Quantum field theory correlators at very short and very long distances

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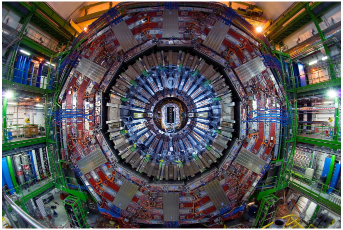
Outline

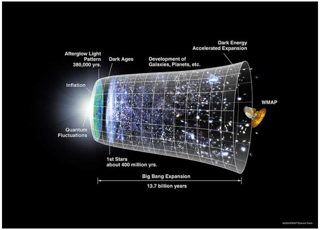
On tiny scales, spacetime looks like Minkowski spacetime. Therefore, questions related to the nature of elementary particles (especially to collider-type experiments) can be dealt with in the framework of this special background



Credit: CERN/CMS collaboration

Observables: (mostly) scattering amplitudes, decay rates, masses etc.

On extremely large scales, the geometry of spacetime is significantly different from Minkowski spacetime–one has an expanding Universe. Therefore, in cosmology (and also the vicinity of black holes), one has to consider quantum field theory on other backgrounds, such as e.g. DeSitter spacetime.



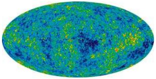
Credit: NASA/WMAP science team Observables: correlations in CMB, Hawking radiation, etc.

A THEORY OF THE VERY SMALL WITH EFFECTS IN THE VERY LARGE

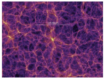
The colossal expansion of the Universe leads to a qualitatively new effect:

> Microscipic quantum fluctuations in the very Early Universe are magnified to macroscopic density perturbations, whose "afterglow" can be seen today in the cosmic microwave background (WMAP, PLANCK).

The macroscopic imprint of such quantum fluctuations has had a major impact on the formation of structure in the Universe (galaxies, clusters, voids, ...).



$\Longrightarrow$



Credit: WMAP

Credit: MPI Garching

How to understand this?

1. FRW metric $ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)$; e.g.

$$a(t) = \begin{cases} e^{tH} & \text{deSitter space (Inflation)} \\ t^{2/3} & \text{matter (dust)} \end{cases}$$

   Hubble rate is defined as $H(t) = \dot{a}(t)/a(t)$.

2. A test field obeys $\Box\phi = 0$; mode decomposition:

$$\ddot{\phi}_{\mathbf{k}} + 3H\, \dot{\phi}_{\mathbf{k}} + \left(\tfrac{k}{a}\right)^2\, \phi_{\mathbf{k}} = 0\ ,$$

   Each mode behaves like a damped harmonic oscillator with time-dependent coefficients.

3. Early universe ("inflation"): $H \gg k/a$ "mode frozen"
   Later universe ("dust"): $H \ll k/a$ "mode oscillates"

4. correctly normalized mode in early Universe $(\Delta\phi_{\mathbf{k}})^2 \sim [a_0^3(k/a_0)]^{-1}$ is amplified by factor $(a_1/a_0)^2 \gg 1$ in late Universe!

5. $(\delta T/T)_{\mathbf{k}} \sim |\Delta\phi_{\mathbf{k}}|^2 \sim H_0^2/k^3$ (fluctuations in CMB)

More accurate analysis: QFT

A more accurate analysis requires the formalism of quantum field theory, because ϕ is a quantum field propagating on an expanding geometry! As in QFT in flat spacetime, one is interested at the end of the day in the redcorrelation functions:

$$\langle \mathscr{O}_1(x_1) \dots \mathscr{O}_n(x_n) \rangle_\Psi \ ,$$

where

- $\mathscr{O}_i$ are local (composite) fields in the theory, evaluated at spacetime points x_i
- Ψ is a quantum state

As in flat space, general form of correlation is restricted by causality (locality), and positivity, as well as further, more sophisticated constraints (see below).

Questions

1. What should I take as Ψ? How sensitive does the final result depend on initial state?
2. How to incorporate interactions, e.g. Lagrange function $L = (\nabla\phi)^2 - m^2\phi^2 - \lambda\phi^4$ instead of $L = (\nabla\phi)^2$? (Renormalization?)
3. How to relate physical observables to correlators?
4. What is the general formalism of QFT in CST?

