N\sqrt{h}\, h^{ik} h^{j\ell} \partial_{[i} A_{j]} \partial_{[k} A_{\ell]} - N\sqrt{h}\, h^{ik} h^{j\ell} \partial_{[i} B_{j]} \partial_{[k} B_{\ell]}$$

It is diffeomorphism invariant [M. Henneaux and C. Teitelboim]

$$\delta A_i = (\xi^j + N^j \xi^0)\big(\partial_j A_i - \partial_i A_j\big) + \frac{N}{\sqrt{h}} \xi^0 h_{ij} \varepsilon^{jk\ell} \partial_k B_\ell$$

Maxwell–Einstein and Dirac duality

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It is diffeomorphism invariant [M. Henneaux and C. Teitelboim]

$$\delta A_\mu \approx \xi^\nu \left(\partial_\nu A_\mu - \partial_\mu A_\nu\right) = \xi^\nu \partial_\nu A_\mu + A_\nu \partial_\mu \xi^\nu - \partial_\mu \left(\xi^\nu A_\nu\right)$$

Maxwell–Einstein and Dirac duality

ADM decomposition of the metric

$$g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\ ,$$

The $U(1)$ invariant Lagrangian

$$\mathcal{L} = \frac{1}{2}\varepsilon^{ijk}\left(\partial_0 A_i\partial_j B_k - \partial_0 B_i\partial_j A_k + 4N^\ell\partial_{[i}A_{\ell]}\partial_j B_k\right) - N\sqrt{h}\,h^{ik}h^{j\ell}\partial_{[i}A_{j]}\partial_{[k}A_{\ell]} - N\sqrt{h}\,h^{ik}h^{j\ell}\partial_{[i}B_{j]}\partial_{[k}B_{\ell]}$$

It is diffeomorphism invariant [M. Henneaux and C. Teitelboim]

$$\delta B_i = (\xi^j + N^j\xi^0)\left(\partial_j B_i - \partial_i B_j\right) - \frac{N}{\sqrt{h}}\xi^0 h_{ij}\varepsilon^{jk\ell}\partial_k A_\ell$$

Duality symmetry

As for example for $SL(2,\mathbb{R})/SO(2)$

$$
\begin{aligned}
\begin{pmatrix} e^{\phi'} & \sigma' e^{\phi'} \\ \varsigma' & e^{-\phi'} + \varsigma'\sigma' \end{pmatrix} &= \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} e^{\phi} & \sigma e^{\phi} \\ \varsigma & e^{-\phi} + \varsigma\sigma \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\
\chi' &= e^{in\alpha}\,\chi \\
\begin{pmatrix} A' \\ B' \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}
\end{aligned}
$$

Duality symmetry

As for example for $SL(2,\mathbb{R})/SO(2)$

$$\begin{pmatrix} e^{\phi'} & \sigma' e^{\phi'} \\ 0 & e^{-\phi'} \end{pmatrix} = \begin{pmatrix} \frac{d-c\sigma}{\sqrt{(d-c\sigma)^2+c^2e^{-4\phi}}} & \frac{-ce^{-2\phi}}{\sqrt{(d-c\sigma)^2+c^2e^{-4\phi}}} \\ \frac{ce^{-2\phi}}{\sqrt{(d-c\sigma)^2+c^2e^{-4\phi}}} & \frac{d-c\sigma}{\sqrt{(d-c\sigma)^2+c^2e^{-4\phi}}} \end{pmatrix} \begin{pmatrix} e^{\phi} & \sigma e^{\phi} \\ 0 & e^{-\phi} \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\chi' = \left(\frac{d - c\sigma - ice^{-2\phi}}{\sqrt{(d-c\sigma)^2 + c^2 e^{-4\phi}}} \right)^n \chi$$

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

Duality symmetry

As for example for $SL(2,\mathbb{R})/SO(2)$, with $\tau \equiv \sigma + ie^{-2\phi}$

$$
\begin{aligned}
\tau' &= \frac{a\tau - b}{d - c\tau} \\
\chi' &= \left(\frac{d - c\tau}{|d - c\tau|}\right)^n \chi \\
\begin{pmatrix} A' \\ B' \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \\
e^{\phi'}(A' + \tau' B') &= \frac{d - c\bar{\tau}}{|d - c\tau|}\, e^{\phi}(A + \tau B)
\end{aligned}
$$

