Spin networks as twisted geometries

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Outline

1. Motivations and overview
   Why do we need discrete geometries in loop quantum gravity?
2. Spin networks, twisted geometries and polyhedra
   Definition and relation to holonomy and fluxes
3. From spinors to twisted geometries
   Spinorial tools and derivation of the holonomy-flux algebra from harmonic oscillators
4. Conclusions

Outline

Motivations and overview

Spin networks, twisted geometries and polyhedra

From spinors to twisted geometries

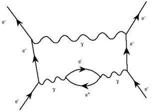
Conclusions

Stating the problem

- LQG is a continuum theory with well-defined and interesting kinematics (spin networks, discrete spectra of geometric operators, etc.)
- Models for the dynamics exist
- Main open problem: how to test the theory and extract low-energy physics from it

Why is it so hard? The quanta are exotic

Stating the problem

	kinematics	dynamics
QFT:	$\|n, p_i, h_i\rangle$	
	quanta: momenta, helicities, etc.	Feynman diagrams
	observables n: # of quantum particles	perturbative expansion degree of the graph $\Downarrow$ order of approximation desired

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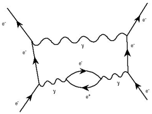
QFT:	$\lvert n, p_i, h_i \rangle$	
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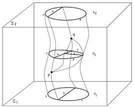
LQG:	$\lvert \Gamma, j_l, i_n \rangle$	
	quanta: areas and volumes	spin foams
	link to classical geometries? meaning of Γ?	what approximation?

Stating the problem

Aim of the talk:

Linking LQG states on a fixed graph with a notion of discrete geometry

Work in collaboration with L. Freidel 1001.2748 and 1006.0199
C. Rovelli 1005.2927
E. Bianchi and P. Doná 1009.3402

Canonical quantum theory: Spin networks

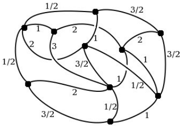
- Gauge-invariant Hilbert space:

$$\mathcal{H} = \bigoplus_{\Gamma} \mathcal{H}_\Gamma$$

- Graph Hilbert space: $(\Gamma = L$ links, N nodes$)$

$$\mathcal{H}_\Gamma = L_2[SU(2)^L/SU(2)^N] = \bigoplus_{j_l} \left[\bigotimes_n \mathcal{H}_n \right]$$

(Peter-Weyl decomposition)



links $\mapsto$ $SU(2)$ irrep: $V^{(j_l)}$

nodes $\mapsto$ $SU(2)$-invariant space: $\mathcal{H}_n \equiv \mathrm{Inv}\left[\bigotimes_{l \in n} V^{(j_l)} \right]$

- Orthogonal basis: spin network states $|\Gamma, j_l, i_n\rangle$

$$\Psi_{(\Gamma, j_l, i_n)}[g_l] = \mathrm{Tr}\left[\otimes_l D^{(j_l)}(g_l) \otimes_n i_n \right]$$

(• graph Γ; • spin j_l on each link; • an intertwiner i_n assigned to each node)

Operator algebra

- Functions

$$\Psi[g_1, \dots g_n] \in \mathcal{H}_\Gamma = L_2[SU(2)^L / SU(2)^N]$$

- Algebra of operators: Holonomy-Flux algebra

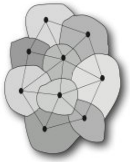
Holonomy:	$g_l = \mathcal{P}e^{\int_l A}$	$\longrightarrow$	multiplicative operator g_l
Flux:	$X_l = \int_{l*} E$	$\longrightarrow$	derivative operator $\vec{L}_l$

- Composite operators $\mathcal{O}(g_{ab}, K_{ab}) \mapsto \hat{O}(\vec{L}_l, g_l)$

  (note: metric composite operator)

Spin networks and quantum geometry

QFT	LQG
$\mathcal{F} = \bigoplus_n \mathcal{H}_n$	$\mathcal{H} = \bigoplus_\Gamma \mathcal{H}_\Gamma$
$\vert n, p_i, h_i\rangle \to$ quanta of fields	$\vert\Gamma, j_e, i_v\rangle \to$ quanta of space



