Ward identities in matrix models arising from noncommutative geometry

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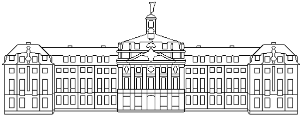

(based on joint work with Harald Grosse)

Introduction

- The Standard Model is a perturbatively renormalisable quantum field theory.
- Scattering amplitudes can be computed as formal power series in coupling constants such as $e^2 \approx \frac{1}{137}$.
  The first terms agree to high precision with experiment.
- The radius of convergence in e^2 is zero!

- Refined summation techniques (e.g. Borel) may establish reasonable domains of analyticity.

- Unfortunately, this also fails for QED due to the Landau ghost problem.

  It is expected to work for non-Abelian gauge theories because of asymptotic freedom.

  But these theories are too complicated.

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  But these theories are too complicated.

QFT's on noncommutative geometries may provide toy models for non-perturbative renormalisation in four dimensions.

These models may have new Ward identities which constrain the renormalisation flow.

General matrix models

- I – set of indices (finite, countable or continuous)
- $\mathcal{M} = \{M = (M_{ab})_{a,b\in I}\}$ space of matrices (with topology)
  product $(MN)_{ab} = \sum_{c\in I} M_{ac}N_{cb}$, trace $\mathrm{tr}(M) = \sum_{a\in I} M_{aa}$
  for continuous I take $(MN)_{ab} = \int_I d\mu(c)\, M_{ac}M_{cb}$
  $\mathcal{M}_* = \{M = M^* \in \mathcal{M}\}$, where $(M^*)_{ab} = \overline{M_{ba}}$ adjoint

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  $\mathcal{M}_* = \{M = M^* \in \mathcal{M}\}$, where $(M^*)_{ab} = \overline{M_{ba}}$ adjoint
- action = non-linear functional S on $\mathcal{M}_*$. We consider

$$S[\phi] = \mathrm{tr}(E\phi^2) + V[\phi]\,, \quad V[\phi] = \mathrm{tr}(P[\phi])$$

where $E \in \mathcal{M}_*$ is a positive external matrix and $P[\phi]$ an (e.g. even) polynomial in ϕ with scalar coefficients.

Euclidean quantum field theory

- action with source term $\longrightarrow$ partition function

$$\mathcal{Z}[J] = \int \mathcal{D}\phi \; \exp(-S[\phi] + \mathrm{tr}(\phi J))$$

where $\mathcal{D}\phi = \prod_{a,b \in I} d\phi_{ab}$

- connected correlation functions obtained from $\mathcal{W}[J] = \ln \mathcal{Z}[J]$ as

$$\langle \varphi_{a_1 b_1} \dots \varphi_{a_n b_n} \rangle = \frac{\partial^n \mathcal{W}[J]}{\partial J_{b_1 a_1} \dots \partial J_{b_n a_n}}\Big|_{J=0}$$

- unless I is finite, the resulting index sums may diverge and require a renormalisation

Ward identity

- unitary transformation $\phi \mapsto \tilde{\phi} = U\phi U^*$
  $U \in \mathcal{M}''$ with $UU^* = U^*U = \mathrm{id}$, leaves $\mathcal{M}_*$ invariant:

$$\int \mathcal{D}\phi\; \exp(-S[\phi] + \mathrm{tr}(\phi J)) = \int \mathcal{D}\tilde{\phi}\; \exp(-S[\tilde{\phi}] + \mathrm{tr}(\tilde{\phi} J))$$

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- measure unitarily invariant: $\mathcal{D}\tilde{\phi} = \mathcal{D}\phi$:

$$0 = \int \mathcal{D}\phi \left[\exp(-S[\phi] + \mathrm{tr}(\phi J)) - \exp(-S[\tilde{\phi}] + \mathrm{tr}(\tilde{\phi} J))\right]$$

note: $[\ \ ] \neq 0$ because $\mathrm{tr}(E\phi^2)$, $\mathrm{tr}(\phi J)$ not unitarily invariant!

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  note: $[\;\;] \neq 0$ because $\mathrm{tr}(E\phi^2)$, $\mathrm{tr}(\phi J)$ not unitarily invariant!