Dilaton-axion Maxwell–Einstein and $SL(2,\mathbb{R})$ duality

Duality covariant complex field strength

$$\mathcal{F}_{ij} \equiv e^{\phi}\big(\partial_i A_j - \partial_j A_i + \tau(\partial_i B_j - \partial_j B_i)\big) \ ,$$

The $SL(2,\mathbb{R})$ invariant Lagrangian [J. Schwarz and A. Sen]

$$\mathcal{L} = \frac{1}{2}\varepsilon^{ijk}\big(\partial_0 A_i \partial_j B_k - \partial_0 B_i \partial_j A_k + 4N^{\ell}\,\partial_{[i}A_{\ell]}\partial_j B_k\big) - \frac{1}{4}N\sqrt{h}\,h^{ik}h^{j\ell}\,\bar{\mathcal{F}}_{ij}\mathcal{F}_{k\ell}$$

leads to the twisted selfduality equation

$$\mathcal{F}_{\mu\nu} - \frac{i}{2\sqrt{-g}}\varepsilon_{\mu\nu}{}^{\sigma\rho}\mathcal{F}_{\sigma\rho} = 0$$

$E_{7(7)}$-invariant action of $\mathcal{N} = 8$ supergravity

Defining $G_{mn} \equiv [\mathcal{V}^t \mathcal{V}]_{mn}$ and the symplectic form Ω_{mn}

$$G^{mn} = \Omega^{mp}\Omega^{nq} G_{pq} \qquad (G^{mp} G_{pn} = \delta^m_n)$$

and the 'complex structure'

$$J_m{}^n \equiv G_{mp}\Omega^{pn} \quad \Rightarrow \quad J_m{}^p J_p{}^n = -\delta^n_m$$

one defines for 56 vectors A^m_i [C. Hillmann]

$$-\mathcal{L}_{\rm vec} = \frac{1}{4}\Omega_{mn}\varepsilon^{ijk}\left(\partial_0 A^m_i + N^l F^m_{il}\right) F^n_{jk} - \frac{1}{4} N\sqrt{h}\, G_{mn} h^{ik} h^{jl} \hat{F}^m_{ij} \hat{F}^n_{kl}$$

$$\hookrightarrow \quad 2\partial_{[i}\left(\partial_0 A^m_{j]} + N^l F^m_{j]l} - \frac{N}{2\sqrt{h}} h_{j]l}\varepsilon^{lhf} J^m{}_n \hat{F}^n_{hf}\right) \approx 0$$

$E_{7(7)}$-invariant action of $\mathcal{N} = 8$ supergravity

One defines for 28 complex field strength

$$F^{ij}_{ij} \equiv u^{ij}{}_{IJ}(\partial_i A^{IJ}_j - \partial_j A^{IJ}_i) + v^{ijIJ}(\partial_i A_{j\,IJ} - \partial_j A_{i\,IJ})$$

The $E_{7(7)}$ invariant Lagrangian

$$-\mathcal{L}_{\rm vec} = \frac{i}{4}\varepsilon^{ijk}\left(\partial_0 A^{IJ}_i \partial_j A_{k\,IJ} - \partial_0 A_{i\,IJ}\partial_j A^{IJ}_k + 4N^l \partial_{[i} A^{IJ}_{l]}\partial_j A_{k\,IJ}\right) - \frac{1}{4} N\sqrt{h}\, h^{ik} h^{jl}\, \hat{F}_{ij\,ij}\hat{F}^{ij}_{kl}$$

leads to the supersymmetric equation [C. Hillmann]

$$\hat{F}^{ij}_{\mu\nu} - \frac{i}{2\sqrt{-g}}\varepsilon_{\mu\nu}{}^{\sigma\rho}\hat{F}^{ij}_{\sigma\rho} = 0$$

