State sum models and the spectral action

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Outline

Induced gravity and the spectral action

State sum models

Geometry – matter couplings

Standard model

Fermions: $\Psi = 8 \times 3$ Dirac spinors

Bosons: $d =$ gravitational Dirac

$A =$ gauge fields

$H =$ Higgs

Generalised Dirac operator

$$D = d + A + yH + m$$

$$y = \text{Yukawa mass matrix}$$

$$m = \text{Majorana mass matrix}$$

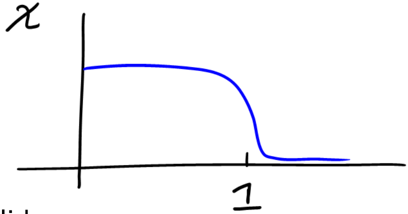
Fermionic action

$$S = \int \overline{\Psi} D \Psi \, \mathrm{d}V$$

Connes-Chamseddine spectral action

$$S_{CC} = \mathrm{Tr}\,\chi(D^2/c^2)$$

c = cutoff energy



- Euclidean
- Spectral
- Asymptotics $\to$ Bosonic SM + gravity

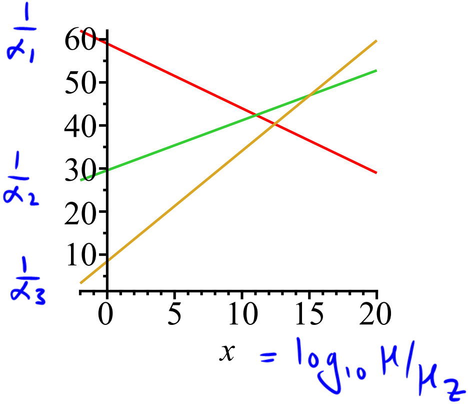
Gauge couplings:

	SU(3)	SU(2)	U(1)
	g_3	g_2	g'

- $g_3 = g_2 = g_1 = \sqrt{\frac{5}{3}} g'$
- etc

Running couplings

$\alpha = \frac{g^2}{4\pi}$



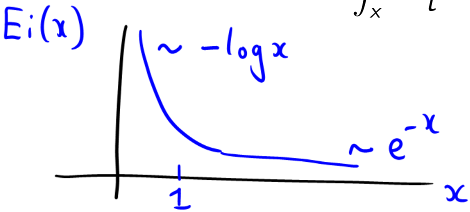
Fermion functional integral

$$\det D = e^{\frac{1}{2}\mathrm{Tr}\log D^2} = e^{-I}$$

- Cutoff energy c (quantum gravity)

Cutoff example (heat kernel): $\log \to -\mathrm{Ei}$

$$\mathrm{Ei}(x) = \int_x^\infty \frac{\mathrm{d}t}{t}\, e^{-t}$$



Hence induced bosonic action

$$I = \frac{1}{2}\mathrm{Tr}\,\mathrm{Ei}(D^2/c^2)$$

… as in Induced Gravity (Sakharov)

$-+++$ metric and N fermion fields. Cutoff $= c$.

If

$$S = \int \left(\overline{\psi} D_c \psi - 2\Lambda_0\right) \mathrm{d}V$$

Integrating over fermion modes gives

$$I = \int \frac{-c^4 N}{32\pi^2} - 2\Lambda_0 + \frac{c^2 N}{192\pi^2} R + \text{etc.} \quad \mathrm{d}V$$

- Correct sign for R (MTW signs)
- Cosmological constant $\Lambda = \Lambda_0 + \frac{c^4 N}{64\pi^2}$
- Effective below fermion mass

Example: induced Yang-Mills term

- A gauge field
- Ψ Dirac fermion, mass $m \neq 0$

gives induced Yang-Mills term

$$\frac{1}{g^2} \int \mathrm{Tr} F^2 \, \mathrm{d}V.$$

Induced coupling constant

$$\frac{1}{\alpha} = \frac{4\pi}{g^2} = \frac{2}{3\pi} \; \overset{\frac{1}{2}}{T(R)} \; \log c/m.$$

Yang Mills couplings

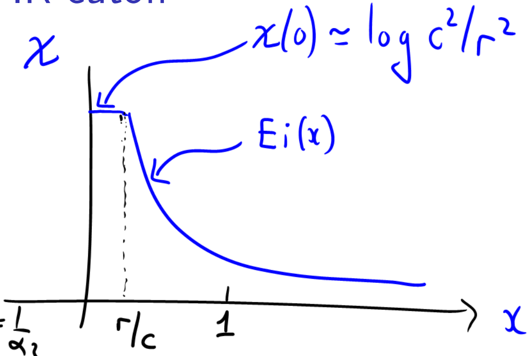
cf ~

$$\frac{1}{\alpha} = \frac{1}{3\pi} \sum_{\text{Weyl fermions}} T(R) \; \log c/m.$$