1. Writing

$$w_\Psi(t_1, \mathbf{x}_1, \dots, t_E, \mathbf{x}_E) = \langle \phi(t_1, \mathbf{x}_1) \dots \phi(t_E, \mathbf{x}_E) \rangle_\Psi$$

observables of interest in cosmology are

$$w_{\{lm\}}(t) = \int_{\vec{\mathbf{k}}} K_{\{lm\}}(\mathbf{k}_1, \dots, \mathbf{k}_{E-1}) \; \widehat{w}_\Psi(t, \mathbf{k}_1; \dots; t, \mathbf{k}_{E-1})$$

where $K_{\{lm\}}$ are certain known kernels. The $w_{\{lm\}}$ have direct interpretation as (a) temperature fluctuations in CMB $(\delta T / T)_{lm} \sim w_{lm}$ für $E = 2$, or (b) "non-Gaussianities" $(E = 3)$ (PLANCK-Mission), etc.

2. I want illustrate the other points in this talk at the example of deSitter space

Long-distance vs. short-distance behavior of correlators

Very roughly speaking, a the curvature of a spacetime (or portion thereof) is characterized by the magnitude of the invariants formed from the Riemann-tensor. Letting this magnitude correspond to a length ℓ ("curvature radius"), one can distinguish heuristically two regimes for the, let's say, 2-point function:

1. "Short distances": Let $|\sigma(x_1, x_2)| \lessapprox \ell^2$ ($\sigma =$ geodesic distance). Then, the 2-point function of a state $\langle \mathscr{O}_1(x_1)\mathscr{O}_2(x_2)\rangle_\Psi$ roughly equal to that of a "corresponding state" in Minkowski spacetime. More precise statement: Operator product expansion (see below).
2. "Long distances": Let $|\sigma(x_1, x_2)| \gg \ell^2$. Then behavior of 2-point function fundamentally different from Minkowski space (IR-effects). Most interesting physical effects fall into this category, as we will now see at the example of DeSitter spacetime.

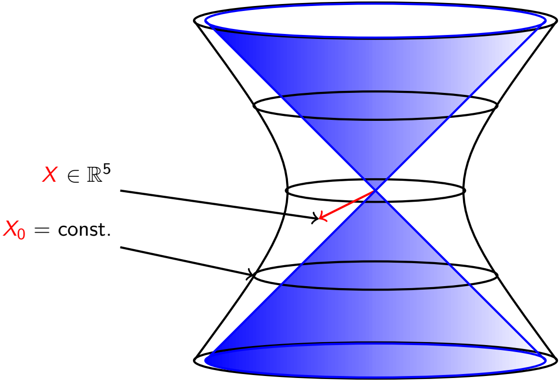
Why deSitter space?

- DeSitter space describes the era of “inflation”, which is characterized by a **very large** cosmological constant Λ of the order of the GUT scale. Λ arises from potential energy of the inflaton field (?).
- It is currently believed that the current accelerated expansion of the universe is described by a deSitter spacetime, with a **very small** cosmological constant Λ. This is believed to be due to some sort of VEV (?) (“dark energy”).

Elementary geometry of deSitter space

DeSitter can be described as 4-dimensional hyperboloid:

$$dS_4 = \{-X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = H^{-2}\} \ .$$



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DeSitter space is:

- A solution to Einstein field equations with *positive* kosmological constant$\Lambda = 3H^2$.
- A space of constant curvature H^2, conformally flat.
- A space of maximal symmetry, $O(4,1)$.
- The geodesic distance σ (with sign) between $X, Y \in dS_4$ is given by:

  $$\cos(H\ \sqrt{\sigma}) = H^2\ X \cdot Y\ \equiv Z.$$

  The quantity Z is also called "point-pair invariant". The relationship between this and causal relations in deSitter is described by:

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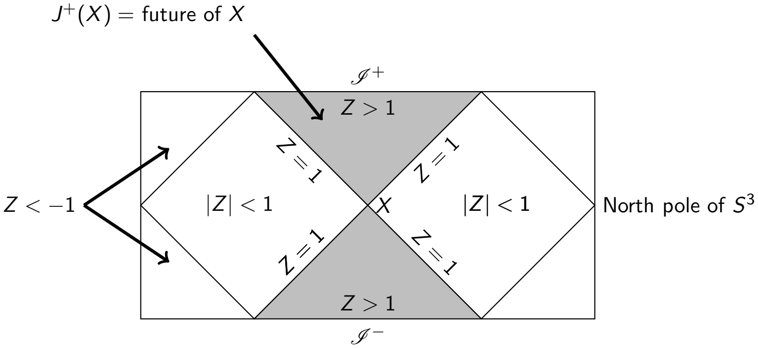