- linearisation $\longrightarrow$ Ward identity

$$0 = \int \mathcal{D}\phi \left[E\phi\phi - \phi\phi E - J\phi + \phi J \right] \exp(-S[\phi] + \mathrm{tr}(\phi J))$$

use functional derivative $\phi_{ab} = \frac{\partial}{\partial J_{ba}}$:

Proposition

The partition function $\mathcal{Z}[J]$ of the matrix model defined by the external matrix E satisfies the $|I| \times |I|$ Ward identities

$$0 = \sum_{m,n \in I} \Big(E_{bn} \frac{\partial^2 \mathcal{Z}}{\partial J_{am} \partial J_{mn}} - E_{ma} \frac{\partial^2 \mathcal{Z}}{\partial J_{mn} \partial J_{nb}} \Big) - \sum_{n \in I} \Big(J_{bn} \frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}}{\partial J_{nb}} \Big) \Big)$$

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Class of examples motivated by NCG

$E_{mn} = E_m \delta_{mn}$ diagonal with $m \mapsto E_m > 0$ injective:

$$0 = \sum_{n \in I} \Big((E_a - E_p) \frac{\partial^2 \mathcal{Z}}{\partial_{an} \partial J_{np}} + J_{pn} \frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}}{\partial J_{np}} \Big)$$

We will see: These Ward identities and the choice of $V[\phi]$ determine the QFT of the matrix model non-perturbatively!

Decomposition into cycles

From perturbative expansion into ribbon graphs for diagonal E: right index of J_{ab} is left index of another J_{bc}, or of the same J_{bb}.

Decomposition of $\mathcal{W}[J]$ for even $V[\phi]$ into J-cycles:

$$\begin{aligned}\mathcal{W}[J] = \mathcal{W}[0] &+ \frac{1}{2}\sum_{p,q\in I} G_{pq}(J_{pq}J_{qp}) + \frac{1}{2}\sum_{p,q\in I} G_{p|q}(J_{pp})(J_{qq}) \\ &+ \frac{1}{4}\sum_{p,q,r,s\in I} G_{pqrs}(J_{pq}J_{qr}J_{rs}J_{sp}) + \frac{1}{3}\sum_{p,q,r,s\in I} G_{pqr|s}(J_{pq}J_{qr}J_{rp})(J_{ss}) \\ &+ \frac{1}{8}\sum_{p,q,r,s\in I} G_{pq|rs}(J_{pq}J_{qp})(J_{rs}J_{sr}) + \frac{1}{4}\sum_{p,q,r,s\in I} G_{pq|r|s}(J_{pq}J_{qp})(J_{rr})(J_{ss}) \\ &+ \frac{1}{24}\sum_{p,q,r,s\in I} G_{p|q|r|s}(J_{pp})(J_{qq})(J_{rr})(J_{ss}) + \mathcal{O}(J^6)\end{aligned}$$

Attention: $G_{pp}J_{pp}J_{pp}$ is topologically different from $G_{p|p}J_{pp}J_{pp}$!

A continuity argument

$$G_{pp} = \lim_{q \to p, q \neq p} G_{pq} \quad \text{versus} \quad G_{p|p} = \lim_{q \to p, q \neq p} G_{p|q}$$

Perturbatively, the G_{pq}, $G_{p|q}$ etc are functions solely of $\{E_m\}$.
We formally extend E_m to any injective C^1-function on a continuous extension of I.

$\longrightarrow$ limits perturbatively well-defined, assumed to remain non-perturbatively true

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We now derive a formula for

$$\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}} = \sum_{n\in I} \Big(\frac{\partial^2 \mathcal{W}[J]}{\partial J_{an}\partial J_{np}} + \frac{\partial \mathcal{W}[J]}{\partial J_{an}} \frac{\partial \mathcal{W}[J]}{\partial J_{np}} \Big) \mathcal{Z}[J]$$

There are again two pieces: A continuous part defined for $p \neq a$ with continuous limit $p \to a$, and a singular part δ_{ap}.