Spin networks are eigenstates of geometric operators such as surface areas

- spins j_l are quantum numbers for areas of surfaces dual to links

  quanta of area $\qquad A(\Sigma) = \gamma\hbar G \sum_{l\in\Sigma} \sqrt{j_l(j_l+1)}$

- intertwiners i_n are quantum numbers for volumes of regions dual to nodes

  quanta of volumes $\qquad V(R) = (\gamma\hbar G)^{3/2} \sum_{n\in R} f(j_e, i_n)$

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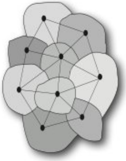

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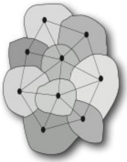
Key result

geometric operators turn out to have discrete spectra
with minimal excitations proportional to the Planck length

The problem of the semiclassical limit



- spins $\mapsto$ quanta of area
- intertwiners $\mapsto$ quanta of volumes

This information is not enough to recover a classical geometry (not even a discrete one) just as the $|q\rangle$ eigenstates in QM do not describe classical states

Three aspects of quantum geometry:

- discrete eigenvalues
- non-commutativity
- graph structure

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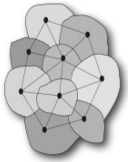

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Three aspects of quantum geometry:

- discrete eigenvalues
- non-commutativity
- graph structure

Semiclassical states: States peaked on a classical geometry with minimal uncertainty

$$\langle\Psi|\hat{g}_{ab}|\Psi\rangle = q_{ab}, \qquad \langle\Psi|\hat{K}_{ab}|\Psi\rangle = K_{ab}, \qquad \text{etc.}$$

Can be built from an holomorphic quantization of the classical phase space.
[Thiemann and collaborators]

BUT:
In practical calculations one works with a fixed graph

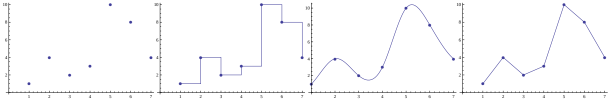
Can we give a meaning to semiclassical states on a fixed graph?

Geometry on a single graph?

$$\{A^i_a(x), E^b_j(y)\} \qquad \longrightarrow \qquad \mathcal{H} = \bigoplus_\Gamma \mathcal{H}_\Gamma, \qquad |\Gamma, j_e, i_v\rangle$$

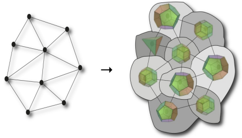
- Consider a single graph Γ, and the associated Hilbert space $\mathcal{H}_\Gamma$.
- This truncation captures only a finite number of degrees of freedom of the theory, thus (semiclassical) states in $\mathcal{H}_\Gamma$ do not represent smooth geometries.
- Can they represent a *discrete* geometry, approximation of a smooth one on the given graph?

The problem is similar to a choice of interpolation:



Can we interpret $\mathcal{H}_\Gamma = \bigoplus_{j_e} \left[\bigotimes_v \mathcal{H}_v \right]$ as the quantization of a space of discrete geometries?

The answer: twisted geometries



Each classical holonomy-flux configuration on a fixed graph can be visualized as a collection of adjacent polyhedra with extrinsic curvature between them

- Each polyhedron is locally flat: curvature emerges at the faces, as in Regge calculus
- They induce a *discontinuous* discrete metric: two neighbouring polyhedra are attached by faces with same area but different shape

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Classical polyhedra

Take F vectors $j_l N_l$ in R^3, subject to the closure condition $C \equiv \sum_l j_l N_l = 0$, and consider the space

$$\mathcal{P}_F = \left\{ N_l \in (S^2)^F \,|\, C = 0 \right\}$$

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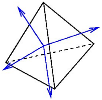
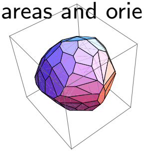
Two key theorems:

1. Minkowski: F (non-coplanar) closed normals identify a *unique* (bounded convex) polyhedron with areas j_l
   $\Longrightarrow P_F$ is a space of shapes of a polyhedron with fixed areas and orientation

$F = 4:$  $F > 4:$ 

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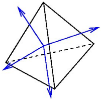
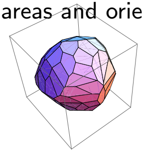
   $F = 4:$  $F > 4:$ 

2. Kapovich and Millson: The space modulo rotations,

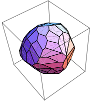
$$\mathcal{S}_F \equiv \mathcal{P}_F / SO(3), \qquad \dim \mathcal{S}_F = 2(F-3)$$

   is a phase space, with Poisson brackets obtained from symplectic reduction of those of the sphere

$$\{N^i, N^j\} = \frac{1}{j_l} \epsilon^{ijk} N^k$$

   $\Longrightarrow S_F$ is a space of shapes of a polyhedron with fixed areas

Geometry of polyhedra



Lasserre '83, E. Bianchi, P. Doná and SS, '10

Explicit reconstruction procedure: $(j_l, N_l) \mapsto$ edge lengths, volume, adjacency matrix

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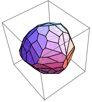

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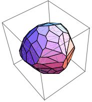

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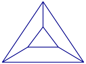
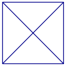
$F = 5$

Dominant:  Codimension 1: 

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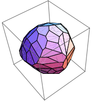

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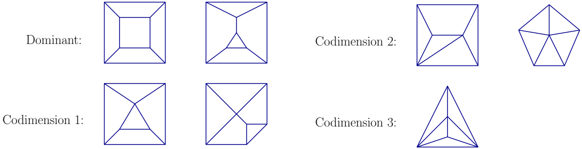

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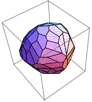

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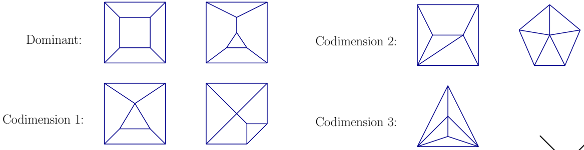
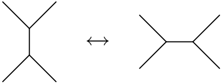


- The classes are all connected by 2-2 Pachner moves (they are all tessellations of the 2-sphere)



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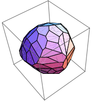

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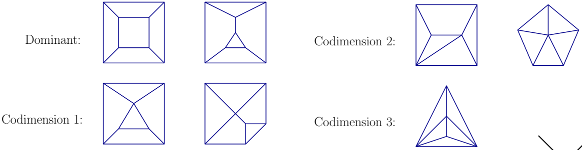

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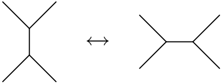
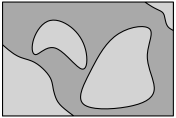


It is the configuration of normals to determine the class

- The phase space $\mathcal{S}_F$ can be mapped in regions corresponding to different classes.
  - *Dominant classes* have all 3-valent vertices.
    [maximal n. of vertices, $V = 3(F-2)$, and edges, $E = 2(F-2)$]
  - *Subdominant classes* are special configurations with lesser edges and vertices, and span measure zero subspaces.
    [lowest-dimensional class for maximal number of triangular faces]



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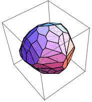

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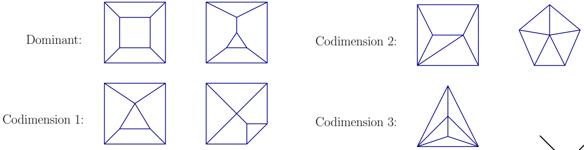

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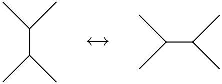

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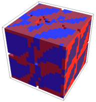