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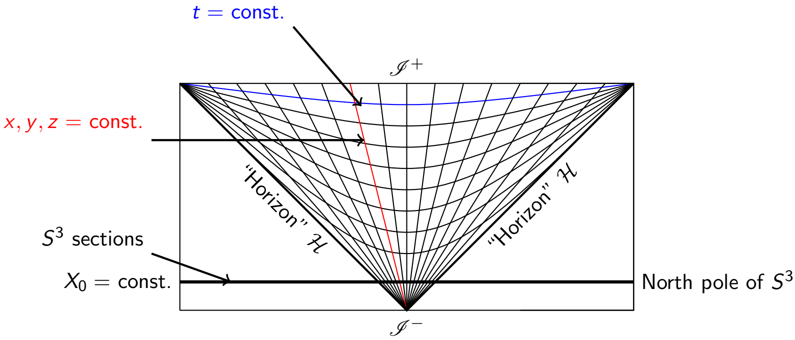
This is a "conformal diagram" of deSitter.

The "Cosmic No-Hair Conjecture" says that, except in the vicinity of black holes, *any* solution to the Einstein equations with Λ will locally approach the exact deSitter spacetime with $\Lambda = 3H^2$ in the far future ($\rightarrow$ "Heat Death of Universe")

An alternative drawing of conformal diagram is:



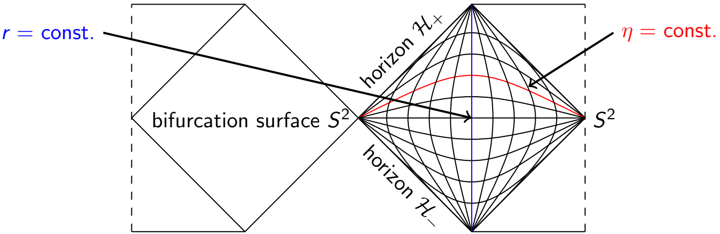
The shaded region is the "COSMOLGICAL CHART", where the metric takes FRW-form

$$ds^2 = -dt^2 + \mathrm{e}^{2Ht}(dx^2 + dy^2 + dz^2) \ .$$

Unusual properties of deSitter:

1. DeSitter has no time-translation symmetry.
2. DeSitter fields ϕ have no conserved energy.
3. No useful *global* notion of particle in deSitter QFT. e.g. deSitter-invariant (= "Bunch-Davies-" = "Vacuum-") state is actually thermal!

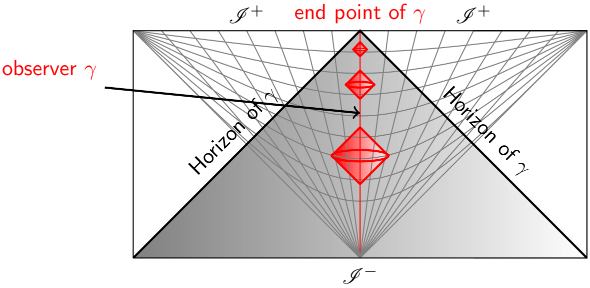
Except in STATIC CHART (shaded):



$$ds^2 = -f(r)\ d\eta^2 + f(r)^{-1} dr^2 + r^2 d\omega_2^2$$

Physical interpretation of "cosmic no-hair"

One can view the "cosmic no-hair" property as saying that deSitter spacetime is classically I(nfra) R(ed) stable. The quantitative meaning of this can e.g. be visualized by an observe moving towards the distant future ($\mathscr{I}^+$) along a worldline γ:



The geometry within each causal diamond of *fixed physical volume* approaches that of exact deSitter at an exponential rate. $\Longrightarrow$ no geometrical "degrees of freedom" left in the distant future, i.e. "no hair".