Main Theorem

For diagonal injective E one has

$$\sum_{n\in I}\frac{\partial^2\mathcal{Z}[J]}{\partial J_{an}\partial J_{np}}$$

$$= \delta_{ap}\Big(\sum_{n\in I} G_{an} + G_{a|a} + \sum_{n,r\in I} G_{anrn}J_{rn}J_{nr} + \sum_{n,s\in I} G_{ann|s}J_{nn}J_{ss}$$

$$+\frac{1}{2}\sum_{n,r,s\in I} G_{an|rs}J_{rs}J_{sr} + \frac{1}{2}\sum_{n,r,s\in I} G_{an|r|s}J_{rr}J_{ss}$$

$$+\frac{1}{2}\sum_{r,s\in I} G_{rs|a|a}J_{rs}J_{sr} + \frac{1}{2}\sum_{r,s\in I} G_{r|s|a|a}J_{rr}J_{ss} + \mathcal{O}(J^4)\Big)\mathcal{Z}[J]$$

$$-\frac{1}{E_a - E_p}\sum_{n\in I}\Big(J_{pn}\frac{\partial\mathcal{Z}[J]}{\partial J_{an}} - J_{na}\frac{\partial\mathcal{Z}[J]}{\partial J_{np}}\Big)$$

The $\mathcal{O}(J^4)$ terms are explicitly known. Injectivity of E is crucial!

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$$(1) \qquad \sum_{n \in I} \frac{\partial^2}{\partial J_{an} \partial J_{np}} \Big(J_{rr} J_{ss} \Big) = 2 \delta_{ap} \delta_{ar} \delta_{as}$$

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$$(2) \qquad \sum_{n\in I} \frac{\partial^2}{\partial J_{an}\partial J_{np}} \Big(J_{rs}J_{st}J_{tu}\dots J_{zr}\Big)$$
$$= \#(J) \times \sum_{n\in I} \frac{\partial}{\partial J_{an}} \Big(J_{pt}J_{tu}\dots J_{zn}\Big)\delta_{nr}\delta_{ps}$$
$$= \#(J) \times \sum_{n\in I} \Big(J_{nu}\dots J_{zn}\Big)\delta_{ap}\delta_{nr}\delta_{as}\delta_{nt} + \mathrm{cont}(a,p)$$

How to use the Ward identity

Functional integration yields, up to irrelevant constant,

$$\mathcal{Z}[J] = e^{-V[\frac{\partial}{\partial J}]} e^{\frac{1}{2}\langle J,J\rangle_E}\;, \qquad \langle J,J\rangle_E := \sum_{m,n\in I} \frac{J_{mn}J_{nm}}{E_m+E_n}$$

Example: G_{ab} (for $a \neq b$)

$$G_{ab} = \frac{1}{\mathcal{Z}[0]} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{ba}\partial J_{ab}}\Big|_{J=0} = \frac{1}{\mathcal{Z}[0]} \Big\{ \frac{\partial}{\partial J_{ba}} e^{-V\left[\frac{\partial}{\partial J}\right]} \frac{\partial}{\partial J_{ab}} e^{\frac{1}{2}\langle J,J\rangle_E} \Big\}_{J=0}$$

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$$= \frac{1}{E_a+E_b} + \frac{1}{(E_a+E_b)\mathcal{Z}[0]} \Big\{ \Big(\phi_{ab} \frac{\partial(-V)}{\partial \phi_{ab}} \Big) \Big[\frac{\partial}{\partial J}\Big] \Big\} \mathcal{Z}[J]\Big|_{J=0}$$

$\frac{\partial(-V)}{\partial \phi_{ab}}$ contains, for any V, the twofold derivative $\frac{\partial^2}{\partial J_{an}\partial J_{np}}$

Results for $V[\phi] = \frac{\lambda}{4}\mathrm{tr}(\phi^4)$

$$G_{ab} = \frac{1}{E_a + E_b} - \frac{\lambda}{E_a + E_b}\Big(G_{ab}\sum_{n\in I} G_{an} + G_{ab}G_{a|a} + G_{aaba} + G_{abab} + \sum_{n\in I} G_{an|ab} + G_{ab|a|a}\Big) + \frac{\lambda}{E_a + E_b}\Big(\sum_{p\in I}\frac{G_{pb} - G_{ab}}{E_p - E_a} + \frac{G_{b|b} - G_{a|b}}{E_b - E_a}\Big)$$