3d slice of $\mathcal{S}_6$, cuboids blue

Polyhedra and intertwiners

The quantization of this classical phase space with its Poisson algebra gives the intertwiner space of LQG,

$$\mathcal{H}_n = \mathrm{Inv}\left[\otimes_{l\in n} V^{j_l}\right]$$

with its SU(2) operator algebra

$$\vec{J}_i, \qquad \vec{J}_i\cdot\vec{J}_j, \qquad \vec{J}_i\cdot\vec{J}_j\wedge\vec{J}_k$$

[Kapovich and Millson '96, '01, Charles '08] [Conrady and Freidel '08, Barrett et al. '08]

Quantization map: $j_i N_i \mapsto \vec{J}_i$

$$\begin{array}{ccc}
\times_{l\in n} S^2_{j_l} & \longrightarrow & \otimes_{l\in n} V^{j_l} \\
\downarrow & & \downarrow \\
\text{S. reduction} & & \text{Q. reduction} \\
C = 0 & & \vec{J} = 0 \\
\downarrow & & \downarrow \\
\mathcal{S}_F & \longrightarrow & \mathcal{H}_n = \mathrm{Inv}\left[\otimes_{l\in n} V^{j_l}\right]
\end{array}$$

Guillemin-Sternberg theorem

Remarks

- Semiclassical states in $\mathcal{H}_n = \mathrm{Inv}\left[\otimes_{l\in n} V^{j_l}\right]$ represents polyhedra with $F(=$ valence of the node) faces
- Intertwiner states correspond to fuzzy polyhedra
- Polyhedral volume operator
  [E. Bianchi, P. Doná and SS 1009.3402]
- Semiclassical evaluation of the volume spectrum using structure of the phase space (4-valent case) $\Rightarrow$ excellent agreement
  [E. Bianchi, H. Haggard 1102.5439]

Polyhedra and intertwiners 2

On a single node:

$$\boxed{\mathcal{S}_F \quad \longrightarrow \quad \mathcal{H}_n = \mathrm{Inv}\left[\otimes_{l\in n} V^{j_l}\right]}$$

How about the full graph?

LQG Hilbert space on a fixed graph:

$$\mathcal{H}_\Gamma = \bigoplus_{j_l}\left[\bigotimes_n \mathcal{H}_n\right]$$

Just as the intertwiners are the building block of the Hilbert space, polyhedra are the building blocks of the classical phase space

Can we make this representation explicit?

The fundamental variables quantized in $\mathcal{H}_\Gamma$ are holonomies and fluxes:

$\Longrightarrow$ Map the holonomies and fluxes into areas and normals

Phase spaces of LQG

Hilbert space: $\mathcal{H} = \bigoplus_\Gamma \mathcal{H}_\Gamma$

- kinematical loop gravity $\Longrightarrow$ $\mathcal{H}_\Gamma = L_2(SU(2)^L)$ $\longrightarrow P_\Gamma$
  twisted geometries

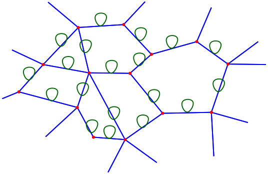
  $\downarrow$ *Gauss law*

- gauge-inv. loop gravity $\Longrightarrow$ $\mathcal{H}_\Gamma = L_2(SU(2)^L/SU(2)^N)$ $\longrightarrow S_\Gamma$
  closed twisted geos and polyhedra

Consider first the *non* gauge-inv. level:
The kinematical Hilber space $L_2[SU(2)]$ with the holonomy-flux algebra is a quantization of the classical phase space $T^*SU(2)$ with its canonical Poisson algebra

Phase space of loop gravity on a fixed graph

$$\boxed{\times_l \, T^*SU(2)}$$



A spinning top for each link of the graph, $T^*SU(2) = R^3 \times S^3 \ni (X_l, g_l)$

Interpretation:

- flux: $X_l = \int_{l^*} E$
- holonomy: $g_l = \exp \int_l A$

Mapping link by link:

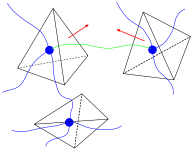
$$(X, g) \in R^3 \times SU(2) \Longrightarrow (N, \tilde{N}, j, \xi) \in S^2 \times S^2 \times R \times S^1$$

$$X = jN, \qquad g = n e^{\xi \tau_3} \tilde{n}^{-1}$$

Gauge-invariance and polyhedra

- On each link:

- On the full graph:



  Each link around a node has associated an area j and a normal N, plus an angle ξ

- Apply closure condition at each node
  $\Longrightarrow$ collection of polyhedra plus the angle ξ conjugated to the area

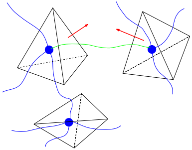
> Each classical holonomy-flux configuration on a fixed graph can be visualized as a collection of adjacent polyhedra with extrinsic curvature between them

- Each polyhedron is locally flat: curvature emerges at the faces, as in Regge calculus

(Draw picture)

- They induce a *discontinuous* discrete metric: two neighbouring polyhedra are attached by faces with same area but different shape

Some technicalities: isomorphism



On each link: $X, g \quad \Longrightarrow \quad N, j, \xi, \tilde{N}$

Isomorphism

$$(N, \tilde{N}, j, \xi) \Longrightarrow (X, g) \ : \qquad \begin{array}{rcl} X & = & jN \\ g & = & n e^{\xi \tau_3} \tilde{n}^{-1} \end{array}$$

$$\text{Hopf map} \quad \begin{array}{rcl} \pi : SU(2) & \mapsto & S^2 \\ n & \mapsto & N = n \tau_3 n^{-1} \end{array}$$

Properties:

- $\tilde{X} = j\tilde{N} = -g^{-1} X g$ $\qquad$ ($\Rightarrow$ left-invariant vector field)
- invertible provided $a \neq 0$
- 2-to-1: $(N, \tilde{N}, j, \xi)$ and $(-N, -\tilde{N}, -j, -\xi)$ give the same (X, g)
- U(1) gauge action inside g corresponding to changes of section $n : S^2 \mapsto SU(2)$

Some technicalities: Poisson brackets on the twisted geometries

- Poisson algebra of $T^*\mathrm{SU}(2)$

$$\{X^i, X^j\} = \epsilon^{ij}{}_k X^k, \qquad \{X^i, \tilde{X}^j\} = 0 \qquad \{X^i, g\} = -\tau^i\, g, \qquad \{\tilde{X}^i, g\} = g\, \tau^i$$

- Isomorphism

$$X = jN, \qquad g = n e^{\xi \tau_3} \tilde{n}^{-1}$$

- Symplectic potential

$$\Theta_{T^*SU(2)} = \mathrm{Tr}[X \mathrm{d}g g^{-1}] \quad = \quad \Theta_{\mathcal{S}^2_j}(N) + \Theta_{\mathcal{S}^2_j}(\tilde{N}) + j\mathrm{d}\xi$$

- Induced Poisson brackets

$$\{N^i, N^j\} = \frac{1}{j}\epsilon^{ij}{}_k N^k, \qquad \{\tilde{N}^i, \tilde{N}^j\} = \frac{1}{j}\epsilon^{ij}{}_k \tilde{N}^k, \qquad \{N^i, \tilde{N}^j\} = 0,$$

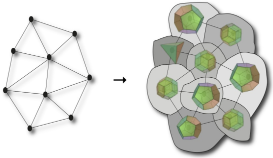
$$\{\xi, j\} = 1, \qquad \{N^i, j\} = 0, \qquad \{\tilde{N}^i, j\} = 0,$$

$$\{\xi, jN^i\} \equiv L^i(N), \qquad \{\xi, j\tilde{N}^i\} \equiv L^i(\tilde{N})$$

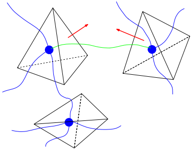
- $L : \mathcal{S}^2 \mapsto \mathbb{R}^3$ unique up to change of section. For the Hopf section, $L^i = (-\bar{z}, z, 1)$

Overview

Twisted geometries	$\Longleftrightarrow$	Loop gravity
$\downarrow$ *closure reduction*		$\downarrow$ *Gauss law reduction*
Closed twisted geometries and polyhedra	$\Longleftrightarrow$	Gauge-inv. loop gravity



Shape-matching conditions

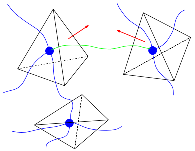


And the connection to Regge calculus?