Free KG-Fields

We would like to consider theories of the form $L = (\nabla\phi)^2 - m^2\phi^2 - \lambda\phi^4$. For simplicity first *free* massive theory $(\lambda = 0)$, $m^2 > 0$. Any correlation function in this theory has properties:

Correlation functions in free theory:

1. Each $\langle\phi(X_1)\dots\phi(X_E)\rangle_\Psi$ satisfies KG-eqn.
2. $\langle\dots[\phi(X_1),\phi(X_2)]\dots\rangle_\Psi = i\Delta(X_1,X_2)\langle\dots\rangle_\Psi$.
3. Short-distance behavior of "Hadamard form".

A state is "Gaussian" if "Wick's theorem" holds.

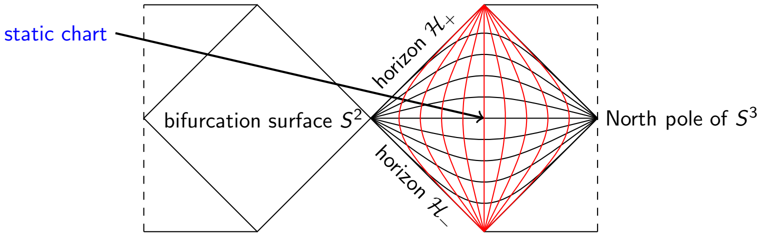
Special state: "Bunch-Davies" state

$$\langle \phi(X_1)\phi(X_2)\rangle_{\mathrm{BD}} = \mathrm{cst.}\ {}_1F_2\left(-c, 3+c; 2; \frac{1+Z}{2}\right)$$

c is dimensionless constant

$$c = -3/2 + (9/4 - m^2/H^2)^{1/2}.$$

This istate is deSitter invariant, and for this reason is often called "vacuum". Actually, it is more like a thermal state in static region, with temperature $T = H/2\pi$!



In a Hilbert-space representation wherein the Bunch-Davies state is given by a "vacuum"-vector $|\mathrm{BD}\rangle$, one can generate more general states by applying "smeared" field operators:

$$|\Psi\rangle = \int f_E(X_1, \dots, X_E)\, \phi(X_1) \cdots \phi(X_E)\, |\mathrm{BD}\rangle\ .$$

f_E ("wave packet") is e.g. of compact support. By [Strohmaier, Verch, Wollenberg 2002], this gives in essence *all* states in Hilbert space!

IR-stability of free field

In each such state we have $\langle \phi \rangle_\Psi = O(\mathrm{e}^{\Re(c) H\tau})$ for large proper times $\tau \to \infty$, where $\Re(c) < 0$. $\Longrightarrow$ IR-stability for deSitter for **free** QFT. .

Interacting KG-fields

We want the correlation funktions $\langle\phi(X_1)\dots\phi(X_E)\rangle_\Psi$ e.g. in QFT described by $L = (\nabla\phi)^2 - m^2\phi^2 - \lambda\phi^4$! $\Rightarrow$ perturbation series in λ.

Questions:

1. Which states Ψ?
2. How to deal with UV-divergences?
3. How to deal with IR-divergences?

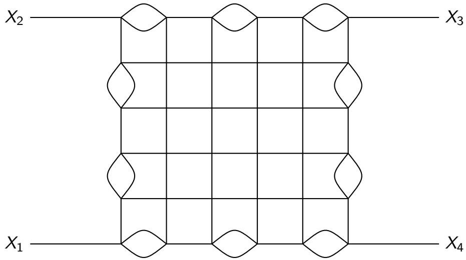
Answers:

1. Probably (?) any state is OK, because we are interested in practice in late time-behavior of correlation functions. For simplicity first $\Psi = |\mathrm{BD}, \lambda\rangle$.
2. [S.H.-Wald 2001-2008, Brunetti et al. 2000, 1998, Radzikowski 1998]
3. Analytic continuation $dS_4 \to S^4$ and back.