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+ \frac{\lambda}{E_a+E_b}\Big(& \sum_{p\in I}\frac{G_{pb}-G_{ab}}{E_p-E_a} + \frac{G_{b|b}-G_{a|b}}{E_b-E_a}\Big)
\end{aligned}
$$

$$
\begin{aligned}
G_{a|b} = -\frac{\lambda}{E_a+E_a}\Big(& G_{a|b}\sum_{n\in I} G_{an} + G_{a|b}G_{a|a} \\
& + G_{aaa|b} + G_{abb|a} + \sum_{n\in I} G_{an|a|b} + G_{a|b|a|a}\Big) \\
+ \frac{\lambda}{E_a+E_a}\Big(& \frac{G_{bb}-G_{ab}}{E_b-E_a} + \sum_{p\in I}\frac{G_{p|b}-G_{a|b}}{E_p-E_a}\Big)
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Genus expansion

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- Our connected functions $G_{ab}, G_{a|b}, G_{abcd}$, etc., involve all these topologies. Accordingly, we expand $G_{ab} = \sum_{g=0}^{\infty} G^{g}_{ab}$, $G_{a|b} = \sum_{g=0}^{\infty} G^{g}_{a|b}$, $G_{abcd} = \sum_{g=0}^{\infty} G^{g}_{abcd}$, etc.

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- The operations leading to the Ward identities increase the genus by 1 whenever two J-cycles are connected by a bridge

We find:

$$G^{g}_{ab} = \frac{\delta_{g0}}{E_a+E_b} - \frac{\lambda}{E_a+E_b}\Big(\sum_{g'+g''=g} G^{g'}_{ab}\sum_{n\in I} G^{g''}_{an} + \sum_{g'+g''=g-1} G^{g'}_{ab}G^{g''}_{a|a}$$
$$+G^{g-1}_{aaba} + G^{g-1}_{abab} + \sum_{n\in I} G^{g-1}_{an|ab} + G^{g-1}_{ab|a|a}\Big)$$
$$+\frac{\lambda}{E_a+E_b}\Big(\sum_{p\in I}\frac{G^{g}_{pb}-G^{g}_{ab}}{E_p-E_a} + \frac{G^{g-1}_{b|b}-G^{g-1}_{a|b}}{E_b-E_a}\Big)$$

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$$+G^{g-1}_{aaba} + G^{g-1}_{abab} + \sum_{n\in I} G^{g-1}_{an|ab} + G^{g-1}_{ab|a|a}\Big)$$

$$+\frac{\lambda}{E_a+E_b}\Big(\sum_{p\in I}\frac{G^g_{pb}-G^g_{ab}}{E_p-E_a} + \frac{G^{g-1}_{b|b}-G^{g-1}_{a|b}}{E_b-E_a}\Big)$$

Theorem

The (unrenormalised!) planar regular two-point function of the (E,ϕ^4)-QFT satisfies, and is determined by, the closed system of non-linear equations

$$G^0_{ab} = \frac{1}{E_a+E_b} - \frac{\lambda}{E_a+E_b}\Big(G^0_{ab}\sum_{n\in I} G^0_{an} - \sum_{p\in I}\frac{G^0_{pb}-G^0_{ab}}{E_p-E_a}\Big)$$

There is a hierarchy such that all other equations are affine in the top degree function.

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example: ϕ^4 on $4D$-Moyal space w/ harmonic oscillator potential

$$S[\phi] = \int d^4x \Big(\frac{Z}{2} \phi \star \big(-\Delta + \Omega^2 \tilde{x}^2 + \mu_0^2\big)\phi + \frac{\lambda_0 Z^2}{4} \phi \star \phi \star \phi \star \phi \Big)(x)$$

Moyal product $\star$ defined by Θ and $\tilde{x} := 2\Theta^{-1} \cdot x$
parameters: $\mu_0^2, \lambda_0, Z \in \mathbb{R}_+$ and $\Omega \in [0,1]$

- renormalisable as formal power series in λ [Grosse-W.]
  means: well-defined perturbative quantum field theory
- β-function vanishes to all orders in λ for $\Omega = 1$
  [Disertori-Gurau-Magnen-Rivasseau]
  means: model is believed to exist non-perturbatively

Up to the sign of μ_0^2, this model arises from a spectral triple.