Consider only 4-valent graphs, dual to triangulations

When closure conditions hold, a triangle acquires two geometries, one from each of the tetrahedra sharing it

Shape-matching conditions



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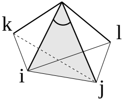
To match the shapes one needs additional
shape-matching constraints:
B.Dittrich and SS 0802.0864

$$F(\phi^n_{ll'}) = 0$$

$\phi^n_{ll'}$ angles between the normals belonging to the two tetrahedra



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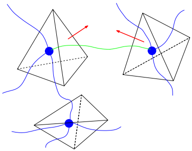

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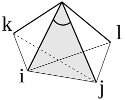

When the gluing conditions hold, we recover Regge calculus

Overview

Twisted geometries	$\Longleftrightarrow$	Loop gravity
$\downarrow$ *closure reduction*		$\downarrow$ *Gauss law reduction*
Closed twisted geometries	$\Longleftrightarrow$	Gauge-inv. loop gravity

Overview

Twisted geometries	$\Longleftrightarrow$	Loop gravity
$\downarrow$ *closure reduction*		$\downarrow$ *Gauss law reduction*
Closed twisted geometries	$\Longleftrightarrow$	Gauge-inv. loop gravity
$\downarrow$ *matching shapes reduction*		
Regge calculus		

Outline

Motivations and overview

Spin networks, twisted geometries and polyhedra

From spinors to twisted geometries

Conclusions

Overview

Twisted geometries	$\Longleftrightarrow$	Loop gravity
$\downarrow$ *closure reduction*		$\downarrow$ *Gauss law reduction*
Closed twisted geometries	$\Longleftrightarrow$	Gauge-inv. loop gravity
$\downarrow$ *matching shapes reduction*		
Regge calculus		

Overview

Spinors		
$\downarrow$ *matching area reduction*		
Twisted geometries	$\Longleftrightarrow$	Loop gravity
$\downarrow$ *closure reduction*		$\downarrow$ *Gauss law reduction*
Closed twisted geometries	$\Longleftrightarrow$	Gauge-inv. loop gravity
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Regge calculus		

Spinors

- $|\mathbf{z}\rangle = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \in \mathbb{C}^2$
- Geometrical meaning: null pole plus null flag: $|\mathbf{z}\rangle \mapsto (X^i, \phi)$

$$|\mathbf{z}\rangle\langle\mathbf{z}| = X^0 \mathbb{1} + X^i \sigma_i, \qquad \phi = \arg z_0 + \arg z_1$$

$$X^0 = \frac{1}{2}\langle\mathbf{z}|\mathbf{z}\rangle, \qquad X^i = \langle\mathbf{z}|\frac{\sigma^i}{2}|\mathbf{z}\rangle$$

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- Poisson brackets

$$\{z_a, \bar{z}_b\} = -i\delta_{ab}$$

- A simple calculation gives

$$\{X^i, X^j\} = \epsilon^{ijk} X^k$$

$$\{X^0, \varphi\} = 1, \qquad \{X^3, \varphi\} = 0, \qquad \{X^\pm, \varphi\} = \frac{X^0}{X^\mp}$$

Link phase space $T^*SU(2)$

Consider two spinors, $|\mathbf{z}\rangle$ and $|\tilde{\mathbf{z}}\rangle$, with canonical Poisson brackets:

$$(z_0, z_1, \tilde{z}_0, \tilde{z}_1) \in \mathbb{C}^4, \qquad \{z_a, \bar{z}_b\} = -i\delta_{ab}, \quad \{\tilde{z}_a, \bar{\tilde{z}}_b\} = -i\delta_{ab}$$

