A class of gauges for the Einstein equations

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Abstract: A class of gauges for the Einstein vacuum equations is introduced, along with three symmetric hyperbolic systems. The first implies the local realizability of the gauge. The second is the dynamical subset of the field equations. The third is used to show that the constraints propagate. The gauges are for an orthonormal frame formalism, with first order, quadratically nonlinear equations. The unknowns are 16 frame components and 28 connection components. After gauge-fixing, a total of 33 remain.

1 Introduction

1.1 The orthonormal frame formalism used in this paper

Introduce the Lie algebra $\Lambda = \mathbb{R}^4 \times (\mathbb{R} \times \mathrm{so}(1,3))$. A quick overview of the formalism is obtained by counting the independent degrees of freedom of its constituents[1]. The number of

- local gauge degrees of freedom
- components of the unknown field
- (first order, quadratically nonlinear) field equations

are equal, respectively, to

- $\dim_{\mathbb{R}} \Lambda = 4 + 7 = 11$
- $\dim_{\mathbb{R}} (\mathbb{R}^4)^* \otimes_{\mathbb{R}} \Lambda = 16 + 28 = 44$
- $\dim_{\mathbb{R}} [((\mathbb{R}^4)^* \wedge_{\mathbb{R}} (\mathbb{R}^4)^*) \otimes_{\mathbb{R}} \Lambda] \,/\, \mathbf{Weyl}_{\mathbb{R}} = 24 + 42 - 10 = 56$

By the Einstein vacuum equations, the curvature vanishes only modulo elements of a subspace $\mathbf{Weyl}_{\mathbb{R}} \subset ((\mathbb{R}^4)^* \wedge_{\mathbb{R}} (\mathbb{R}^4)^*) \otimes_{\mathbb{R}} \mathrm{so}(1,3)$, $\dim_{\mathbb{R}} \mathbf{Weyl}_{\mathbb{R}} = 10$.

The algebraic structure of the principal part[2] *of the equations is central to this paper.* This structure is already in the linear model problem (LMP)

- $F \mapsto F + \mathrm{d}\xi$ are the infinitesimal local gauge transformations
- $U \in \mathbf{Weyl}_{\mathbb{R}}$ (pointwise) are the field equations; here $U = \mathrm{d}F$
- $\mathrm{d}U = 0$ are differential identities for U

with Λ-valued differential forms ξ, F (the unknown), U of degree 0, 1, 2.

[1] Essentially: coordinate transformations + $(\mathbb{R}_+ \times \mathrm{O}(1,3))$ frame transformations; frame + connection; torsion + curvature modulo Weyl. Torsion=0 is here a field equation!

[2] Principal part: the union of terms *with first derivatives* of the unknown.
Non-principal part: the union of terms *without derivatives* of the unknown.

More details about this formalism are given as needed, in Section 4.

1.2 Gauge-fixing and symmetric hyperbolicity

The gauges in this paper are fields with values in a vector space, on which $\mathrm{SL}(2,\mathbb{C})$ acts. These gauges impose 11 linear conditions, that restrict the unknown field to a subspace with $\dim_{\mathbb{R}} = 33$, see $(*)$ in Section 3 and $(*_{\mathrm{GR}})$ in Subsection 4.3. Accordingly, there are 33 dynamical equations, and $56-33 = 23$ constraint equations.

These gauges are introduced with the explicit goal of obtaining the three *symmetric hyperbolic* systems[3 4 5] that are mentioned in the abstract. Their dimensions are 11×11 and 33×33 and 23×23.

Symmetric hyperbolicity is a property of *the algebraic structure of just the principal part of the equations*. Thus, the algebraic manipulations, used to derive symmetric hyperbolic systems, can be presented within the context of a linear model problem on $\mathbb{R}^4$ (LMP), in self-contained Sections 2 and 3. Section 4 explains, why the LMP results are results about general relativity.

1.3 Comparison with the Newman-Penrose-Friedrich formalism

The Newman-Penrose [NP] formalism differs from Subsection 1.1 by the addition of the Weyl curvature to the list of unknowns, and the *differential Bianchi equations* to the list of (still first order, quadratically nonlinear) field equations.

Friedrich [Fr] derived differential identities for the field equations, and showed that by gauge-fixing, the field equations can be reduced to symmetric hyperbolic systems. Other examples of gauges that admit such a symmetric hyperbolic reduction have since been found. The essential strategy is to: gauge-fix frame and connection; derive 'simple' equations[6] for frame and connection; bring the differential Bianchi equations for the Weyl curvature into symmetric hyperbolic form. One way or another, the last step uses the Bel-Robinson tensor.

[3] Symmetric hyperbolic systems were used, in this context, by H. Friedrich [Fr].

[4] See [Tay] for the general theory of local existence and uniqueness for quasilinear symmetric hyperbolic systems, and finite speed of propagation.

[5] Example: The familiar $(\partial_0 + i\,\mathrm{curl})\mathbf{v} = 0$ on $\mathbb{R}^4$ is a linear, constant coefficient, symmetric hyperbolic system. Here $\mathbf{v} = \mathbf{E} + i\mathbf{B}$ is a complex vector field, $\mathbf{v} = (v^1, v^2, v^3)$. In this example, symmetric hyperbolicity means that the 3×3 matrix $\theta(\partial_0 + i\,\mathrm{curl})$ is *Hermitian for all* real one-forms $\theta = (\theta_0, \boldsymbol{\theta})$ and *positive for some* such θ. Here, it is positive iff θ is future timelike, $\theta_0 > |\boldsymbol{\theta}|_{\mathbb{R}^3}$.

[6] Here, 'simple' refers to equations that are easily symmetric hyperbolic, e.g. with diagonal matrix differential operator, a 'coupled system of ordinary differential equations'.

In this approach, gauge-fixing and the derivation of symmetric hyperbolic systems can be carried out almost independently: the first concerns only frame and connection, the second only the Weyl curvature / differential Bianchi equations. By contrast, in the present paper, gauge-fixing and the derivation of symmetric hyperbolic systems is correlated: both concern frame and connection.

2 Notation and definitions

- $z \in \mathbb{C}$ has complex conjugate $\overline{z}$ and real part $\operatorname{Re} z = \frac{1}{2}(z + \overline{z})$.
  Fields in this paper are complex. Reality conditions are spelled out.
- Small Latin indices $a, b, \ldots$ take values in the (spinor) index set $\mathcal{S} = \{0, 1\}$.
- ε^{ab}, ε_{ab} are defined by $\varepsilon^{ab} = -\varepsilon^{ba}$ and $\varepsilon_{ab} = -\varepsilon_{ba}$ and $\varepsilon^{01} = \varepsilon_{01} = 1$.
  $\mathbf{S}^{a_1 \ldots a_k}_{b_1 \ldots b_k} = (k!)^{-1} \sum_{\pi \in S_k} \delta^{a_1}_{b_{\pi(1)}} \cdots \delta^{a_k}_{b_{\pi(k)}}$, total symmetrization of $\mathcal{S}$-indices[7].
  Observe that $\varepsilon^{ka}\varepsilon_{kb} = \delta^a_b$ and $\varepsilon^{ak}\varepsilon_{kb} = -\delta^a_b$ and $\delta^i_a \delta^j_b = \mathbf{S}^{ij}_{ab} + \frac{1}{2}\varepsilon^{ij}\varepsilon_{ab}$.
- Small Greek indices $\alpha, \beta, \ldots$ take values in the (coordinate) index set $\mathcal{C}$ with $|\mathcal{C}| = 4$. The index set $\mathcal{C}$ is assumed to be disjoint from $\mathcal{S}$.
- Capital Latin indices $A, B, \ldots$ take values in $\mathcal{C} \cup (\mathcal{S} \times \mathcal{S} \times \mathcal{S} \times \mathcal{S})$. Namely, (f^A) is short for $(f^\alpha) \oplus (f_{ab}{}^{cd})$, and (f_A) is short for[8] $(f_\alpha) \oplus (f^{ab}{}_{cd})$.
- $\mathbf{\Pi}$ is a projection: $\mathbf{\Pi}_{a_1 \ldots a_k}{}^{A b_1 \ldots b_k}{}_B$ vanishes if $A \in \mathcal{C}$ or $B \in \mathcal{C}$, otherwise