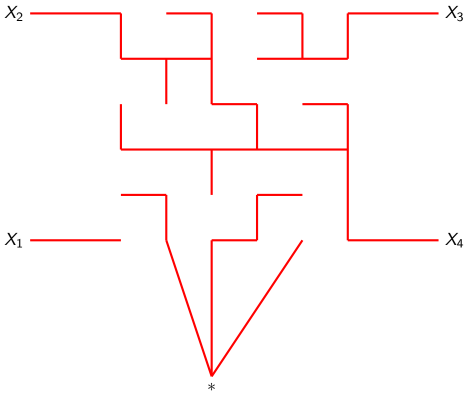
Perturbation series has general form:

$$\langle \phi(X_1)\cdots\phi(X_E)\rangle^C_{\mathrm{BD},\lambda} = \sum_{V\geq 0}\ \sum_{\mathrm{Graphs}\ G} \lambda^V \ \ I_G(X_1,\dots,X_E)$$

f. “connected” correlation functions. I_G is the contribution of an individual Feynman graph G, mit V inner vertices and E external legs, e.g.:



An elegant closed form expression for I_G, and therefore correlation functions, was given in [Hollands 2010, 2011]. The formula is a multiple "Mellin-Barnes" integral ("generalized H-function" [Sarivastava et al. 1973]). Integration parameters w_F are 1-1 correspondence with "forests" F in an associated graph G^* as e.g. in



DeSitter Feynman integral $I_G(X_1, \dots, X_E)$ given by [Hollands 2010,2011 (massless)]

$$\boxed{I_G = K_{c,G} \int_{\vec{w}} \Gamma_{c,G}(\vec{w}) \prod_{1 \le i \ne j \le E} (1 - Z_{ij})^{\sum_F w_F} \,,}$$

wobei:

- Sum $\sum_F$ in exponent over all forests connecting X_i, X_j.
- $K_{c,G}$: a constant,
- Z_{ij} are point-pair invariants $H^2 X_i \cdot X_j$. $\Gamma_{c,G}(\vec{w})$ is kernel given in terms of gamma-fcts.:

$$\Gamma_{c,G}(\vec{w}) := \frac{\Gamma(\frac{5}{2} + \sum_F w_F) \prod_F \Gamma(-w_F)}{\Gamma(\frac{5}{2} + \sum_{(ij) \notin \Phi} \sum_{F \ni \Phi} w_F - \sum_{(ij) \in \Phi} \sum_{F \not\ni (ij)} w_F)} \cdot \frac{\prod_{(ij) \notin \Phi} \Gamma(-c + \sum_{F \ni (ij)} w_F) \Gamma(3 + c + \sum_{F \ni (ij)} w_F) \Gamma(1 - \sum_{F \ni (ij)} w_F)}{\prod_{(ij) \in \Phi} \Gamma(\frac{5}{2} + \sum_{F \not\ni (ij)} w_F)}$$

Application of master formula

Using the formula for I_G one can:

- Calculate correlation functions and hence e.g. $w_{\{lm\}}(t)$ ($\rightarrow$ CMB!)
- Long-distance expansion, including very important

Cosmic-no-hair/IR-stability in interacting QFT

For generic quantum state Ψ, we have:

$$\langle \phi \rangle_\Psi \sim O(\mathrm{e}^{\Re(c) H \tau})$$

to arbitrary orders in perturbation theory and $m^2 > 0$. As above $c = -3/2 + (9/4 - m^2/H^2)^{1/2}$, which has **negative** real part.

(Similar result: [Marolf & Morrison 2010, ... & Higuchi 2011])

General features of correlators

As in flat Minkowski spacetime, we could say that a QFT on a curved spacetime (M, g) is specified by a collection of correlators

$$\langle \mathcal{O}_{i_1}(x_1) \dots \mathcal{O}_{i_n}(x_n) \rangle_\Psi$$

where:

1. There is one set of correlators for each state Ψ.
2. The $\mathcal{O}_i$'s are composite fields, e.g. $\phi, T_{\mu\nu}, J_\mu, \dots$.
3. We expect there to be singularities if x_i is on x_j's lightcone.
4. We expect that fields should (anti-) commute if x_i and x_{i+1} are spacelike.
5. We want the state to be "positive" ("unitarity"): $\langle A^* A \rangle_\Psi \geq 0$ for any expression such as (f a smearing function)

$$A = \int f(x_1, \dots, x_n) \prod \phi(x_i) \ .$$

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But:

1. There are many examples of pathological states satisfying these criteria (e.g. α-states in deSitter, "instantaneous ground state" in FRW-universe,...).
2. How to recognize that correlators belong to *different state* but *same theory*? (Operator-product-expansion, OPE)
3. Does not incorporate any notion of positive energy (microlocal spectrum condition).
4. How to formulate notion of "general covariance" (OPE)?

