Boundary Quantum Field Theory and Conformal Field Theory

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Mainly based on papers with Y. Kawahigashi, K.H. Rehren and a joint work with E. Witten

Things to discuss

- Local conformal nets (Y. Kawahigashi, R.L.)
- Algebraic Boundary Conformal Field Theory (K.H. Rehren, R.L.)
- Inner functions and Beurling-Lax theorem
- Real Hilbert subspaces
- Models of Boundary QFT (E. Witten, R.L.)
- Recent work and work in progress

2-dimensional CFT

$M = \mathbb{R}^2$ Minkowski plane.

$\begin{pmatrix} T_{00} & T_{10} \\ T_{01} & T_{11} \end{pmatrix}$ conserved and traceless stress-energy tensor.

As is well known, $T_L = \frac{1}{2}(T_{00} + T_{01})$ and $T_R = \frac{1}{2}(T_{00} - T_{01})$ are chiral fields,

$$T_L = T_L(t + x), \quad T_R = T_R(t - x).$$

Left and right movers.

Two-dimensional conformal fields and nets

Ψ_k family of conformal fields on M: T_{ij} + *relatively local fields*
$\mathcal{O} = I \times J$ double cone, I, J intervals of the chiral lines $t \pm x = 0$

$$\mathcal{A}(\mathcal{O}) = \{e^{i\Psi_k(f)}, \mathrm{supp} f \subset \mathcal{O}\}''$$

then by relative locality

$$\mathcal{A}(\mathcal{O}) \supset \mathcal{A}_L(I) \otimes \mathcal{A}_R(J)$$

$\mathcal{A}_L, \mathcal{A}_R$ chiral fields on $t \pm x = 0$ generated by T_L, T_R and other chiral fields

(completely) rational case: $\mathcal{A}_L(I) \otimes \mathcal{A}_R(J) \subset \mathcal{A}(\mathcal{O})$ finite Jones index

Local conformal nets

A local *Möbius covariant net* $\mathcal{A}$ on S^1 is a map

$$I \in \mathcal{I} \to \mathcal{A}(I) \subset B(\mathcal{H})$$

$\mathcal{I} \equiv$ family of proper intervals of S^1, that satisfies:

- **A.** *Isotony*. $I_1 \subset I_2 \implies \mathcal{A}(I_1) \subset \mathcal{A}(I_2)$
- **B.** *Locality*. $I_1 \cap I_2 = \varnothing \implies [\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}$
- **C.** *Möbius covariance*. $\exists$ unitary rep. U of the Möbius group Möb on $\mathcal{H}$ such that

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI), \quad g \in \text{Möb}, \ I \in \mathcal{I}.$$

- **D.** *Positivity of the energy*. Generator L_0 of rotation subgroup of U (conformal Hamiltonian) is positive.
- **E.** *Existence of the vacuum*. $\exists!$ U-invariant vector $\Omega \in \mathcal{H}$ (vacuum vector), and Ω is cyclic for $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I)$.

First consequences

- *Irreducibility*: $\bigvee_{I\in\mathcal{I}}\mathcal{A}(I)=B(H)$.
- *Reeh-Schlieder theorem*: Ω is cyclic and separating for each $\mathcal{A}(I)$.
- *Bisognano-Wichmann property*: Tomita-Takesaki modular operator Δ_I and conjugation J_I of $(\mathcal{A}(I),\Omega)$, are

$$U(\Lambda_I(2\pi t))=\Delta_I^{it},\ t\in\mathbb{R}, \qquad \text{dilations}$$
$$U(r_I)=J_I \qquad \text{reflection}$$

  (Frölich-Gabbiani, Guido-L.)
- *Haag duality*: $\mathcal{A}(I)'=\mathcal{A}(I')$
- *Factoriality*: $\mathcal{A}(I)$ is III_1-factor (in Connes classification)
- *Additivity*: $I\subset\cup_i I_i \implies \mathcal{A}(I)\subset\vee_i\mathcal{A}(I_i)$ (Fredenhagen, Jorss).

Local conformal nets

$\mathrm{Diff}(S^1) \equiv$ group of orientation-preserving smooth diffeomorphisms of S^1

$\mathrm{Diff}_I(S^1) \equiv \{g \in \mathrm{Diff}(S^1) : g(t) = t \ \forall t \in I'\}$.

A local conformal net $\mathcal{A}$ is a Möbius covariant net s.t.

F. *Conformal covariance*. $\exists$ a projective unitary representation U of $\mathrm{Diff}(S^1)$ on $\mathcal{H}$ extending the unitary representation of Möb s.t.

$$\begin{aligned} U(g)\mathcal{A}(I)U(g)^* &= \mathcal{A}(gI), \quad g \in \mathrm{Diff}(S^1), \\ U(g)xU(g)^* &= x, \quad x \in \mathcal{A}(I),\ g \in \mathrm{Diff}_{I'}(S^1), \end{aligned}$$

$\longrightarrow$ unitary representation of the *Virasoro algebra*

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}$$

$[L_n, c] = 0,\ L_n^* = L_{-n}$.