$$\mathbf{\Pi}_{a_1 \ldots a_k c_1 c_2}{}^{c_3 c_4 b_1 \ldots b_k d_1 d_2}{}_{d_3 d_4} = \delta^{d_1}_{c_1} \delta^{c_3}_{d_3} \varepsilon^{c_4 e} \varepsilon_{d_4 f} \mathbf{S}^{b_1 \ldots b_k d_2 f}_{a_1 \ldots a_k c_2 e}$$

- Define the prefix $\sharp$ by $f^A = g^{\sharp A}$ iff $f^\alpha = g^\alpha$, $f_{ij}{}^{k\ell} = g_{ji}{}^{\ell k}$. Similar for $g_{\sharp A}$.
  Define the prefix & by[9] $f^{\& A} = \mathbf{\Pi}^A{}_B \mathbf{\Pi}^{\sharp B}{}_{\sharp C} f^C$ and $f_{\& C} = f_A \mathbf{\Pi}^A{}_B \mathbf{\Pi}^{\sharp B}{}_{\sharp C}$.
- $\mathbf{Weyl} = \big\{ \, (f_{ab}{}^A) \, \big| \, \mathbf{\Pi}_{ab}{}^{Acd}{}_B f_{cd}{}^B = f_{ab}{}^A \text{ and } f_{ab}{}^{\& A} = 0 \big\}$, $\dim_{\mathbb{C}} \mathbf{Weyl} = 5$.

2.1 $\mathrm{SL}(2, \mathbb{C})$, unprimed and primed $\mathcal{S}$-indices

The constructions in this paper are invariant under the action[10] of $\mathrm{SL}(2, \mathbb{C})$, the group of complex 2×2 matrices A with $\varepsilon_{ab} A^a{}_m A^b{}_n = \varepsilon_{mn}$, equivalently $A^a{}_m A^b{}_n \varepsilon^{mn} = \varepsilon^{ab}$. The matrix $A \in \mathrm{SL}(2, \mathbb{C})$ acts: trivially on $\mathcal{C}$ indices; as

[7]Here, S_k is the symmetric group on $\{1, \ldots, k\}$.

[8]Example of a summation that involves capital Latin indices:
$f^K g^L h_{iKL} = f^\alpha g^\beta h_{i\alpha\beta} + f^\alpha g_{mn}{}^{pq} h_{i\alpha}{}^{mn}{}_{pq} + f_{ab}{}^{ef} g^\beta h_i{}^{ab}{}_{ef\beta} + f_{ab}{}^{ef} g_{mn}{}^{pq} h_i{}^{ab}{}_{ef}{}^{mn}{}_{pq}$.

[9]Observe that $\mathbf{\Pi}^A{}_B \mathbf{\Pi}^{\sharp B}{}_{\sharp C} = \mathbf{\Pi}^{\sharp A}{}_{\sharp B} \mathbf{\Pi}^B{}_C$.

[10]The 'global action'. That is, the $\mathrm{SL}(2, \mathbb{C})$-matrix is a constant, not a function.

A on unprimed $\mathcal{S}$ indices; as $\overline{A}$ on primed $\mathcal{S}$ indices[11]. The basic fields in this paper are $\varepsilon^{ab}, \varepsilon^{a'b'}, \varepsilon_{ab}, \varepsilon_{a'b'}$ and

$$L_{ab'} \qquad F_{ab'}{}^{A} \qquad \xi^{A} \qquad U_{a'b'}{}^{A} \qquad G^{ab'}{}_{A'B} \qquad \eta^{A} \qquad \mathbf{U}_{a'b'}{}^{A} \qquad \zeta^{A} \tag{1}$$

The placement of primes determines the action of $\mathrm{SL}(2,\mathbb{C})$ on each of these fields. $\mathcal{S}$-indices internal to A and A' are unprimed or primed as in $mn'rs'$ and $m'nr's$, respectively[12]. *Primes are suppressed in this paper.* They can be restored by referring to (1).

Let (m,n) be the irreducible representation of $\mathrm{SL}(2,\mathbb{C})$ with $\dim_{\mathbb{C}} = (m+1)(n+1)$, equivalent to the action on tensors that are totally symmetric in m unprimed and totally symmetric in n primed $\mathcal{S}$-indices. Then:

- ξ^{A} transforms as $4(0,0) \oplus ((0,0) \oplus (2,0) \oplus (0,2) \oplus (2,2))$. The condition $\xi^{\&A} = 0$ removes $(2,2)$. The additional condition $\overline{\xi^{A}} = \xi^{\sharp A}$ selects a *real* $\mathrm{SL}(2,\mathbb{C})$-invariant subspace, $\dim_{\mathbb{R}} = 11$, which can be thought of as Λ from Subsection 1.1.

- $(U_{ab}{}^{A}) \in \mathbf{Weyl}$ if and only if U transforms as $(0,4)$.

The constructions in this paper are invariant under simultaneous complex conjugation of all fields in (1). This extends the action to $\mathrm{SL}(2,\mathbb{C}) \rtimes_{\varphi} \mathbb{Z}_2$, with complex conjugation $\varphi : \mathbb{Z}_2 \to \mathrm{Aut}(\mathrm{SL}(2,\mathbb{C}))$. The action on the $\dim_{\mathbb{R}} = 4$ vector space of all v^{ab} with $\overline{v^{ba}} = v^{ab}$ (a unprimed, b primed) leaves $(v,w) \mapsto -\varepsilon_{ab}\varepsilon_{mn}v^{am}w^{bn}$ invariant, and yields a homomorphism from $\mathrm{SL}(2,\mathbb{C}) \rtimes_{\varphi} \mathbb{Z}_2$ *onto* the orthochronous Lorentz group $\mathrm{O}^{+}(1,3)$. Its kernel $\{\pm \mathbb{1}_2\}$ acts trivially on the fields in (1), because they have an even number of $\mathcal{S}$-indices. Thus, $\mathrm{O}^{+}(1,3)$ acts on (1).

3 Linear model problem (LMP) and gauge-fixing

The constituents of LMP on $\mathbb{R}^4$ are[13]:

(i) Four linearly independent, complex vector fields $L_{ab} = L_{ab}{}^{\sigma}\frac{\partial}{\partial x^{\sigma}}$, with constant coefficients, and $\overline{L_{ba}} = L_{ab}$. They are fixed for the discussion[14].

(ii) The unknown field $F_{ab}{}^{A}$, subject to[15] $\overline{F_{ab}{}^{A}} = F_{ba}{}^{\sharp A}$ and $F_{ab}{}^{\&A} = 0$.

[11] Example: $f \mapsto Af$ with $(Af)^{ab'}{}_{c}{}^{\mu} = A^{a}{}_{i}\overline{A^{b'}{}_{j'}}(A^{-1})^{k}{}_{c}f^{ij'}{}_{k}{}^{\mu}$.