The $E_{7(7)}/SU(8)$ coset

The algebra decomposes as $\mathfrak{e}_{7(7)} \cong \mathfrak{su}(8) \oplus \mathbf{70}$

$$\Lambda^i{}_j L_i{}^j + \phi^{ijkl} Y_{ijkl} = \begin{pmatrix} 2\delta^{[i}_{[k} \Lambda^{j]}{}_{l]} & \phi^{ijkl} \\ \bar{\phi}_{ijkl} & -2\delta^{[k}_{[i} \Lambda^{l]}{}_{j]} \end{pmatrix}$$

with the condition

$$\bar{\phi}_{ijkl} = \frac{1}{24} \varepsilon_{ijklmnpq} \phi^{mnpq}$$

$E_{7(7)}$-invariant gauge-fixing

To avoid technicalities associated to $SU(8)$ local anomalies [di Vecchia, Ferrara, Girardello and de Wit, Grisaru]

$$
\begin{aligned}
u^{ij}{}_{KL} &= \cosh(\phi)^{ij}{}_{kl} = \delta^{ij}_{kl} + \frac{1}{2}\phi^{ijpq}\bar{\phi}_{pqkl} + \dots \\
v^{ijKL} &= \sinh(\phi)^{ijkl} = \phi^{ijkl} + \frac{1}{6}\phi^{ijmn}\bar{\phi}_{mnpq}\phi^{pqkl} + \dots
\end{aligned}
$$

With the notation $f(\Phi) * X \equiv f(\mathrm{ad}_\Phi)X$ for $\Phi \equiv \phi^{ijkl} Y_{ijkl} \in \mathfrak{e}_{7(7)}$

$$
\delta^{\mathfrak{e}_{7(7)}}\Phi = [C_{\mathfrak{k}}, \Phi] - \frac{\Phi}{\tanh \Phi} * C_{\mathfrak{p}} \ , \qquad \delta^{\mathfrak{e}_{7(7)}}\psi = \delta^{\mathfrak{su}(8)}\Big(C_{\mathfrak{k}} + \tanh(\Phi/2) * C_{\mathfrak{p}}\Big)\psi
$$

and the supersymmetry variation is

$$
\delta^{Susy}\Phi = \frac{\Phi}{\sinh \Phi} * [\bar{\epsilon}\chi]
$$

$E_{7(7)}$-invariant gauge-fixing

The BRST operator s for local supersymmetry, diffeomorphism, and abelian gauge invariance

$$\{s, \delta^{\mathfrak{e}_{7(7)}}\} = 0$$

Gauge-fixing functional $E_{7(7)}$ invariant.

↪ The action with sources solution to two consistent Slavnov–Taylor identities.

Coupling to the $E_{7(7)}$ Noether current

↪ Local Slavnov–Taylor identity for $\mathfrak{e}_{7(7)}$.

Anomalies

The 1PI generating functional Γ

$$s\Gamma = \mathcal{A}_s \ , \quad \delta^{\mathfrak{e}_{7(7)}}\Gamma = \mathcal{A}_{\mathfrak{e}_{7(7)}}$$

for local functionals satisfying Wess–Zumino

$$s\mathcal{A}_s = s\mathcal{A}_{\mathfrak{e}_{7(7)}} + \delta^{\mathfrak{e}_{7(7)}}\mathcal{A}_s = \delta^{\mathfrak{e}_{7(7)}}\mathcal{A}_{\mathfrak{e}_{7(7)}} = 0$$

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No superdiffeomorphism anomaly in $D = 4$

$$s\mathcal{A}_{\mathfrak{e}_{7(7)}} = \delta^{\mathfrak{e}_{7(7)}}\mathcal{A}_{\mathfrak{e}_{7(7)}} = 0$$

$\hookrightarrow$ Cohomology of $\delta^{\mathfrak{e}_{7(7)}}$ in complex of supersymmetric $\mathscr{F}$

Wess–Zumino consistency condition

According to the quantum action principle

$\hookrightarrow$ Consistent anomalies $\Leftrightarrow$ cohomology classes

For a compact group K in 4 dimensions

$$\mathcal{H}^1(\mathfrak{k}|d) \cong \mathcal{H}^5(K, \mathbb{R})$$

Such that

$$\mathrm{Tr}\ dC_{\mathfrak{k}}\left(B_{\mathfrak{k}} dB_{\mathfrak{k}} + \frac{1}{2} B_{\mathfrak{k}}{}^3\right) \Leftrightarrow \mathrm{Tr}\ (g^{-1} dg)^5$$

Wess–Zumino consistency condition

Riemannian symmetric G/K is topologically trivial

$\hookrightarrow$ Trivial equivariant cohomology

$$\mathcal{H}^n_K(d) \cong \{0\} \text{ for } n > 0$$

So we have the homotopy equivalence

$$E_{7(7)} \cong SU(8)/\mathbb{Z}_2 \times \mathbb{R}^{70}$$

and in particular $\mathcal{H}^5(E_{7(7)}, \mathbb{R}) \cong \mathcal{H}^5(SU(8), \mathbb{R})$