Representations

A representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a map

$$I \in \mathcal{I} \mapsto \pi_I, \text{ normal rep. of } \mathcal{A}(I) \text{ on } B(\mathcal{H})$$

$$\pi_{\tilde{I}} \restriction \mathcal{A}(I) = \pi_I, \quad I \subset \tilde{I}$$

π is automatically diffeomorphism *covariant*: $\exists$ a projective, pos. energy, unitary rep. U_π of $\mathrm{Diff}^{(\infty)}(S^1)$ s.t.

$$\pi_{gI}(U(g)xU(g)^*) = U_\pi(g)\pi_I(x)U_\pi(g)^*$$

for all $I \in \mathcal{I}$, $x \in \mathcal{A}(I)$, $g \in \mathrm{Diff}^{(\infty)}(S^1)$ (Carpi & Weiner)

DHR argument: given I, there is an endomorphism of $\mathcal{A}$ localized in I equivalent to π; namely ρ is a representation of $\mathcal{A}$ on the vacuum Hilbert space $\mathcal{H}$, unitarily equivalent to π, such that $\rho_{I'} = \mathrm{id} \restriction_{\mathcal{A}(I')}$.

- Rep($\mathcal{A}$) is a *braided tensor category* (Doplicher-Haag-Roberts, Frölich, Fredenagen-Rehren-Schroer, L.)

Split property

$\mathcal{A}$ satisfies the *split* property if the von Neumann algebra

$$\mathcal{A}(I_1) \vee \mathcal{A}(I_2) \simeq \mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$$

(natural isomorphism) if $\bar{I}_1 \cap \bar{I}_2 = \varnothing$.

$$\mathrm{Tr}(e^{-tL_0}) < \infty, \ \forall t > 0 \implies \text{ split } .$$

Complete rationality

I_1, I_2 intervals $\bar{I}_1 \cap \bar{I}_2 = \varnothing$, $E \equiv I_1 \cup I_2$.

$$\mu\text{-}index: \quad \mu_{\mathcal{A}} \equiv [\mathcal{A}(E')' : \mathcal{A}(E)]$$

(Jones index). $\mathcal{A}$ conformal:

$$\mathcal{A} \textit{ completely rational} \stackrel{\text{def}}{=} \mathcal{A} \text{ split } \& \ \mu_{\mathcal{A}} < \infty$$

Thm. (Y. Kawahigashi, M. Müger, R.L.) $\mathcal{A}$ completely rational: then

$$\mu_{\mathcal{A}} = \sum_i d(\rho_i)^2$$

sum over all irreducible sectors. (F. Xu in $SU(N)$ models);

• $\mathcal{A}(E) \subset \mathcal{A}(E')' \sim$ LR inclusion (quantum double);

• Representations form a modular tensor category (i.e. non-degenerate braiding).

Classification of local conformal nets, $c = 1 - \frac{6}{m(m+1)}$

Local conformal nets with $c < 1$ are classified by pair of Dynkin diagrams $A - D_{2n} - E_{6,8}$ s.t. difference of Coxeter numbers is 1. (Kawahigashi,L.)

m	Labels for Z
n	(A_{n-1}, A_n)
$4n+1$	(A_{4n}, D_{2n+2})
$4n+2$	(D_{2n+2}, A_{4n+2})
11	(A_{10}, E_6)
12	(E_6, A_{12})
29	(A_{28}, E_8)
30	(E_8, A_{30})

(Z: Cappelli-Itzykson-Zuber modular invariant)

One *new example* (A_{28}, E_8), most probably not constructable as coset.

Case $c = 1$: (Xu, Carpi)

Boundary CFT

Stress-energy tensor left/right movers $T_L = \frac{1}{2}(T_{00} + T_{01})$ and $T_R = \frac{1}{2}(T_{00} - T_{01})$: $T_L = T_L(t + x)$, $T_R = T_R(t - x)$.

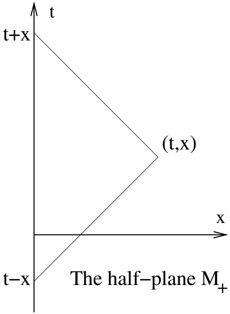
Boundary condition: no energy flow across the boundary:

$$T_{01}(t, x = 0) = 0 \qquad \Leftrightarrow \qquad T_L = T_R \equiv T.$$

so $T_{10} = T_{01}$, $T_{11} = T_{00}$ are of the form

$$T_{00}(t,x) = T(t{+}x){+}T(t{-}x), \qquad T_{01}(t,x) = T(t{+}x){-}T(t{-}x),$$

i.e., *bi-local* expressions in terms of T

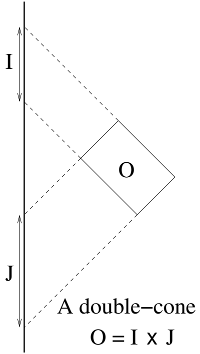


The chiral fields of a boundary CFT generate a net

$$O \mapsto A_+(O).$$

$A_+(O)$ is generated by chiral fields smeared in the variable $t+x$ over the interval I and in the variable $t-x$ over the interval J, where $O = I \times J$, $I > J$, is an open double-cone in M_+. The bi-localized structure translates into the form of the local algebras

$$A_+(O) = A(I) \vee A(J) \qquad (O = I \times J, \;\; I > J).$$



Definition of Boundary CFT

A *boundary CFT (BCFT)* associated with A is a local, isotonous net $O \mapsto B_+(O)$ over the double-cones within the half-space M_+, represented on a Hilbert space $\mathcal{H}_B$ such that