[12] For example, $F_{ab'k\ell'}{}^{mn'}$ and $G^{ab'k'\ell}{}_{m'n}{}^{pq'}{}_{rs'}$.

[13] All fields are infinitely differentiable.

[14] Example: $L_{00} = \partial_0 + \partial_1,\ L_{11} = \partial_0 - \partial_1,\ L_{01} = \partial_2 + i\partial_3,\ L_{10} = \partial_2 - i\partial_3$.

[15] $F_{ab}{}^{A}$ is a field on $\mathbb{R}^4$ with values in a $\dim_{\mathbb{R}} = 44$ vector space.

(iii) The field $U_{mn}{}^A = \mathbf{S}^{ij}_{mn} \varepsilon^{ab} L_{ai}(F_{bj}{}^A)$ associated to the unknown F.

It follows that:

(iv) U is invariant under *local gauge transformations* $F_{ab}{}^A \mapsto F_{ab}{}^A + L_{ab}(\xi^A)$, for any ξ with $\overline{\xi^A} = \xi^{\sharp A}$ and $\xi^{\& A} = 0$, abbreviated $F \mapsto F + L(\xi)$.

(v) $\varepsilon^{mn} U_{mn}{}^A = 0$ and $U_{mn}{}^{\& A} = 0$.

(vi) $\overline{\varepsilon^{mn} L_{im}(U_{nj}{}^A)} - \varepsilon^{mn} L_{jm}(U_{ni}{}^{\sharp A}) = 0$, differential identities[16] for U.

By definition, F is a solution to LMP if and only if it solves:

(vii) Partial differential field equations: $U \in \mathbf{Weyl}$, pointwise.

Definition (Gauge-fixing) *Suppose $G^{ab}{}_{AB}$ is constant and:*

(G1) (Hermitian) $G^{ab}{}_{A\&B} = 0$ *and* $\overline{G^{ba}{}_{BA}} = G^{ab}{}_{AB}$.

(G2) (Main algebraic condition) $G^{ab}{}_{AB} \mathbf{\Pi}_b{}^{Bc}{}_C = 0$.

(G3) (Positivity) $\overline{v_a{}^A} G^{ab}{}_{AB} v_b{}^B \geq 0$ *for all* v.
Assuming $v_b{}^{\& B} = 0$, *equality holds if and only if* $\mathbf{\Pi}_a{}^{Ab}{}_B v_b{}^B = v_a{}^A$.

A field F as in (ii) is gauge-fixed with respect to G if and only if, pointwise,

$$\boxed{\forall \eta \text{ with } \overline{\eta^A} = \eta^{\sharp A} \text{ and } \eta^{\& A} = 0 : \quad \operatorname{Re}\left(\overline{\eta^A} G^{ab}{}_{AB} F_{ab}{}^B \right) = 0} \qquad (*)$$

Gauge-fixing $(*)$ leads to three symmetric hyperbolic systems, introduced in the next sections. The field $G^{ab}{}_{AB}$ is fixed for the discussion. One can read (G4), (G5), (G6) only when they are referred to, later on.

Lemma

(G4) $\overline{w_a{}^A} G^{ab}{}_{AB} w_b{}^B > 0$ *if* $w_a{}^A = s_a \eta^A$, $\eta \neq 0$, $\eta^{\& A} = 0$, $\overline{\eta^A} = \eta^{\sharp A}$, $s \neq 0$.

If t^{ab} *satisfies* $\overline{t^{ba}} = t^{ab}$ *and* $\overline{s_a} t^{ab} s_b > 0$ *for all* $s \neq 0$, *then:*

(G5) $\overline{w_{ma}{}^A} (t^{mn} G^{ab}{}_{AB}) w_{nb}{}^B > 0$ *if* $w \neq 0$, $w_{ab}{}^{\& A} = 0$, $\overline{w_{ab}{}^A} = w_{ba}{}^{\sharp A}$.

(G6) $\overline{w_{ma}{}^A} (t^{mn} G^{ab}{}_{AB}) w_{nb}{}^B \geq 0$ *if* $w_{ab}{}^{\& A} = 0$, $\overline{\varepsilon^{ab} w_{ab}{}^A} = \varepsilon^{ab} w_{ab}{}^{\sharp A}$.
Equality if and only if $w \in \mathbf{Weyl}$.

[16] Derivation: $\varepsilon^{mn} L_{im}(U_{nj}{}^A) = \varepsilon^{mn} \varepsilon^{pq} [L_{im}(L_{pj}(F_{qn}{}^A)) - \frac{1}{2} L_{ij}(L_{pm}(F_{qn}{}^A))]$ by (iii) and the identity $\mathbf{S}^{ij}_{ab} = \mathbf{S}^{ij}_{ba} = \delta^i_a \delta^j_b - \frac{1}{2} \varepsilon^{ij} \varepsilon_{ab}$. Now use (i), (ii) and $[L_{ab}, L_{cd}] = 0$.

Proof. In (G4), (G5), (G6): ${w_{*a}}^{\&A} = 0$, where $*$ is nothing or one $\mathcal{S}$-index. By (G3), the bilinear expression is ≥ 0, with equality iff[17] ${\mathbf{\Pi}_a}^{Ab}{}_B {w_{*b}}^B = {w_{*a}}^A$. Assume equality. Then ${\mathbf{\Pi}^A}_B {w_{*a}}^B = {w_{*a}}^A$. Thus in (G4), ${\mathbf{\Pi}^A}_B \eta^B = \eta^A$, in turn ${\mathbf{\Pi}^{\sharp A}}_{\sharp B} \eta^B = \eta^A$, that is $\eta^A = \eta^{\&A} = 0$, contradicting $\eta \neq 0$. In (G5), ${\mathbf{\Pi}^A}_B {w_{ab}}^B = {w_{ab}}^A$, ${\mathbf{\Pi}^{\sharp A}}_{\sharp B} {w_{ab}}^B = {w_{ab}}^A$, that is ${w_{ab}}^A = {w_{ab}}^{\&A} = 0$, contradicting $w \neq 0$. In (G6), ${\mathbf{\Pi}^A}_B {w_{ab}}^B = {w_{ab}}^A$ and ${w_{ab}}^{\&A} = 0$ imply ${\mathbf{\Pi}^{\sharp A}}_{\sharp B} {w_{ab}}^B = 0$. Multiply with ε^{ab}, conjugate, find $\varepsilon^{ab} {w_{ab}}^A = 0$. To summarize, ${w_{ab}}^A = {w_{ba}}^A$ and ${\mathbf{\Pi}_a}^{Ab}{}_B {w_{kb}}^B = {w_{ka}}^A$. The group of permutations of $\{1,2,3,4\}$ is generated by the transposition (12) and the six stabilizers of 1. Thus, in (G6), equality implies ${\mathbf{\Pi}_{ab}}^{Acd}{}_B {w_{cd}}^B = {w_{ab}}^A$. The converse holds by (G2). □

3.1 First system: the gauge is locally realizable

Assume F as in (ii) is given. Do not assume $(*)$ or (vii). The condition for $F + L(\xi)$ in (iv) to satisfy $(*)$ is a partial differential equation for ξ:

$$\boxed{\forall \eta \text{ as in } (*): \qquad \operatorname{Re}\big(\overline{\eta^A} {G^{ab}}_{AB} L_{ab}(\xi^B) \big) = -\operatorname{Re}\big(\overline{\eta^A} {G^{ab}}_{AB} {F_{ab}}^B \big)} \tag{2}$$

This is a symmetric hyperbolic system for ξ ($\mathbb{R}$-dimension 11×11), by (G4).