Compare with Connes-Chamseddine

$$\frac{1}{\alpha_{CC}} = \frac{1}{6\pi} \sum_{\text{Weyl fermions}} T(R) \, \chi(0)$$

CC as IR cutoff



$\frac{1}{\alpha_1} = \frac{1}{\alpha_2} = \frac{1}{\alpha_3}$

ind of fermion

$$\frac{1}{\alpha_{CC}} = \frac{1}{6\pi} \sum_{\text{Weyl}} T(R)\, \chi(0) = \frac{1}{3\pi} \sum_{\text{Weyl}} T(R)\, \log c/r$$

- Scaling $c \to \infty$, $r \to \infty$, eventually $r > m$

Partition function

$$Z = \int \mathrm{d}D \, \mathrm{d}\Psi \, \mathrm{d}\overline{\Psi} \; e^{iS(D,\psi)}$$

as ‘usual’. (State sum model?)

Below fermion masses, fermions can be removed, renormalising the bosonic couplings (fermion decoupling theorem)

Induced standard model (speculative)

$$S = \int (\overline{\Psi} D \Psi - 2\Lambda_0) \, \mathrm{d}V$$

- Bosonic SM+gravity action induced...

Possible scenarios

- Extra generation of heavy fermions (e.g. S_{CC})
- RH neutrino
- New physics

Induced Holst action

$$\int R^{ab} \wedge e^c \wedge e^d \, \varepsilon_{abcd} \quad + \frac{1}{\gamma} R^{ab} \wedge e_a \wedge e_b$$

- D contains parts of torsion
- Induced action is ECSK action, plus Holst term
- Immirzi parameter for S_{CC}

$$\frac{1}{\gamma} = k \, \frac{N_L - N_R}{N_L + N_R}$$

State sum models

- Discrete functional integrals on a triangulated manifold
- Input: gauge group (or category $\mathcal{C}$)
- Optional input: data for a local observable
- Output: partition function $Z \in \mathbb{C}$

Examples from physics

- lattice gauge theory
- 2d BF
- 3d quantum gravity
- 4d quantum gravity models

Ponzano-Regge $G = SU(2)$

Turaev-Viro $G = U_q\, sl2$

Category $\mathcal{C}$

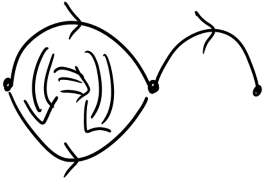
Manifold dimension n

Requires (n-1)-category $\mathcal{C}$:

- Morphisms have duals
- Diagrams have diffeomorphism invariance on S^{n-1}
- Linearity over $\mathbb{C}$. (Quantum Mechanics)
- Finiteness, semisimplicity

e.g.
spin networks

3-category



A weakish 3-category $\mathcal{C}$ contains

- Objects (0-morphisms)
- 1-morphisms mapping objects
- 2-morphisms mapping 1-morphisms (between the same objects)
- 3-morphisms mapping 2-morphisms (between the same 1-morphisms)
- 3 composition laws
- Structural maps for associativity, compatibility, units

Examples:

- A braided monoidal category (0- 1-morphisms trivial)
- Rep(crossed module) (0-morphisms trivial)
- Bicat

State sum model definition

$(n \equiv d)$

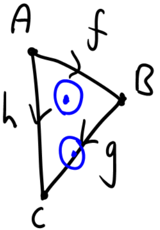
Labelling: k-simplexes $\longrightarrow$ k-morphisms   k=0,1,2,...,d-1

Weight of simplex σ: $W(\sigma) = S^{n-1}$ diagram $\in \mathbb{C}$

State sum:

$$Z = \sum_{\text{labellings}} \prod_{\sigma} W(\sigma)^{\pm 1}$$

$+1$	$d-1$
-1	$d-2$
$+1$	$d-3$
$\vdots$	$\vdots$



$$w(\sigma_2) = \mathrm{tr}\, f g h^*$$

$$w(\sigma_1) = \mathrm{tr}\, g g^*$$

Triangulation independence

constraints (?)

Summing over a suitable basis of all morphisms $\longrightarrow$ triangulation independence.