Similarly, we prove that $\mathcal{H}^n_K(\delta^{\mathfrak{g}}|d) \cong \{0\}$ for $n > 0$

$$\mathcal{H}^1(\delta^{\mathfrak{g}}|d) \cong \mathcal{H}^1(\delta^{\mathfrak{k}}|d)$$

$SL(2,\mathbb{R})$ anomaly

With 2 Weyl fermions and one vector field there is no cubic anomaly,

however there is a $U(1)$ gravitational anomaly

$$dJ^{\mathfrak{u}(1)} = \frac{1}{32\pi^2} R^{ab}{}_{\wedge} R_{ab}$$

Which corresponds to the non-linear anomaly

$$\mathbf{f}\Gamma^{\text{1-loop}} = \frac{1}{32\pi^2} \int e^{-2\phi} R^{ab}{}_{\wedge} R_{ab}$$

where

$$\mathbf{f}\tau = -\tau^2 \quad \mathbf{h}\tau = 2\tau \quad \mathbf{e}\tau = 1$$

No $E_{7(7)}$ anomaly

The unique algebraic possibility is the cubic Adler–Bardeen anomaly.

$$\int \text{Tr}\ dC_{\mathfrak{k}} \left(B_{\mathfrak{k}} dB_{\mathfrak{k}} + \frac{1}{2} B_{\mathfrak{k}}{}^3 \right)$$

The coefficient is defined by the family's index.
[O. Alvarez, I. M. Singer and B. Zumino]

$$\text{ch}[\text{Ind}(D)] = -\frac{3}{(2\pi)^3} \int_{S^2 \times M} \hat{\text{A}}_0 \text{Tr} \left(\square_A{}^{-1} [\delta A^\mu, \delta A_\mu] F_\wedge F + \delta A_\wedge \delta A_\wedge F \right)$$

Marcus computation establishes the absence of $\mathfrak{su}(8)$ anomaly.

$$(-3) \times 1 + 2 \times 4 + (-1) \times 5 = 0$$

Vector contribution via Pauli–Villars

The ‘axial-axial-axial’ anomaly is

$$\mathcal{A}^{\mu\nu}_{\rm vec}(p_1,p_2) = -\frac{1}{28}\lim_{M\to\infty}\Big[M\int\frac{dk^4}{(2\pi)^4}\mathrm{Tr}\;J\Big(\Delta(k+p_1,M)\Upsilon^\mu(k+p_1,k)\Delta(k,\text{-}M)\Upsilon^\nu(k,k+p_1)\Delta(k+p_1,M)\Upsilon_5(2k+p_1\text{-}p_2) + \Delta(k+p_1,M)R^{\mu\nu}\Delta(k\text{-}p_2,M)\Upsilon_5(2k+p_1\text{-}p_2)\Big)\Big]$$

and one computes

$$\mathcal{A}^{0i}_{\rm vec} = 0 = \mathcal{A}^{i0}_{\rm vec}$$

$$\mathcal{A}^{ij}_{\rm vec} = \frac{1}{6\pi^2}\varepsilon^{ijk}(p_{1\,0}p_{2\,k} - p_{2\,0}p_{1\,k}) - \frac{1}{6\pi^2}\varepsilon^{ijk}(p_{1\,0}+p_{2\,0})(p_{1\,k}-p_{2\,k})$$

Vector contribution via Pauli–Villars

With the finite renormalisation

$$-\mathcal{L}_0[B] = \frac{1}{2}\Omega_{mn}\varepsilon^{ijk}\left(\partial_0 A_i^m + B_0{}^m{}_p A_i^p\right)\left(\partial_j A_k^n + B_j{}^n{}_q A_k^q\right)$$
$$+\frac{1}{2}G_{mn}(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk})\left(\partial_i A_j^m + B_i{}^m{}_p A_j^p\right)\left(\partial_k A_l^n + B_k{}^n{}_q A_l^q\right) + b_m\left(\partial_i A_i^m + B_i{}^m{}_n A_i^n\right)$$
$$+\frac{1}{6\pi^2}\varepsilon^{ijk} J^m{}_n B_0{}^n{}_p B_i{}^p{}_q \partial_j B_k{}^q{}_m$$

one gets

$$\mathcal{A}^{\mu\nu}_{\text{vec}}(p_1, p_2) = \frac{1}{6\pi^2}\varepsilon^{\mu\nu\sigma\rho} p_{1\,\sigma} p_{2\,\rho} \ .$$

$E_{7(7)}$ Ward identities

$E_{7(7)}$ Ward identities for the 1PI correlation functions at all orders in perturbation theory.