3.2 Second system: dynamical subset of the field equations

From now on, condition $(*)$ is assumed. By (G4), the $\mathbb{R}$-linear functional associated to any $\eta \neq 0$ in $(*)$ is non-trivial. Thus, $(*)$ restricts F to a vector space with $\dim_{\mathbb{R}} = 44 - 11 = 33$, that depends only on G. Fix a (constant, $\mathbb{R}$-linear) parametrization $\Phi \mapsto F = P(\Phi)$, with $\mathbb{R}^{33}$-valued Φ. For $P(\Phi)$ to solve the field equations (vii), it is *necessary* that

$$\boxed{\forall \Psi \text{ real}: \qquad \operatorname{Re}\Big(\overline{{P(\Psi)_{qa}}^A} {G^{ab}}_{AB} \varepsilon^{pq} \varepsilon^{mn} L_{mp}\big({P(\Phi)_{nb}}^B\big) \Big) = 0} \tag{3a}$$

or equivalently[18]

$$\forall \Psi \text{ real}: \qquad \operatorname{Re}\Big(\overline{{P(\Psi)_{qa}}^A} {G^{ab}}_{AB} \varepsilon^{pq} {U_{pb}}^B \Big) = 0 \tag{3b}$$

because ${G^{ab}}_{AB} {U_{pb}}^B = {G^{ab}}_{AB} {\mathbf{\Pi}_{pb}}^{Bk\ell}{}_C {U_{k\ell}}^C = 0$, by (vii) and (G2). Equation (3a) is a symmetric hyperbolic system for Φ ($\mathbb{R}$-dimension 33×33), by (G5)[19].

[17] To see this for (G5), (G6), pick e_a, f_a with $\overline{e_a} t^{ab} e_b = \overline{f_a} t^{ab} f_b = 1$ and $\overline{e_a} t^{ab} f_b = 0$. Then ${w_{na}}^A = e_n {p_a}^A + f_n {q_a}^A$, with ${p_a}^{\&A} = 0$ and ${q_a}^{\&A} = 0$.

[18] Replace $L_{mp}({P(\Phi)_{nb}}^B) = \delta^i_p \delta^j_b L_{mi}({P(\Phi)_{nj}}^B) = (\mathbf{S}^{ij}_{pb} + \frac{1}{2}\varepsilon^{ij}\varepsilon_{pb}) L_{mi}({P(\Phi)_{nj}}^B)$ in (3a). Then $\mathbf{S}^{ij}_{pb}$ yields (3b), but $\frac{1}{2}\varepsilon^{ij}\varepsilon_{pb}$ contributes nothing, because $P(\Psi)$ satisfies $(*)$.

[19] As an aside, consider (G5) with $t^{mn} = \overline{s^m} s^n$, $s \neq 0$, and suppose ${w_{ab}}^A$ satisfies $(*)$. To analyze equality in (G5) in this *degenerate* case, pick r^a with $\varepsilon_{ab} s^a r^b = 1$. Set $s_a = \varepsilon_{ab} s^a$,

3.3 Third system: the constraints propagate

Fix a solution Φ to (3a), equivalently (3b). Here and below, U is the field associated to $P(\Phi)$ through (iii). Recall $\varepsilon^{ab}U_{ab}{}^{A} = 0$ from (v). Set

$$\mathbf{U}_{ab}{}^{A} = U_{ab}{}^{A} + \varepsilon_{ab}\zeta^{A} \tag{4}$$

with $\overline{\zeta^{A}} = \zeta^{\sharp A}$ and $\zeta^{\& A} = 0$. Irrespective of the eventual choice of ζ:

(v') $\overline{\varepsilon^{ab}\mathbf{U}_{ab}{}^{A}} = \varepsilon^{ab}\mathbf{U}_{ab}{}^{\sharp A}$ and $\mathbf{U}_{ab}{}^{\& A} = 0$ by (v).

(vi') $\overline{\varepsilon^{mn}L_{im}(\mathbf{U}_{nj}{}^{A})} - \varepsilon^{mn}L_{jm}(\mathbf{U}_{ni}{}^{\sharp A}) = 0$ by (vi).

In addition, (3b) holds with U replaced by $\mathbf{U}$, because $P(\Psi)$ satisfies $(*)$. Uniquely fix ζ by strengthening to:

$$\forall F \text{ as in}^{20} \text{ (ii)}: \quad \mathrm{Re}\left(\overline{F_{qa}{}^{A}} G^{ab}{}_{AB} \varepsilon^{pq} \mathbf{U}_{pb}{}^{B} \right) = 0 \tag{$**$}$$

Equation $(**)$ is an $\mathbb{R}$-linear map[21] (U satisfying (3b)) $\mapsto \zeta$, that depends *only* on G. The kernel of this map contains **Weyl**. Therefore:

(vii') $\mathbf{U} \in \mathbf{Weyl}$ is equivalent to the field equations $U \in \mathbf{Weyl}$ in (vii).

$\mathbf{U}$ takes values in a vector space of $\dim_{\mathbb{R}} = 33$, by (v') and $(**)$. Recall that $\dim_{\mathbb{R}} \mathbf{Weyl} = 10$. Fix a (constant, $\mathbb{R}$-linear) parametrization $\widehat{\Phi} \mapsto \mathbf{U} + \mathbf{Weyl} = \widehat{\mathbf{P}}(\widehat{\Phi})$ of the $\dim_{\mathbb{R}} = 23$ quotient, with $\mathbb{R}^{23}$-valued $\widehat{\Phi}$. Then

$$\boxed{\forall \widehat{\Psi} \text{ real:} \qquad \mathrm{Re}\left(\overline{\widehat{\mathbf{P}}(\widehat{\Psi})_{qa}{}^{A}} G^{ab}{}_{AB} \varepsilon^{pq} \varepsilon^{mn} L_{pm}(\widehat{\mathbf{P}}(\widehat{\Phi})_{nb}{}^{B}) \right) = 0} \tag{5a}$$

To see this, first observe that the left hand side of (5a) is well defined by (G2): adding **Weyl**-valued fields to $\widehat{\mathbf{P}}(\widehat{\Psi})$ or $\widehat{\mathbf{P}}(\widehat{\Phi})$ does not affect the outcome. Thus, $\widehat{\mathbf{P}}(\widehat{\Phi})$ can be replaced by $\mathbf{U}$. Equation (5a) is equivalent to[22]

$$\forall \widehat{\Psi} \text{ real:} \quad \mathrm{Re}\left(\overline{\widehat{\mathbf{P}}(\widehat{\Psi})_{qa}{}^{A}} G^{ab}{}_{AB} \varepsilon^{pq} \left\{ \varepsilon^{mn} L_{pm}(\mathbf{U}_{nb}{}^{B}) - \overline{\varepsilon^{mn} L_{bm}(\mathbf{U}_{np}{}^{\sharp B})} \right\} \right) = 0 \tag{5b}$$

$r_a = \varepsilon_{ab} r^b$. Then $\delta_a^b = r_a s^b - s_a r^b$. Expand $w_{ab}{}^{A} = s_a \overline{s_b} \alpha^A + s_a \overline{r_b} \beta^A + r_a \overline{s_b} \gamma^A + r_a \overline{r_b} \delta^A$ with $\alpha^{\& A} = \beta^{\& A} = \gamma^{\& A} = \delta^{\& A} = 0$ and $\overline{\alpha^A} = \alpha^{\sharp A}$, $\overline{\gamma^A} = \beta^{\sharp A}$, $\overline{\delta^A} = \delta^{\sharp A}$. Equality implies $\mathbf{\Pi}_a{}^{Ab}{}_B s^n w_{nb}{}^{B} = s^n w_{na}{}^{A}$ by (G3), implies $\delta^A = 0$, $\gamma^\sigma = 0$, $\gamma_{am}{}^{bn} = z \delta_a^b \overline{s_m s^n}$, $\beta^A = \overline{\gamma^{\sharp A}}$, $z \in \mathbb{C}$; now $(*)$ determines α^A, see (G4). Conversely, they imply equality, for all $z \in \mathbb{C}$.