All states should satisfy the operator product expansion:

[Wilson,Kadanoff], [Zimmermann], [Keller et al.],[Belavin et al.],...,[curved space: S.H.]

$$\langle \mathscr{O}_{j_1}(x_1) \cdots \mathscr{O}_{j_n}(x_n) \rangle_\Psi \sim \sum \underbrace{C^i_{j_1 \dots j_n}(x_1, \dots, x_n; y)}_{\text{state indep.}} \underbrace{\langle \mathscr{O}_i(y) \rangle_\Psi}_{\text{state dep.}}$$

- **Physical idea:** Behavior of the theory at short distances "factorizes" from the behavior at long distances/from properties of specific state.
- **Covariance of theory:** Coefficients are local functionals of the metric, curvature etc.
- **Convergence:** The OPE converges (!) even at finite distances [S.H. & Kopper 2011], due to following thm. $\Longrightarrow$ All information about n-pt. functions is contained in OPE coefficients + 1-pt. functions only!

Convergence of OPE:

At least in Euclidean 4-dimensional space, the OPE is not only *asymptotic* at short distances, but even *converges* at finite (!) distances. This was shown for $L = (\partial\phi)^2 + m^2\phi^2 + \lambda\phi^4$ to *all loop orders*. The convergence follows from bound (at l loops), where f_{p_i} are test-functions ("wave packets") that are compactly supported in momentum space near $p_i \in \mathbb{R}^4$:

$$
\begin{aligned}
&\left| \left\langle \mathcal{O}_a(x)\,\mathcal{O}_b(0)\,\phi(f_{p_1})\cdots\phi(f_{p_n}) \right\rangle - \sum_c C_{ab}^c(x) \left\langle \mathcal{O}_c(0)\,\phi(f_{p_1})\cdots\phi(f_{p_n}) \right\rangle \right| \\
&\leq \quad \sqrt{[a]![b]!}\ \tilde{K}^{[a]+[b]} \prod_i \sup|\hat{f}_{p_i}|\ m^{[a]+[b]+n} \\
&\times \quad \sup(1, \frac{|\vec{p}|_n}{m})^{2([a]+[b])(n+2l+1)+3n} \sum_{\lambda=0}^{n/2+2l} \frac{\log^\lambda \sup(1, \frac{|\vec{p}|_n}{m})}{2^\lambda\,\lambda!} \\
&\times \quad \frac{1}{\sqrt{\Delta!}} \left(\tilde{K}\ m\,|x|\ \sup(1, \frac{|\vec{p}|_n}{m})^{n+2l+1} \right)^\Delta .
\end{aligned}
$$

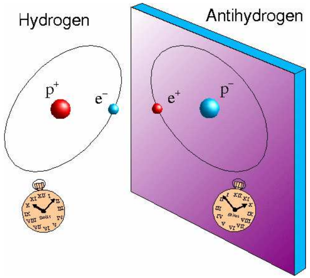
where the OPE is carried out up to dimension Δ. Proof: Wilson-Polchinsky RG-flow equations [S.H. & Kopper 2011]

Fundamental theorems

Fundamental theorems about QFT on curved spacetimes e.g. **P**arity-**T**ime-**C**harge [S.H. 2004, & Wald 2008]:

But: What actually is "P" and "T" in a spacetime without reflection symmetries, such as e.g. expanding FRW:

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2) \quad ?$$



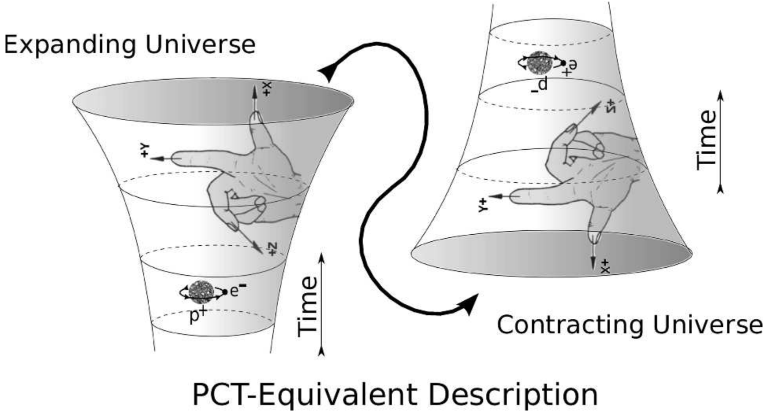
Minkowski spacetime:

$U|\mathrm{in, momenta}\rangle = |\mathrm{out}, -\mathrm{momenta}\rangle^C$.