The linear $\mathfrak{su}(8)$ Ward identities

$$\sum_{i\in I}\delta^{(4)}(x-x_i)\Big\langle\big[X,\Phi(x_i)\big]\prod_{j\neq i}\Phi(x_j)\Big\rangle=\partial_\mu\Big\langle J^\mu_{\mathfrak{k}}(X,x)\prod_{i\in I}\Phi(x_i)\Big\rangle$$

and the $\mathfrak{e}_{7(7)}\ominus\mathfrak{su}(8)$ Slavnov–Taylor identities

$$\sum_{J\subset I}\Big\langle\phi_A(x)\prod_{i\in J}\Phi(x_i)\Big\rangle\Big\langle\Big[\frac{\Phi}{\tanh(\Phi)}(x)*X\Big]^A\prod_{j\in I\setminus J}\Phi(x_j)\Big\rangle$$
$$=\sum_{J\subset I}\partial_\mu\Big\langle J^\mu_{\mathfrak{p}}(X,x)\prod_{i\in J}\Phi(x_i)\Big\rangle\Big\langle\prod_{j\in I\setminus J}\Phi(x_j)\Big\rangle$$

Linearised supersymmetry invariants and $SU(2,2|8)$

[J. M. Drummond, P. J. Heslop, P. S. Howe and S. F. Kerstan]

The 1/2 BPS ultra-short $D^4|$ 

$$\phi^{ijkl} \rightsquigarrow C_{\alpha\beta\gamma\delta} \oplus \bar{C}_{\dot\alpha\dot\beta\dot\gamma\dot\delta}$$

The 1/2 BPS operator $D^{16}|$ 

$$\phi^{ijkl}\phi^{mnop}\phi^{qrst}\phi^{uvwx} - \text{traces} \rightsquigarrow (\bar{C}C)^2 + \ldots$$

The 1/4 BPS operator $D^{24}|$ 

$$\phi^{ijrs}\phi^{kltu}\bar\phi_{mnrs}\bar\phi_{pqtu} - \text{traces} \rightsquigarrow (\partial^2\bar{C}C)^2 + \ldots$$

The 1/8 BPS operator $D^{28}|$ 

$$\phi^{imnp}\phi^{jqrs}\bar\phi_{kmns}\bar\phi_{lpqr} - \text{traces} \rightsquigarrow (\partial^3\bar{C}C)^2 + \ldots$$

Superdiffeomorphism invariant R^4

One could think that there exists a 1/2 BPS harmonic measure

$$\int dx^4 d^{16}\theta du \mathcal{E}(x,\theta,u) F(U,V) \doteqdot \int d^4x \Big(e\, f_{42}(\phi)(\bar{C}C)^2 + \dots \Big)$$

But the superspace torsion

$$T^{i\,j\,\dot\gamma k}_{\alpha\beta} = \varepsilon_{\alpha\beta}\bar{\chi}^{\dot\gamma ijk}$$

is an obstruction to the existence of such measure.

Only the body component of a closed superform

$$dL_4 = 0 \quad \hookrightarrow \int L_{(4,0)} = \int d^4x \sqrt{-g}\Big(f_{42}(\phi)(\bar{C}C)^2 + \dots \Big)$$

Superdiffeomorphism invariant R^4

Incompatible with supersymmetry Ward identities

[G. Bossard, P.S. Howe and K.S. Stelle]

$\hookrightarrow$ Supersymmetry non-renormalisation theorem

Incompatible with $E_{7(7)}$ Ward identities

[H. Elvang and M. Kiermaier]

$\hookrightarrow$ Duality invariance non-renormalisation theorem

This explains the absence of logarithm divergence at 3-loop

[Bern, Carrasco, Dixon, Johansson, Kosower and Roiban]