[20] F is a dummy, completely unrelated to $P(\Phi)$, and *does not* have to satisfy $(*)$.

[21] Given a U that satisfies (3b), interpret $(**)$ as 44 $\mathbb{R}$-linear equations for ζ. The 33 coming from $F \in \mathrm{image}\, P$ hold by (3b). Denote by $[F] = F + \mathrm{image}\, P$ elements of the quotient. The main observation is that $([F], \zeta) \mapsto \mathrm{Re}(\overline{F_{ba}{}^{A}} G^{ab}{}_{AB} \zeta^B)$ is a well defined $\mathbb{R}$-bilinear pairing between vector spaces of equal $\dim_{\mathbb{R}} = 11$, *non-degenerate* by (G4).

[22] The expression $\mathrm{Re}(\overline{\widehat{\mathbf{P}}(\widehat{\Psi})_{qa}{}^{A}} G^{ab}{}_{AB} \varepsilon^{pq} X_{pb}{}^{B})$ with $X_{ab}{}^{\& A} = 0$ is invariant under $X_{pb}{}^{B} \rightsquigarrow -\overline{X_{bp}{}^{\sharp B}}$, because $\widehat{\mathbf{P}}(\widehat{\Psi})$ satisfies $(**)$, in the role of $\mathbf{U}$. The arrow $\rightsquigarrow$ is used again later.

But (5b) holds, by (vi'). Therefore, (5a) holds as well: a symmetric hyperbolic system for $\widehat{\Phi}$ ($\mathbb{R}$-dimension 23×23), by (G6).

For later reference, record for both $\bigstar = \delta_p^\ell \mathbf{\Pi}_b{}^{Bc}{}_C$ and $\bigstar = \delta_b^c \mathbf{\Pi}_p{}^{\sharp B\ell}{}_{\sharp C}$, and for all Y with $Y_{ab}{}^{\& A} = 0$, the identity[23]

$$\forall \widehat{\Psi} \text{ real:} \qquad \mathrm{Re}\left(\overline{\widehat{\mathbf{P}}(\widehat{\Psi})_{qa}{}^{A}} G^{ab}{}_{AB} \varepsilon^{pq} \bigstar Y_{\ell c}{}^{C} \right) = 0 \tag{6}$$

4 From LMP to general relativity (GR)

This section is logically organized as follows. Subsections 4.1 and 4.2 gently replace LMP by GR; the field equations are now quasilinear and contain non-principal terms. Subsection 4.3 introduces a more general, inhomogeneous gauge condition. In the light of these modifications, Subsections 4.4, 4.5, 4.6 revisit the derivation of symmetric hyperbolic systems.

4.1 Translation of LMP to fields with 'real four-indices'

The $\dim_{\mathbb{R}} = 4$ vector space of all v^{ab} with $\overline{v^{ba}} = v^{ab}$ (a unprimed, b primed), equipped with $(v, w) \mapsto -\varepsilon_{ab}\varepsilon_{mn} v^{am} w^{bn}$, is $(-,+,+,+)$ Minkowski space. *Thus, each Hermitian unprimed-primed $\mathcal{S}$-index pair is one 'real four-index'.* Below, such pairs are sometimes bracketed, $(am), (bn), \ldots$ When space is short, boldface $\mathbf{a}, \mathbf{b}, \ldots$ are used as placeholders for pairs. Set $k_{(am)(bn)} = -\varepsilon_{ab}\varepsilon_{mn}$. Observe that, for any f, the condition $f^{\& A} = 0$ is equivalent to

$$f_{(am)}{}^{(ck)} k_{(ck)(bn)} + f_{(bn)}{}^{(ck)} k_{(ck)(am)} - \tfrac{1}{2} f_{(ck)}{}^{(ck)} k_{(am)(bn)} = 0$$

Items (i), (ii) require no translation. To translate (iii), (v), (vi), (vii), a new field $U_{(am)(bn)}{}^{A}$ is introduced instead of $U_{mn}{}^{A}$:

(iii") $U_{(am)(bn)}{}^{A} = L_{(am)}(F_{(bn)}{}^{A}) - L_{(bn)}(F_{(am)}{}^{A})$, equivalent[24] to $U_{mn}{}^{A}$.

(v") $U_{(am)(bn)}{}^{A} = -U_{(bn)(am)}{}^{A}$, $U_{(am)(bn)}{}^{\& A} = 0$ and $\overline{U_{(am)(bn)}{}^{A}} = U_{(ma)(nb)}{}^{\sharp A}$.

(vi") $L_{(am)}(U_{(bn)(ck)}{}^{A}) + L_{(bn)}(U_{(ck)(am)}{}^{A}) + L_{(ck)}(U_{(am)(bn)}{}^{A}) = 0$.

(vii") $(U_{(am)(bn)}{}^{A}) \in \mathbf{Weyl}_{\mathbb{R}}$.

[23]For the first $\bigstar$ identity use (G2). For the second, use (G2) and $\rightsquigarrow$ from a previous footnote: $-(\delta_p^\ell \mathbf{\Pi}_b{}^{Bc}{}_C)\overline{Y_{c\ell}{}^{\sharp C}} \rightsquigarrow \overline{(\delta_b^\ell \mathbf{\Pi}_p{}^{\sharp Bc}{}_C)\overline{Y_{c\ell}{}^{\sharp C}}} = (\delta_b^\ell \mathbf{\Pi}_p{}^{\sharp Bc}{}_C) Y_{c\ell}{}^{\sharp C} = (\delta_b^c \mathbf{\Pi}_p{}^{\sharp B\ell}{}_{\sharp C}) Y_{\ell c}{}^{C}$.

[24]In fact, $U_{mn}{}^{A} = \frac{1}{2}\varepsilon^{ab} U_{ambn}{}^{A}$ and $U_{ambn}{}^{A} = \varepsilon_{ab} U_{mn}{}^{A} + \varepsilon_{mn} \overline{U_{ba}{}^{\sharp A}}$. For the second, write $U_{ambn}{}^{A} = (\mathbf{S}^{ij}_{ab} + \frac{1}{2}\varepsilon_{ab}\varepsilon^{ij})(\mathbf{S}^{pq}_{mn} + \frac{1}{2}\varepsilon_{mn}\varepsilon^{pq}) U_{ipjq}{}^{A}$, then exploit $U_{ambn}{}^{A} + U_{bnam}{}^{A} = 0$.

Here, $\mathbf{Weyl}_{\mathbb{R}}$ is the vector space of all $(f_{(am)(bn)}{}^{A})$ with $\overline{f_{(am)(bn)}{}^{A}} = f_{(ma)(nb)}{}^{\sharp A}$ and $f_{\mathbf{ab}}{}^{A} = -f_{\mathbf{ba}}{}^{A}$ and $f_{\mathbf{ab}}{}^{\& A} = 0$ and $f_{\mathbf{ab}}{}^{\sigma} = 0$ and $f_{\mathbf{ijk}}{}^{\mathbf{a}} + f_{\mathbf{jki}}{}^{\mathbf{a}} + f_{\mathbf{kij}}{}^{\mathbf{a}} = 0$ and $f_{\mathbf{ika}}{}^{\mathbf{a}} = 0$ and $f_{\mathbf{iak}}{}^{\mathbf{a}} = 0$. Observe that $\dim_{\mathbb{R}} \mathbf{Weyl}_{\mathbb{R}} = 10$.