$\mathrm{tr} \int B \wedge F_A + \Lambda B \wedge B$

Examples:

- 3d quantum gravity
- Crane-Yetter model for $\mathcal{C} = \mathrm{Rep}(\mathrm{U}_q G)$
- Quantum flat space $\mathcal{C} = \mathrm{Rep}(\text{Poincare 2-group})$

Constraints or extra data destroy triangulation independence:

- 4d quantum gravity models
- Lattice gauge theory

? $B = *(e \wedge e)$

‘Diffeomorphism’ invariance

M, N triangulated manifolds, $d \leq 4$.

$$\begin{aligned}\text{‘Diffeomorphism’} &= \text{PL homeomorphism}\colon M \to N \\ &= \text{Simplicial isomorphism}\colon M' \to N'\end{aligned}$$

for some subdivisions M', N'.

For $Z(M)$:

triangulation independence $\leftrightarrow$ ‘diffeomorphism’ invariance

Proof: subdivide.

4d quantum gravity models

Models with a geometric interpretation:

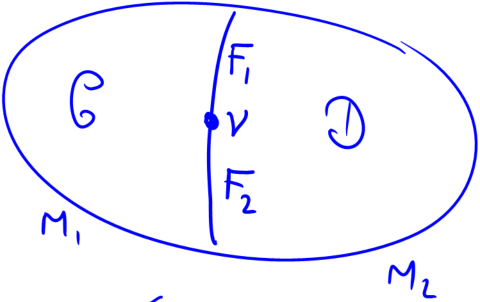
- $\mathcal{C} = \text{Rep}(\mathrm{SU}(2))$ with constraints: Plebanski gravity
- $\mathcal{C} = \text{Rep}(\mathrm{SU}(2) \times \mathrm{SU}(2))$ with constraints: 4d Euclidean gravity models
- $\mathcal{C} = \text{Rep}(\mathrm{SO}(3,1))$ with constraints: 4d Lorenzian gravity models $(k, p) \in \frac{1}{2}\mathbb{Z} \times \mathbb{R}$ $\quad p = \gamma k$, $k > 0$
- Cosmological constant: $G \to \mathrm{U}_q G$

Properties of models

- Geometry is discrete ✓ area quantized γk
- Approximates quantized Holst GR in limit(?) $w(\sigma^4)$ ✓
- Triangulation independence lost manifold ?

Geometry - matter coupling

- Matter as defects in a state sum model



$F : \mathcal{D} \to \mathcal{C}$

$\nu : F_1 \to F_2$

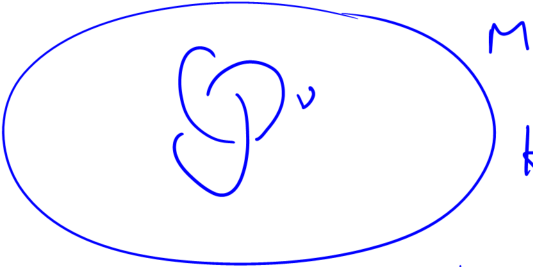
$m : \nu_1 \to \nu_2$

$$Z = \langle Z(M_1), F(Z(M_2)) \rangle$$

Example: 3d, $\mathcal{C} = \mathrm{Rep}(U_q G)$

$$\nu \colon \mathrm{id} \to \mathrm{id}$$

$$\nu \in \mathrm{Rep}(DU_q G)$$



knot invariant

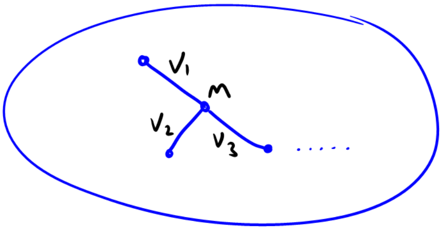
"Jones poly"

3d gravity, $\mathcal{C} = \mathrm{Rep}(U_q\mathrm{sl}2)$

$$\nu \colon \mathrm{id} \to \mathrm{id}$$

$$\nu \in \mathrm{Rep}(DU_q sl2) = \mathrm{Rep}(U_q\mathrm{sl}2 \times \mathrm{U}_q\mathrm{sl}2) = (j, j), \qquad j \in \frac{1}{2}\mathbb{Z}$$

planar graph



Z = probability for random vertices on $S^3_{r/2\pi}$ with distance(edge)$=j+1/2$.

State sum models - my wish list (speculative)

Constructed from $\mathcal{C} \in$ 3-cat

- Diffeomorphism-invariant TQFT on 4-manifolds
- State sum model for geometries D
- Discrete geometry at Planck scale
- Fermionic matter couplings Ψ
- Gaussian integral for Ψ
- Matter modes cut off at Planck energy c.
- Cosmological constant $(\Lambda_0 < 0)$
- Low energy limit: induced SM +gravity