Duality covariance of the R^4 invariant

Not duality invariant but duality covariant

$\hookrightarrow$ The corrected action functional $S = S^{(0)} + I_{R^4} + \cdots$ satisfies the Gaillard–Zumino constraint

$$\delta^{\mathfrak{g}} S_{f_{42}}[F^m_{\mu\nu}] = S_{\delta^{\mathfrak{g}} f_{42}}[F^m_{\mu\nu}] + \frac{1}{8}\int d^4x \Big(\varepsilon^{\mu\nu\sigma\rho} X^{\bar{m}}{}_n \delta_{\bar{m}m} F^m_{\mu\nu} F^n_{\sigma\rho} - \frac{4}{\text{-}g} \varepsilon_{\mu\nu\sigma\rho} X^m{}_{\bar{n}} \delta^{\bar{n}n} \frac{\delta S}{\delta F^m_{\mu\nu}} \frac{\delta S}{\delta F^n_{\sigma\rho}} \Big)$$

with $X^m{}_n = \begin{pmatrix} X^m{}_n & X^m{}_{\bar{n}} \\ X^{\bar{m}}{}_n & X^{\bar{m}}{}_{\bar{n}} \end{pmatrix} \in \mathfrak{e}_{7(7)}$

$$\delta^{\mathfrak{g}} F^m_{\mu\nu} = X^m{}_n F^n_{\mu\nu} - X^m{}_{\bar{n}} \delta^{\bar{n}n} \frac{1}{\text{-}g} \varepsilon_{\mu\nu\sigma\rho} \frac{\delta S}{\delta F^n_{\sigma\rho}}$$

Gaillard–Zumino constraint		Diff. master equation
Duality covariance	$\Longleftrightarrow$	
Cov. Formulation		Henneaux–Teitelboim

Duality covariance of the R^4 invariant

$$I_{R^4} = \int d^4x \sqrt{-g} \Big(T_{\mu\nu\sigma\rho} T^{\mu\nu\sigma\rho} + \nabla_\mu F_\nu{}^\kappa{}_{ij} T^{\mu\nu\sigma\rho} \nabla_\sigma F^{ij}_{\rho\kappa} + \dots \Big)$$

where the Bel–Robinson tensor

$$T^{\mu\nu\sigma\rho} \equiv C^{\mu\kappa\sigma\lambda} C^\nu{}_\kappa{}^\rho{}_\lambda - \frac{3}{2} g^{\mu[\nu} C^{\kappa\lambda]\sigma\vartheta} C_{\kappa\lambda}{}^\rho{}_\vartheta$$

Duality covariance of the R^4 invariant

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Consider the Maxwell–Einstein example

$$\mathcal{F}^-_{\mu\nu} + \nabla_\sigma T_{[\mu}{}^{\sigma\rho\lambda} \nabla_\rho \mathcal{F}^+_{\nu]\lambda} = 0$$

with the definition

$$\begin{aligned} \mathcal{F}_{\mu\nu} &\equiv \partial_\mu A_\nu - \partial_\nu A_\mu + i(\partial_\mu B_\nu - \partial_\nu B_\mu) \\ F_{\mu\nu} &\equiv \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned}$$

Duality covariance of the R^4 invariant

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Then

$$S = -\int d^4x \sqrt{-g} \left(\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \sum_{n=1}^{\infty} F^{\mu\nu} (\Delta^n)_{\mu\nu}{}^{\sigma\rho} F_{\sigma\rho} \right)$$

for

$$\Delta_{\mu\nu}{}^{\rho\sigma} \equiv \nabla_\kappa T_{[\mu}{}^{\kappa\lambda[\sigma} \nabla_\lambda \delta^{\rho]}_{\nu]}$$

Duality covariance of the R^4 invariant

$$\Delta_{\mu\nu}{}^{\rho\sigma} \equiv \nabla_\kappa T_{[\mu}{}^{\kappa\lambda[\sigma} \nabla_\lambda \delta^{\rho]}_{\nu]}$$

Because

$$\frac{1}{2\sqrt{-g}} \varepsilon_{\mu\nu}{}^{\kappa\lambda} \Delta_{\kappa\lambda}{}^{\sigma\rho} = -\Delta_{\mu\nu}{}^{\kappa\lambda} \frac{1}{2\sqrt{-g}} \varepsilon_{\kappa\lambda}{}^{\sigma\rho}$$