Translations as in this subsection are implicit in the discussion below.

4.2 Orthonormal frame formalism for GR

The discussion about GR is local on $\mathbb{R}^4$. Set $(\text{ii})_{\text{GR}} = (\text{ii})$, $(\text{v})_{\text{GR}} = (\text{v''})$. The first implies, in particular, $F_{ab}{}^{\sigma} = \overline{F_{ba}{}^{\sigma}}$.

$(\text{i})_{\text{GR}}$ $F_{(ab)} = F_{(ab)}{}^{\sigma}\frac{\partial}{\partial x^{\sigma}}$ are linearly independent ('non-degenerate frame').

$(\text{iii})_{\text{GR}}$ Set $U_{(am)(bn)}{}^{A} = F_{(am)}(F_{(bn)}{}^{A}) - F_{(bn)}(F_{(am)}{}^{A}) + p(F,F)$ for a polynomial p. Explicitly, with $\mathbf{A}^{\mathbf{a}_1\ldots\mathbf{a}_k}_{\mathbf{b}_1\ldots\mathbf{b}_k} = \sum_{\pi\in S_k}\text{sgn}(\pi)\,\delta^{\mathbf{a}_1}_{\mathbf{b}_{\pi(1)}}\cdots\delta^{\mathbf{a}_k}_{\mathbf{b}_{\pi(k)}}$:

$$
\begin{aligned}
U_{\mathbf{ij}}{}^{\sigma} &= \mathbf{A}^{\mathbf{bc}}_{\mathbf{ij}}\big(F_{\mathbf{b}}(F_{\mathbf{c}}{}^{\sigma}) - F_{\mathbf{bc}}{}^{\boldsymbol{\ell}}F_{\boldsymbol{\ell}}{}^{\sigma}\big)\\
U_{\mathbf{ijm}}{}^{\mathbf{n}} &= \mathbf{A}^{\mathbf{bc}}_{\mathbf{ij}}\big(F_{\mathbf{b}}(F_{\mathbf{cm}}{}^{\mathbf{n}}) + F_{\mathbf{cm}}{}^{\boldsymbol{\ell}}F_{\mathbf{b}\boldsymbol{\ell}}{}^{\mathbf{n}} - F_{\mathbf{bc}}{}^{\boldsymbol{\ell}}F_{\boldsymbol{\ell}\mathbf{m}}{}^{\mathbf{n}}\big)
\end{aligned}
$$

$(\text{iv})_{\text{GR}}$ For all (φ,Δ), with φ a diffeomorphism (of open subsets of $\mathbb{R}^4$) and $\Delta^{(am)}{}_{(bn)} = \Theta^2 A^{a}{}_{b}\overline{A^{m}{}_{n}}$ (where the field A is $\text{SL}(2,\mathbb{C})$-valued and the field $\Theta > 0$), the local gauge transformation $F \mapsto \widetilde{F}$ given by

$$
\begin{aligned}
\widetilde{F}_{\mathbf{i}}{}^{\sigma}\circ\varphi &= \Delta^{\mathbf{a}}{}_{\mathbf{i}}\,F_{\mathbf{a}}(\varphi^{\sigma})\\
\widetilde{F}_{\mathbf{im}}{}^{\mathbf{n}}\circ\varphi &= \Delta^{\mathbf{a}}{}_{\mathbf{i}}\Big(F_{\mathbf{ak}}{}^{\boldsymbol{\ell}}\Delta^{\mathbf{k}}{}_{\mathbf{m}}(\Delta^{-1})^{\mathbf{n}}{}_{\boldsymbol{\ell}} + (\Delta^{-1})^{\mathbf{n}}{}_{\boldsymbol{\ell}}F_{\mathbf{a}}(\Delta^{\boldsymbol{\ell}}{}_{\mathbf{m}})\Big)
\end{aligned}
$$

implies the transformation $U \mapsto \widetilde{U}$ given by, with $U_{\mathbf{ab}} = U_{\mathbf{ab}}{}^{\sigma}\frac{\partial}{\partial x^{\sigma}}$:

$$
\begin{aligned}
\widetilde{U}_{\mathbf{ij}}{}^{\sigma}\circ\varphi &= \Delta^{\mathbf{a}}{}_{\mathbf{i}}\Delta^{\mathbf{b}}{}_{\mathbf{j}}\,U_{\mathbf{ab}}(\varphi^{\sigma})\\
\widetilde{U}_{\mathbf{ijm}}{}^{\mathbf{n}}\circ\varphi &= \Delta^{\mathbf{a}}{}_{\mathbf{i}}\Delta^{\mathbf{b}}{}_{\mathbf{j}}\Big(U_{\mathbf{abk}}{}^{\boldsymbol{\ell}}\Delta^{\mathbf{k}}{}_{\mathbf{m}}(\Delta^{-1})^{\mathbf{n}}{}_{\boldsymbol{\ell}} + (\Delta^{-1})^{\mathbf{n}}{}_{\boldsymbol{\ell}}U_{\mathbf{ab}}(\Delta^{\boldsymbol{\ell}}{}_{\mathbf{m}})\Big)
\end{aligned}
$$

$(\text{vi})_{\text{GR}}$ $F_{(am)}(U_{(bn)(ck)}{}^{A}) + F_{(bn)}(U_{(ck)(am)}{}^{A}) + F_{(ck)}(U_{(am)(bn)}{}^{A}) + q(F\oplus\partial F, U) = 0$, with q a polynomial, are differential identities for U. Explicitly:

$$
0 = \tfrac{1}{2}\mathbf{A}^{\mathbf{bcd}}_{\mathbf{ijk}}\big(F_{\mathbf{b}}(U_{\mathbf{cd}}{}^{\sigma}) - U_{\mathbf{cd}}(F_{\mathbf{b}}{}^{\sigma}) - 2F_{\mathbf{bc}}{}^{\boldsymbol{\ell}}U_{\boldsymbol{\ell}\mathbf{d}}{}^{\sigma} + U_{\mathbf{cdb}}{}^{\boldsymbol{\ell}}F_{\boldsymbol{\ell}}{}^{\sigma}\big) \qquad (7a)
$$

$$
\begin{aligned}
0 = \tfrac{1}{2}\mathbf{A}^{\mathbf{bcd}}_{\mathbf{ijk}}\big(&F_{\mathbf{b}}(U_{\mathbf{cdm}}{}^{\mathbf{n}}) - U_{\mathbf{cd}}(F_{\mathbf{bm}}{}^{\mathbf{n}}) + U_{\mathbf{cdm}}{}^{\boldsymbol{\ell}}F_{\mathbf{b}\boldsymbol{\ell}}{}^{\mathbf{n}}\\
&- F_{\mathbf{bm}}{}^{\boldsymbol{\ell}}U_{\mathbf{cd}\boldsymbol{\ell}}{}^{\mathbf{n}} - 2F_{\mathbf{bc}}{}^{\boldsymbol{\ell}}U_{\boldsymbol{\ell}\mathbf{dm}}{}^{\mathbf{n}} + U_{\mathbf{cdb}}{}^{\boldsymbol{\ell}}F_{\boldsymbol{\ell}\mathbf{m}}{}^{\mathbf{n}}\big)
\end{aligned} \qquad (7b)
$$

$(\text{vii})_{\text{GR}}$ Field equations: $(U_{(am)(bn)}{}^{A}) \in \mathbf{Weyl}_{\mathbb{R}}$.
Note that gauge transformations $(\text{iv})_{\text{GR}}$ map solutions to solutions.