- Vector-phase parametrization:

$$(z_A, \tilde{z}_A) \mapsto (X_i, \phi, \tilde{X}_i, \tilde{\phi})$$

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$$(z_A, \tilde{z}_A) \mapsto (X_i, \phi, \tilde{X}_i, \tilde{\phi})$$

- *Norm-matching constraint*:

$$H = |X_i| - |\tilde{X}_i| = 0$$

- The constraint generates a U(1) action:

$$\{H, z_A\} = \frac{i}{2} z_A, \qquad \{H, \tilde{z}_A\} = -\frac{i}{2}\tilde{z}_A, \qquad (|\mathbf{z}\rangle, |\tilde{\mathbf{z}}\rangle) \mapsto (e^{i\frac{\theta}{2}}|\mathbf{z}\rangle, e^{-i\frac{\theta}{2}}|\tilde{\mathbf{z}}\rangle),$$

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- Phase space reduction: $\mathbb{C}^4 : 8d \xrightarrow{H=0} 7d \xrightarrow{/U(1)} 6d : T^*SU(2)$

$$\boxed{\text{Symplectic reduction by } H = 0 \text{ gives } T^*SU(2)}$$

Nothing fancy is going on: this is simply the classical version of the familiar Schwinger "double harmonic oscillator" representation of the angular momentum!

H–reduction

The variables of $T^*SU(2)$ can be described in terms of spinors as the quantities Poisson-commuting with H on the $H=0$ surface

- in terms of the standard holonomy-flux parametrization:

$$X^i(z_A) \equiv \langle \mathbf{z}|\frac{\sigma^i}{2}|\mathbf{z}\rangle, \qquad g(z_A, \tilde{z}_A) \equiv \frac{|\mathbf{z}\rangle[\tilde{\mathbf{z}}| - |\mathbf{z}]\langle\tilde{\mathbf{z}}|}{\sqrt{\langle\mathbf{z}|\mathbf{z}\rangle\langle\tilde{\mathbf{z}}|\tilde{\mathbf{z}}\rangle}}$$

- in terms of the twisted geometry parametrization:

$$j = \frac{1}{2}\langle\mathbf{z}|\mathbf{z}\rangle, \qquad \xi_A \equiv i\left(\ln\frac{z_A}{\bar{z}_A} + \ln\frac{\tilde{z}_A}{\bar{\tilde{z}}_A}\right)$$

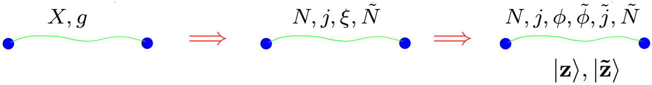
- and the correct Poisson brackets are induced

We obtain a spinorial parametrization of holonomies and fluxes:

$$\boxed{\mathbb{C}^4 \ni (|\mathbf{z}\rangle, |\tilde{\mathbf{z}}\rangle) \xrightarrow{/H=0} \big(X(\mathbf{z}), g(\mathbf{z})\big) \in T^*SU(2)}$$

Interpretation of H

Interpretation of $\mathbb{C}^4_e$: twisted geometries with areas non matching:



But why twisted geometries...

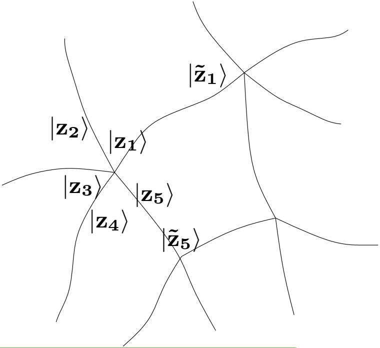
Remark: from the two spinors I can define a twistor

$\Rightarrow H = 0$ is a condition that the twistor is null

Spinor representation

[Freidel and SS '10, Livine and Tambornino '11]