one has

$$\begin{aligned} \int d^4x \frac{2}{-g} \varepsilon_{\mu\nu\sigma\rho} \frac{\delta S}{\delta F_{\mu\nu}} \frac{\delta S}{\delta F_{\sigma\rho}} &= \int d^4x \frac{1}{2} \varepsilon^{\mu\nu\sigma\rho} \Big(\delta^{\kappa\lambda}_{\mu\nu} + 2 \sum_{n\geq 1} \Delta^{n\,\kappa\lambda}{}_{\mu\nu} \Big) F_{\kappa\lambda} \Big(\delta^{\theta\tau}_{\sigma\rho} + 2 \sum_{n\geq 1} \Delta^{n\,\theta\tau}{}_{\sigma\rho} \Big) F_{\theta\tau} \\ &= \int d^4x \frac{1}{2} \varepsilon^{\mu\nu\sigma\rho} F_{\mu\nu} \Big(\delta^{\kappa\lambda}_{\sigma\rho} + 2 \sum_{n\geq 1} (\text{-}\Delta)^{n\,\kappa\lambda}{}_{\sigma\rho} \Big) \Big(\delta^{\theta\tau}_{\kappa\lambda} + 2 \sum_{n\geq 1} \Delta^{n\,\theta\tau}{}_{\kappa\lambda} \Big) F_{\theta\tau} \\ &= \int d^4x \frac{1}{2} \varepsilon^{\mu\nu\sigma\rho} F_{\mu\nu} F_{\sigma\rho} \quad . \end{aligned}$$

Duality covariance of the R^4 invariant

$$\Delta_{\mu\nu}{}^{\rho\sigma} \equiv \nabla_\kappa T_{[\mu}{}^{\kappa\lambda[\sigma} \nabla_\lambda \delta^{\rho]}_{\nu]}$$

Because
$$\int d^4x \frac{2}{\text{-}g} \varepsilon_{\mu\nu\sigma\rho} \frac{\delta S}{\delta F_{\mu\nu}} \frac{\delta S}{\delta F_{\sigma\rho}} = \int d^4x \frac{1}{2} \varepsilon^{\mu\nu\sigma\rho} F_{\mu\nu} F_{\sigma\rho} \quad ,$$

the Gaillard–Zumino constraint is satisfied

$$\begin{aligned} \delta^{\mathfrak{u}(1)} S &= \int d^4x \frac{1}{\text{-}g} \varepsilon_{\mu\nu\sigma\rho} \frac{\delta S}{\delta F_{\mu\nu}} \frac{\delta S}{\delta F_{\sigma\rho}} \\ &= \frac{1}{8} \int d^4x \Big(\varepsilon^{\mu\nu\sigma\rho} F_{\mu\nu} F_{\sigma\rho} + \frac{4}{\text{-}g} \varepsilon_{\mu\nu\sigma\rho} \frac{\delta S}{\delta F_{\mu\nu}} \frac{\delta S}{\delta F_{\sigma\rho}} \Big) \end{aligned}$$

Superdiffeomorphism invariant $(\nabla^3 R^2)^2$

The integrability condition holds

$$u^1{}_i u^1{}_j T^{i\,j\,A}_{\alpha\beta} = u^1{}_i u^j{}_8 T^i{}_{\alpha\dot\beta j}{}^A = u^i{}_8 u^j{}_8 T_{\dot\alpha i \dot\beta j}{}^A = 0$$

By Frobenius theorem there exist a measure such that a superfield

$$\mathcal{F} = u^1{}_i u^1{}_j u^k{}_8 u^l{}_8\, L^{ij}{}_{kl}$$

which is Grassmann-analytic

$$u^1{}_i D^i_\alpha \mathcal{F} = u^i{}_8 D_{\dot\alpha i}\mathcal{F} = 0$$

integrates to a supersymmetry invariant

$$I = \int d\mu_{(8,1,1)}\mathcal{F} \equiv \int d^4x d^{28}\theta d^{26}u \mathcal{E}(x,\theta,u)\,\mathcal{F}$$

Superdiffeomorphism invariant $(\nabla^3 R^2)^2$

For the $E_{7(7)}/SU(8)$ superfield

$$\begin{pmatrix} U^{ij}{}_{IJ} & V^{ijKL} \\ \bar{V}_{klIJ} & \bar{U}_{kl}{}^{KL} \end{pmatrix} \in E_{7(7)}/SU(8)$$

the superfield

$$\mathcal{F} = u^1{}_i u^1{}_j u^k{}_8 u^l{}_8 \, \bar{V}^{imIJ} \bar{V}^{jnKL} V_{kmKL} V_{lnIJ}$$

is Grassmann-analytic

$$u^1{}_i D^i_\alpha \mathcal{F} = u^i{}_8 D_{\dot{\alpha} i} \mathcal{F} = 0$$

and integrates to a supersymmetry invariant

$$I^{(6)} = \int d\mu_{(8,1,1)} \mathcal{F} = \int d^4x \sqrt{-g} \Big(f_{60}(\phi) \nabla^\mu \nabla^\nu \nabla^\sigma T^{\rho\kappa\lambda\vartheta} \nabla_\mu \nabla_\nu \nabla_\sigma T_{\rho\kappa\lambda\vartheta} + \dots \Big)$$