These formulas are derived in [RT]. In the notation of [RT], $(\mathrm{ii})_{\mathrm{GR}}$: $\lozenge^1 \in \mathcal{P}^1$. $(\mathrm{iii})_{\mathrm{GR}}$: $\lozenge^2 = \frac{1}{2}[\![\lozenge^1, \lozenge^1]\!] \in \mathcal{P}^2$ where $[\![\,\cdot\,,\,\cdot\,]\!] : \mathcal{P}^k \times \mathcal{P}^\ell \to \mathcal{P}^{k+\ell}$ is a Lie superbracket. $(\mathrm{iv})_{\mathrm{GR}}$: equation (7.8) [RT]. $(\mathrm{vi})_{\mathrm{GR}}$: $[\![\lozenge^1, \lozenge^2]\!] = 0$ by the super Jacobi identity. $(\mathrm{vii})_{\mathrm{GR}}$: $\lozenge^2 \in \mathcal{P}^2_{\mathrm{vac}}$ where $\mathcal{P}^2_{\mathrm{vac}} \subset \mathcal{P}^2$ is the Weyl sector. See Proposition 4.5 [RT]. F and U are the components of $\lozenge^1$ and $\lozenge^2$.

This formalism is a subformalism of, and equivalent to[25], the Newman-Penrose-Friedrich [NP], [Fr] formalism, with ${F_{(ab)}}^A$ the frame ($A \in \mathcal{C}$) and connection ($A \notin \mathcal{C}$), ${U_{(am)(bn)}}^A$ the torsion ($A \in \mathcal{C}$) and curvature ($A \notin \mathcal{C}$).

4.3 Gauge-fixing revisited for GR

In the LMP discussion, it was assumed that ${G^{ab}}_{AB}$ and the parametrizations P and $\widehat{\mathbf{P}}$ are constant. *These assumptions are now dropped.* Furthermore, the homogeneous $(*)$ is replaced by the inhomogeneous

$$\boxed{\forall \eta \text{ with } \overline{\eta^A} = \eta^{\sharp A} \text{ and } \eta^{\& A} = 0 : \quad \mathrm{Re}\left(\overline{\eta^A} {G^{ab}}_{AB} {F_{ab}}^B\right) = \overline{\eta^A}\, i_A} \qquad (*_{\mathrm{GR}})$$

where the field i_A, with $i_{\& A} = 0$ and $\overline{i_A} = i_{\sharp A}$, is fixed beforehand, like ${G^{ab}}_{AB}$. The inhomogeneity i_A has 11 real components, and is important for $(\mathrm{i})_{\mathrm{GR}}$.

The parametrization becomes $\Phi \mapsto F = F_0 + P(\Phi)$, where F_0 is chosen beforehand and satisfies $(\mathrm{ii})_{\mathrm{GR}}$ and $(*_{\mathrm{GR}})$. On the other hand, $\Phi \mapsto P(\Phi)$ still parametrizes $(\mathrm{ii})_{\mathrm{GR}} = (\mathrm{ii})$ and the homogeneous $(*)$.

4.4 First system revisited: the gauge is locally realizable in GR

Assume F as in $(\mathrm{ii})_{\mathrm{GR}}$ is given. Don't assume $(*_{\mathrm{GR}})$ or $(\mathrm{vii})_{\mathrm{GR}}$. The condition for $\widetilde{F}$ in $(\mathrm{iv})_{\mathrm{GR}}$ to satisfy $(*_{\mathrm{GR}})$ is a partial differential equation for (φ, Δ):

$$\forall \eta \text{ as in } (*_{\mathrm{GR}}) : \qquad \mathrm{Re}\left(\overline{\eta^A} ({G^{ab}}_{AB} \circ \varphi)\, \bigstar\right) = \overline{\eta^A} (i_A \circ \varphi) \tag{8}$$

where $\bigstar = {\widetilde{F}_{ab}}{}^B \circ \varphi$ is expressed in terms of F, φ, Δ using $(\mathrm{iv})_{\mathrm{GR}}$. The principal part consists only of the terms involving $F_{\mathbf{a}}(\varphi^\sigma)$ and $F_{\mathbf{a}}({\Delta^{\boldsymbol{\ell}}}_{\mathbf{m}})$. Parametrize[26]

$$\varphi^\sigma = \xi^\sigma \qquad {\Delta^{\mathbf{a}}}_{\mathbf{b}} = {\exp(\xi)^{\mathbf{a}}}_{\mathbf{b}} = \textstyle\sum_{k=0}^\infty \frac{1}{k!} {\xi_{\mathbf{b}}}^{\mathbf{n}_1} {\xi_{\mathbf{n}_1}}^{\mathbf{n}_2} \cdots {\xi_{\mathbf{n}_{k-1}}}^{\mathbf{a}}$$

where $\overline{\xi^A} = \xi^{\sharp A}$ and $\xi^{\& A} = 0$. For every ξ, the map $\kappa \mapsto \mathcal{D}(\xi, \kappa)$ given by

$$\mathcal{D}(\xi,\kappa)^\sigma = \kappa^\sigma \qquad {\mathcal{D}(\xi,\kappa)_{\mathbf{m}}}^{\mathbf{n}} = {\exp(-\xi)^{\mathbf{n}}}_{\boldsymbol{\ell}}\, \tfrac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} {\exp(\xi + s\kappa)^{\boldsymbol{\ell}}}_{\mathbf{m}}$$

[25] The solution spaces are one-to-one, assuming sufficient differentiability of the solutions.
[26] A local parametrization around ${\xi_{\mathbf{m}}}^{\mathbf{n}} = 0$ suffices for this subsection.

is an invertible $\mathbb{R}$-linear map on the vector space given by $\overline{\kappa^A} = \kappa^{\sharp A}$, $\kappa^{\& A} = 0$. Therefore, equation (8) is equivalent to

$$\boxed{\forall \eta \text{ as in } (*_{\mathrm{GR}}): \ \mathrm{Re}\left(\overline{\mathcal{D}(\xi,\eta)^A}\Delta^{mn}{}_{ab}(G^{ab}{}_{AB}\circ\varphi)F_{mn}{}^{\sigma}\mathcal{D}(\xi,\tfrac{\partial}{\partial x^{\sigma}}\xi)^B\right) = (\mathrm{npt})} \tag{9}$$

where the non-principal terms (npt) are *without* derivatives of ξ. This is a symmetric hyperbolic system for ξ. It is analogous to (2), but quasilinear.

4.5 Second system revisited

Assume $(*_{\mathrm{GR}})$. Use $\Phi \mapsto F = F_0 + P(\Phi)$ from Subsection 4.3. Then $U_{mn}{}^A = \mathbf{S}^{ij}_{mn}\varepsilon^{ab}F_{ai}(P(\Phi)_{bj}{}^A) + (\mathrm{npt})$ by $(\mathrm{iii})_{\mathrm{GR}}$. The non-principal terms (npt) are without derivatives of Φ. The field equations $(\mathrm{vii})_{\mathrm{GR}}$ imply (3b), equivalently

$$\boxed{\forall \Psi \text{ real:} \qquad \mathrm{Re}\left(\overline{P(\Psi)_{qa}{}^A}G^{ab}{}_{AB}\varepsilon^{pq}\varepsilon^{mn}F_{mp}\big(P(\Phi)_{nb}{}^B\big)\right) = (\mathrm{npt})} \tag{10}$$

analogous to (3a). The equation is quasilinear, because F_{mp} depends on Φ.