Curved spacetime: **No "same Universe" formulation of PCT:** Theorem connects "in"-state in one universe with "out state" in universe with opposite space and time orientations!

PCT=symmetry of OPE-coefficients!!!

THE PCT-THEOREM IN CURVED SPACE



The PCT theorem connects the QFT in one spacetime (e.g. expanding universe) to that of the spacetime with opposite time- and space- orientation (contracting universe).

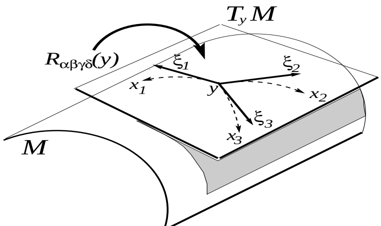
The PCT-theorem in curved space

As a **technical statement** about the OPE-coefficients, the PCT-theorem can be stated as saying that

$$C^{j}_{i_1 \dots i_n}[T, o] = \overline{C^{\bar{j}}_{\bar{i}_n \dots \bar{i}_1}[-T, -o]}$$

where o, T are the orientation (o) and time orientation (T) of the underlying spacetime (M, g).
The **proof** of the PCT-theorem relies on fundamental properties of the OPE:

• Curvature expansion: (e.g. free KG-field 2-point OPE-coefficient $\phi(x_1)\phi(x_2) \sim [1/\xi^2 + R_{\mu\nu}\xi^\mu\xi^\nu/\xi^2 + \dots]\ \mathbf{1} + \dots$)



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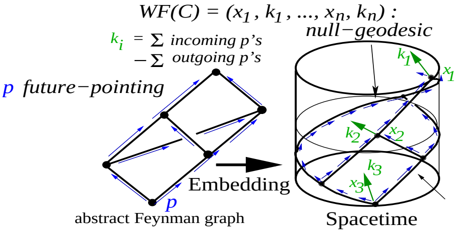
- μ-local spectrum condition: [Brunetti et al. 1998,2000, S.H. 2006]



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The **proof** of the PCT-theorem relies on fundamental properties of the OPE:

• Associativity of OPE (="crossing symmetry"="consistency"): E.g. $\sum_k C^k_{ij} C^m_{kl} = C^m_{ijl}$ if a subset of the points (x_1, x_2, x_3) is closer to each other than to the remaining point.

Summary

In this talk I have tried to explain:

1. How, in simple terms, the microscopic quantum fluctuations in the early cosmos give rise to macroscopic density contrast in late cosmos. [Mukhanov, Guth, Steinhard-Bardeen-Turner 80's]
2. Said to what are the corresponding quantum observables.
3. Explained how one can compute correlators e.g. at long distances in deSitter space, in renormalized perturbation theory.
4. Explained what this has to do with "cosmic no-hair", see however [Polyakov 2009,2010]
5. Explained the role of the OPE in curved spacetime and some of its properties and uses for the calculation of correlation functions at short distances.

Summary

There are many more contributions to these topics:

1. Theorems about QFT in CST: Brunetti, Fewster, Fröhlich & Birke, Fredenhagen, Haag, Kay, Radzikowski, Verch, Wald,...
2. Perturbation theory in curved spacetimes: Brunetti & Fredenhagen, Hollands & Wald, Bunch,...
3. deSitter: M. Anderson, Allen, Balasubramanian, Bros & Moschella, Einhorn, Friedrich, Higuchi, Jacobson, Jaekel & Gerard, Kitazawa et al., Larsen, Minic, Marolf & Morrison, Mottola, Moretti, Dappiaggi & Pinamonti, Polyakov, Randall, Schomblond & Spindel, Strominger, Spradlin, Tsamis, Urakawa et al., Verlinde, Volovic, S. Weinberg, Woodard,...