Superdiffeomorphism invariant $(\nabla^3 R^2)^2$

$$I^{(6)} = \int d^4x \sqrt{-g} \Big(f_{60}(\phi) \nabla^\mu \nabla^\nu \nabla^\sigma T^{\rho\kappa\lambda\vartheta} \nabla_\mu \nabla_\nu \nabla_\sigma T_{\rho\kappa\lambda\vartheta} + \dots \Big)$$

is not duality invariant

Incompatible with $E_{7(7)}$ Ward identities

Duality and divergences

The BPS invariants for known log divergences

- ⋆ R^4 $SL(2)\times SL(3)$invariant in 8D $\Leftrightarrow$ 1-loop divergence
- ⋆ $\nabla^4 R^4$ $SL(5,\mathbb{R})$ invariant in 7D $\Leftrightarrow$ 2-loop divergence
- ⋆ $\nabla^6 R^4$ $SO(5,5)$ invariant in 6D $\Leftrightarrow$ 3-loop divergence

Superspace integrals for suspected log divergences

- ⋆ $\nabla^{12} R^4$ $E_{6(6)}$ invariants in 5D $\Leftrightarrow$ 6-loop divergence

Non-standard BPS invariant

- ⋆ $\nabla^8 R^4$ $E_{7(7)}$ invariant in 4D $\Leftrightarrow$ 7-loop divergence?

Checked by explicit computation until 4-loop
[Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson and R. Roiban]

The vanishing volume

Normal coordinate expansion in harmonic superspace.

Short expansion of the supervielbein superdeterminant

$$\begin{aligned} \mathcal{V} &\equiv \int d^4x\, d^{32}\theta\, E(x,\theta) \\ &= \int d\mu_{(8,1,1)}\, d^4\zeta \left(1 - \frac{1}{6}\, u^1{}_i u^j{}_8\, \bar{\chi}^{ikl}_{\dot\beta} \chi_{\alpha\, jkl}\, \zeta^\alpha \zeta^{\dot\beta}\right) \\ &= 0 \end{aligned}$$

But there exists a duality invariant $\nabla^8 R^4$ candidate counterterm

$$\begin{aligned} &\int d\mu_{(8,1,1)}\, u^1{}_i u^1{}_j u^k{}_8 u^l{}_8\, \bar{\chi}^{imn}_{\dot\beta} \chi_{\alpha\, kmn}\, \bar{\chi}^{\dot\beta\, jpq} \chi^{\alpha}_{lpq} \\ = &\int d^4x \sqrt{-g} \left(\nabla^\mu \nabla^\nu \nabla^\sigma \nabla^\rho T^{\kappa\lambda\vartheta\tau} \nabla_\mu \nabla_\nu \nabla_\sigma \nabla_\rho T_{\kappa\lambda\vartheta\tau} + \ldots \right) \end{aligned}$$

Conclusion

$\mathcal{N} = 8$ supergravity amplitudes have excellent UV behaviour

- ⋆ No log divergence before 7-loop
- ⋆ Even 7-loop may be protected
- ⋆ Amplitude behave like $\mathcal{N} = 4$ Yang–Mills until 4-loop

Large set of supersymmetry at the quantum level

- ⋆ $E_{7(7)}$ duality symmetry
- ⋆ 32 local supersymmetry

↪ 4-point kinematics behaves like for a real scalar field.

Outlook

How the theory can be constrained beyond 8-loop?

- ⋆ Non-perturbative information
  - ↪ $E_{7(7)}(\mathbb{Z})$-invariant effective action
    - ∗ Instanton corrections
    - ∗ String theory limit
- ⋆ Symmetries of the amplitudes
  - ↪ A square of $\mathcal{N} = 4$ Yangian symmetry
    - ∗ Super-BMS?
    - ∗ $E_{9(9)}$?