Rewriting as a matrix model

- Moyal algebra has a basis in which $\star$-product becomes matrix product for $I = \mathbb{N}^2$ (in d dimensions: $I = \mathbb{N}^{\frac{d}{2}}$)

- The kinetic term is of non-local form $\mathrm{Tr}(\mathcal{E}(\phi \otimes \phi))$. For $\Omega = 1$ it reduces to $\mathrm{tr}(E\phi\phi)$, with

$$E_{mn} = \delta_{m_1 n_1}\delta_{m_2 n_2} Z\left(m_1 + m_2 + \frac{\mu_0^2}{2}\right), \qquad m = (m_1, m_2) \in \mathbb{N}^2$$

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- But G_{mn}, etc. only depend on $|m| = m_1 + m_2$, and each sum over m_1, m_2 with $|m| = m_1 + m_2$ yields a measure factor $(|m| + 1)$.

  $\left[\text{In } d \text{ dimensions, the measure is } \binom{|m| + \frac{d}{2} - 1}{\frac{d}{2} - 1}\right]$

With cut-off N and $a \equiv |a|$, the equation becomes

$$G^0_{ab} = \frac{1}{Z(a+b+\mu_0^2)} - \frac{\lambda}{(a+b+\mu_0^2)}\Big(Z G^0_{ab}\sum_{n=0}^{N}(n+1)G^0_{an} - \sum_{p=0}^{N}(p+1)\frac{G^0_{pb}-G^0_{ab}}{p-a}\Big)$$

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- We pass to 1PI functions $G^0_{ab} = (Z(a+b+\mu_0^2) - \Gamma_{ab})^{-1}$ and Taylor-expand $\Gamma_{ab} = Z\mu_0^2 - \mu^2 + (Z-1)(a+b) + \Gamma^{ren}_{ab}$
- normalisation conditions: $\Gamma^{ren}_{00} = 0$ and $(\partial\Gamma^{ren})_{00} = 0$ determine μ_0^2 and Z.
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After introduction of continuous variables $a = \mu^2\frac{\alpha}{1-\alpha}$ and elimination of Z, μ_0, the limit $N \to \infty$ exists:

Theorem [Grosse-W., 2009]

The renormalised planar connected two-point function $G_{\alpha\beta}$ of self-dual n.c. ϕ^4_4-theory satisfies (and is determined by)

$$G_{\alpha\beta} = 1 + \lambda\Big(\frac{1-\alpha}{1-\alpha\beta}(\mathcal{M}_\beta - \mathcal{L}_\beta - \beta\mathcal{Y}) + \frac{1-\beta}{1-\alpha\beta}(\mathcal{M}_\alpha - \mathcal{L}_\alpha - \alpha\mathcal{Y}) + \frac{(1-\alpha)(1-\beta)}{1-\alpha\beta}(G_{\alpha\beta} - 1)\mathcal{Y} - \frac{\alpha(1-\beta)}{1-\alpha\beta}(\mathcal{L}_\beta + \mathcal{N}_{\alpha\beta} - \mathcal{N}_{\alpha 0}) + \frac{1-\beta}{1-\alpha\beta}\Big(\frac{G_{\alpha\beta}}{G_{0\alpha}} - 1\Big)(\mathcal{M}_\alpha - \mathcal{L}_\alpha + \alpha\mathcal{N}_{\alpha 0})\Big)$$

with $\alpha, \beta \in [0,1)$ and

$$\mathcal{L}_\alpha := \int_0^1 d\rho \, \frac{G_{\alpha\rho} - G_{0\rho}}{1-\rho} \qquad \mathcal{M}_\alpha := \int_0^1 d\rho \, \frac{\alpha\, G_{\alpha\rho}}{1-\alpha\rho}$$

$$\mathcal{N}_{\alpha\beta} := \int_0^1 d\rho \, \frac{G_{\rho\beta} - G_{\alpha\beta}}{\rho - \alpha} \qquad \mathcal{Y} = \lim_{\alpha\to 0} \frac{\mathcal{M}_\alpha - \mathcal{L}_\alpha}{\alpha}$$