4.6 Third system revisited

Fix a solution Φ to (3b), where U is the field associated to $F_0 + P(\Phi)$ through $(\mathrm{iii})_{\mathrm{GR}}$. Adopt (4) and $(**)$ verbatim to define $\mathbf{U}$. Then $(\mathrm{v'})_{\mathrm{GR}} = (\mathrm{v'})$ and

$(\mathrm{vi'})_{\mathrm{GR}}$ $\overline{\varepsilon^{mn}F_{im}(\mathbf{U}_{nj}{}^A)} - \varepsilon^{mn}F_{jm}(\mathbf{U}_{ni}{}^{\sharp A}) = \overline{\varepsilon^{mn}F_{im}(U_{nj}{}^A)} - \varepsilon^{mn}F_{jm}(U_{ni}{}^{\sharp A}) =$ (npt) by $(\mathrm{vi})_{\mathrm{GR}}$. The terms (npt) are without derivatives of U or $\mathbf{U}$.

and $(\mathrm{vii'})_{\mathrm{GR}} = (\mathrm{vii'})$. Parametrize $\widehat{\Phi} \mapsto \mathbf{U} + \mathbf{Weyl} = \widehat{\mathbf{P}}(\widehat{\Phi})$. Equation (5b) holds if L is replaced by F and appropriate non-principal terms (npt) from $(\mathrm{vi'})_{\mathrm{GR}}$ are added. The resulting system is equivalent to

$$\boxed{\forall \widehat{\Psi} \text{ real:} \qquad \mathrm{Re}\left(\overline{\widehat{\mathbf{P}}(\widehat{\Psi})_{qa}{}^A}G^{ab}{}_{AB}\varepsilon^{pq}\varepsilon^{mn}F_{pm}(\widehat{\mathbf{P}}(\widehat{\Phi})_{nb}{}^B)\right) = (\mathrm{npt})} \tag{11}$$

analogous to (5a). The non-principal terms (npt) are without derivatives of, and linear homogeneous in, U or $\mathbf{U}$, see $(\mathrm{vi})_{\mathrm{GR}}$ and $(\mathrm{vi'})_{\mathrm{GR}}$. *However, to use* (11) *to show that the constraints propagate, it is essential that the* (npt) *in* (11) *descend to linear homogeneous functions of* $\widehat{\mathbf{P}}(\widehat{\Phi})$.

To see that this is the case, hypothetically add in $(\mathrm{vi})_{\mathrm{GR}}$ an arbitrary field $X_{\mathbf{ab}}{}^A$ with values in $\mathbf{Weyl}_{\mathbb{R}}$ to $U_{\mathbf{ab}}{}^A$. Now, (7a) and (7b) are potentially violated. But (7a) is not violated, by the definition of $\mathbf{Weyl}_{\mathbb{R}}$. The right hand side of (7b) can be rewritten $\frac{1}{2}\mathbf{A}^{\mathbf{bcd}}_{\mathbf{ijk}}Y_{\mathbf{bcdm}}{}^{\mathbf{n}}$ where $Y_{\mathbf{bcdm}}{}^{\mathbf{n}} = F_{\mathbf{b}}(X_{\mathbf{cdm}}{}^{\mathbf{n}}) +$

$F_{\mathbf{b\ell}}{}^{\mathbf{n}} X_{\mathbf{cdm}}{}^{\boldsymbol{\ell}} - F_{\mathbf{bm}}{}^{\boldsymbol{\ell}} X_{\mathbf{cd\ell}}{}^{\mathbf{n}} - F_{\mathbf{bc}}{}^{\boldsymbol{\ell}} X_{\mathbf{\ell dm}}{}^{\mathbf{n}} - F_{\mathbf{bd}}{}^{\boldsymbol{\ell}} X_{\mathbf{c\ell m}}{}^{\mathbf{n}}$. For fixed index $\mathbf{b}$, the field $Y_{\mathbf{bcdm}}{}^{\mathbf{n}}$ is in $\mathbf{Weyl}_{\mathbb{R}}$. Thus, in the notation of Subsection 2.1, the field Y is in $(1,1) \otimes_{\mathbb{C}} ((4,0) \oplus (0,4)) = (5,1) \oplus (3,1) \oplus (1,3) \oplus (1,5)$. In turn, $\frac{1}{2}\mathbf{A}_{\mathbf{ijk}}^{\mathbf{bcd}} Y_{\mathbf{bcdm}}{}^{\mathbf{n}}$, being antisymmetric in $\mathbf{ijk}$, is in $(3,1) \oplus (1,3)$. Such violations of (7b) do not contribute to (11): those in $(3,1)$ do not contribute by the second $\bigstar$ in (6), those in $(1,3)$ do not contribute by the first $\bigstar$ in (6).

A different proof uses $[\![\Diamond^1, \mathcal{P}^2_{\mathrm{vac}}]\!] \subset \mathcal{P}^3_{\mathrm{vac}}$ from Lemma 5.1 [RT], then (6).

4.7 Systematic choice of $G^{ab}{}_{AB}$

Under the $\mathrm{SL}(2,\mathbb{C})$-action in Subsection 2.1, $G^{ab}{}_{AB}$ transforms as a direct sum of irreducible representations, all of type $(1,1)$, $(1,3)$, $(3,1)$, $(3,3)$. The trivial representation $(0,0)$ does not appear. Thus, no nonzero $G^{ab}{}_{AB}$ is $\mathrm{SL}(2,\mathbb{C})$-invariant.

However, to every subgroup $H \subset \mathrm{SL}(2,\mathbb{C})$ one can associate the subspace of all H-invariant $G^{ab}{}_{AB}$. *The positive*[27] *elements of this subspace are the natural candidates for $G^{ab}{}_{AB}$, if a given problem or physical situation is symmetric with respect to*[28] *H.*

For example, consider physical situations in which only the timelike vector δ^{ab} (a unprimed, b primed) is distinguished. The stabilizer group is $\mathrm{SU}(2) \subset \mathrm{SL}(2,\mathbb{C})$. A particular example of an $\mathrm{SU}(2)$-invariant gauge that satisfies (G1), (G2), (G3) is given by $G^{ab}{}_{AB} = \delta^{pq}\delta_{PQ}\boldsymbol{\pi}_p{}^{Pa}{}_A \boldsymbol{\pi}_q{}^{Qb}{}_B$, with the projection operator $\boldsymbol{\pi}_a{}^{Ab}{}_B = (\delta^c_a\delta^A_C - \boldsymbol{\Pi}_a{}^{Ac}{}_C)(\delta^b_c\delta^C_B - \delta^b_c\boldsymbol{\Pi}^C{}_E\boldsymbol{\Pi}^{\sharp E}{}_{\sharp B})$.

References

[Fr] Friedrich, H., Proc. R. Soc. Lond. A **375**, (1981) 169-184

[NP] Newman, E. and Penrose, R., J. Math. Phys. **3**, (1962) 566-578

[RT] Reiterer, M. and Trubowitz, E., arxiv.org/abs/0910.4666, v1

[Tay] Taylor, M. E., *Partial Differential Equations III* (Springer, 1997) 360-370

[27] Positivity in the sense of (G3).

[28] The discussion is oversimplified. Lorentz transformations that change the orientation, and / or flip future and past light cones, are ignored. $\mathcal{C}$-indices are ignored. The discussion is point-by-point. The field character of $G^{ab}{}_{AB}$ (Subsection 4.3) is ignored. And so forth.