We can describe semiclassical spin networks with a collection of spinors on each half-link



$$\times_n (\mathbb{C}^2)^{V(n)}$$

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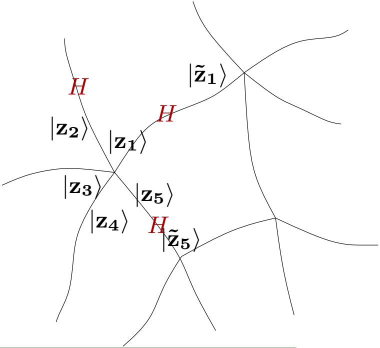

$$\times_n (\mathbb{C}^2)^{V(n)}$$
$$\downarrow$$
U(1) reduction
$$H = 0$$
$$\downarrow$$
$$\times_l T^* SU(2)$$

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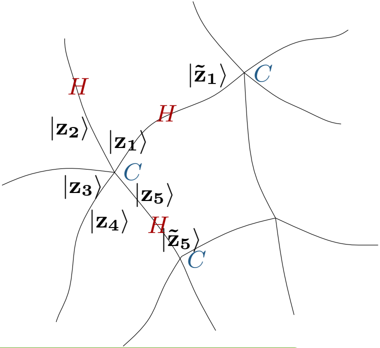

$$\times_n (\mathbb{C}^2)^{V(n)}$$

$$\downarrow$$

U(1) reduction

$$H = 0$$

$$\downarrow$$

$$\times_l T^* SU(2) \quad \longrightarrow \quad \times_l T^* S^1 \times_n \mathcal{S}_n$$

C=0 SU(2) reduction

Spinor representation

[Freidel and SS '10, Livine and Tambornino '11]

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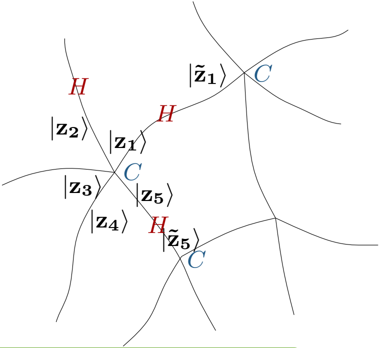


$$\begin{array}{ccc} \times_n (\mathbb{C}^2)^{V(n)} & \xrightarrow[\text{C=0 SU(2) reduction}]{} & U(N) \\ \downarrow & & \downarrow \\ \text{U(1) reduction} & & \text{U(1) reduction} \\ H=0 & & H=0 \\ \downarrow & & \downarrow \\ \times_l T^*SU(2) & \xrightarrow[\text{C=0 SU(2) reduction}]{} & \times_l T^*S^1 \times_n \mathcal{S}_n \end{array}$$

$U(N)$ framework [Livine et al.]

Overview

Spinors $\times_n (\mathbb{C}^2)^{V(n)}$ $\Longrightarrow$ Twistors?

$\downarrow$ *matching area reduction*

Twisted geometries $\Longleftrightarrow$ Loop gravity

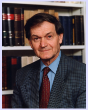
$\downarrow$ *closure reduction* $\downarrow$ *Gauss law reduction*

Closed twisted geometries $\Longleftrightarrow$ Gauge-inv. loop gravity

$\downarrow$ *matching shapes reduction*

Regge calculus

Overview



$$\swarrow \qquad \searrow$$

Spin networks Twistors

$$\searrow \qquad \swarrow$$

Twisted geometries

$$\downarrow$$

Regge geometries

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Conclusions

- It is possible to visualize the truncation $\mathcal{H}_\Gamma$ as capturing a discretization of 3-geometries
- These are the assignment to each triangle of its oriented area, the two unit normals as seen from the two tetrahedra sharing it, and an additional angle related to the extrinsic curvature $(N, \tilde{N}, A, \xi) \Longleftrightarrow (X, g)$
- The 3-geometries are piecewise-flat but in general *discontinuous*
- At the saddle point of the EPRL model the shape-matching conditions are satisfied $\Rightarrow$ Regge action
  Barrett et al. '08
- The twisted geometries can be easily derived from spinors associated to half-edges through the area-matching constraints $\Rightarrow$ introduction of spinorial techniques