Theorem

The renormalised planar 1PI four-point function $\Gamma_{\alpha\beta\gamma\delta}$ of self-dual n.c. ϕ^4_4-theory satisfies (and is determined by)

$$\Gamma_{\alpha\beta\gamma\delta} = \lambda \cdot \frac{\Big(1 - \frac{(1-\alpha)(1-\gamma\delta)(G_{\alpha\delta}-G_{\gamma\delta})}{G_{\gamma\delta}(1-\delta)(\alpha-\gamma)} + \int_0^1 \rho\, d\rho \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}G_{\delta\rho}}{(1-\beta\rho)(1-\delta\rho)} \frac{\Gamma_{\rho\beta\gamma\delta}-\Gamma_{\alpha\beta\gamma\delta}}{\rho-\alpha}\Big)}{G_{\alpha\delta}+\lambda\Big((\mathcal{M}_\beta-\mathcal{L}_\beta-\mathcal{Y})G_{\alpha\delta}+\int_0^1 d\rho \frac{G_{\alpha\delta}G_{\beta\rho}(1-\beta)}{(1-\delta\rho)(1-\beta\rho)} + \int_0^1 \rho\, d\rho \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}}{(1-\beta\rho)(1-\delta\rho)} \frac{(G_{\rho\delta}-G_{\alpha\delta})}{(\rho-\alpha)}\Big)}$$

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Corollary

$\Gamma_{\alpha\beta\gamma\delta} = 0$ is not a solution!

We have a non-trivial (interacting) QFT in four dimensions!

Discussion

These integral equations determine the $\phi^{\star 4}$-Euclidean QFT non-perturbatively. The main difficulty is the non-linearity of the equation for $G_{\alpha\beta}$.

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- We were able to solve the non-linearity perturbatively up to order λ^3. There appear iterated integrals which evaluate to polylogarithms and ζ-functions.
- In 2D, where the renormalised equation for $G_{\alpha\beta}$ is quadratic, we were able to exactly solve the reduced equation with the quadratic term omitted. The solution exists for all $\lambda \in \mathbb{R}$, but is more regular for $\lambda > -1$.
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Thus, constructing this interacting renormalised QFT amounts to solve the single non-linear integral equation for $G_{\alpha\beta}$

Reduced equation in 2D

Let $\beta > 0$, or $\lambda > -1$ in case of $\beta = 0$. For $f, g \in C(]0,1[)$, with $g_\sigma, \log(1-\sigma) f_\sigma$ integrable, the integral equation

$$\mathcal{J}_{\alpha\beta} = \int_0^1 d\sigma \frac{f_\sigma + (1-\alpha) g_\sigma}{1-\alpha\sigma} + \lambda \frac{(1-\alpha)(1-\beta)}{1-\alpha\beta} \int_0^1 d\rho \frac{\mathcal{J}_{\alpha\beta} - \mathcal{J}_{\rho\beta}}{\alpha - \rho}$$

has the solution

$$\mathcal{J}_{\alpha\beta} = \int_0^1 d\sigma \Big\{ \frac{f_\sigma + (1-\alpha) g_\sigma}{1-\alpha\sigma} - \Big(\frac{1-\sigma_\beta}{1-\alpha\sigma_\beta} - \frac{1-\sigma}{1-\alpha\sigma} \Big) \frac{\lambda(1-\beta)\frac{\log(1-\sigma)}{\sigma}(\sigma f_\sigma - (1-\sigma) g_\sigma)}{\sigma - \beta - \lambda(1-\beta)(1-\sigma)\log(1-\sigma)} \Big\},$$

where σ_β is a function of $\beta \in [0,1[$ and $\lambda \in \mathbb{R}$ given implicitly by the unique solution in $[0,1[$ of the equation

$$0 = \sigma_\beta - \beta - \lambda(1-\beta)(1-\sigma_\beta)\log(1-\sigma_\beta)$$