(i) there is a unitary representation $\mathcal{U}$ of the covering of the Möbius group $PSL(2,\mathbb{R})$ with positive generator for the subgroup of translations, such that

$$\mathcal{U}(g)B_+(O)\mathcal{U}(g)^* = B_+(gO)$$

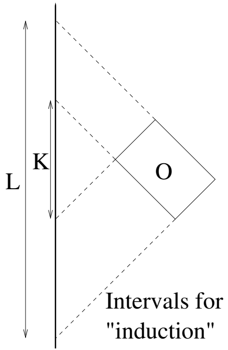
(ii) There is a representation π of A on $\mathcal{H}_B$ such that $B_+(O)$ contains $\pi(A_+(O))$ and π is U-covariant.

(iii) *"Joint irreducibility":* For each double-cone O, $B_+(O) \vee \pi(A_+)''$ is irreducible on $\mathcal{H}_B$ (almost automatic)

chiral extension → boundary condition

If $I \mapsto B(I)$ is an irreducible chiral extension of $I \mapsto A(I)$ (possibly non-local, but relatively local with respect to A), then the *induced net* is defined by

$$O \mapsto B_+^{ind}(O) := B(L) \cap B(K)'.$$



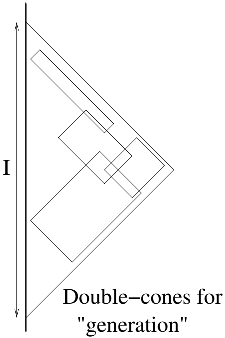
The observables of the induced BCFT localized in O belong to $B(L)$ and commute with $B(K)$.

BCFT $\rightarrow$ non-local chiral net

A boundary CFT $O \mapsto B_+(O)$ generates a chiral net $I \mapsto B^{gen}(I)$ (the associated *boundary net*) on $\mathcal{H}_B$, by

$$B^{gen}(I) := \bigvee_{O \subset W_L} B_+(O) \equiv B_+(W_L)$$

where W_L is the left wedge spanned by I



The observables of the associated chiral boundary net localized in I are generated by BCFT observables localized in double cones

(i) In the special case $B = A$, the induced net is the dual net A_+^{dual}:

$$B^{dual}(O) \equiv B(O')'$$

so $A_+(O) \subset A_+^{dual}(O)$ is the 2-interval inclusion.

(ii) If B is a chiral extension of A, then

$$(B_+^{ind})^{gen} = B$$

Conversely

$$(B_+^{gen})^{ind} = B_+^{dual}$$

(iii) Every induced net $B\,\mathrm{ind}_+$ is self-dual (Haag dual).

conclusion:

non-local chiral extensions of $A \leftrightarrow$ local extensions of A_+

Classification of non-local extensions

Kawahigashi, Penning, Rehren, L. All irreducible (non-local) extensions of nets Vir_c, $c < 1$, are classified

$$\Downarrow$$

All conformal (local) Boundary CFT with $c < 1$ are classified

Boundary and non boundary nets

$A(I)$ completely rational chiral net

$B(I)$ irreducible (possibly non-local) chiral extension

$B_+^{ind}(O)$, be the induced Haag dual boundary CFT net on M_+

$B_2^\alpha(O)$ two-dimensional local net on M extending $A \otimes A$, obtained from B by α-induction construction

Then the local subfactors

$$A(I) \vee A(J) \subset B_+^{ind}(O) \qquad \text{and} \qquad A(I) \otimes A(J) \subset B_2^\alpha(O)$$

are *isomorphic*.

Note: The isomorphism is not natural, same invariants. A natural, geometric proof, by letting the boundary go to infinity, is given in a subsequent paper by Rehren, L.

Remarkable properties

Let B chiral extension of A, and B_+^{ind} the induced BCFT net.

(i) The index of $\pi(A_+(O)) \subset B_+^{\mathrm{ind}}(O)$ equals the μ-index μ_A of A. This index is thus the same for each chiral extension

(ii) When B_+ is Haag dual, then $\mu_{B_+} = 1$, and B_+ satisfies Haag duality also for the disconnected regions of the form $E = O_1 \cup O_2$

(iii) A Haag dual boundary CFT net B_+ has the no nontrivial DHR sectors.

The semigroup $\mathcal{E}(\mathcal{A})$

Let $\mathcal{A}$ be a local Möbius covariant net of von Neumann algebras on $\mathbb{R}$

$$I \subset \mathbb{R} \text{ interval} \to \mathcal{A}(I)$$

T one-parameter unitary translation group. Then $T(t)\mathcal{A}(I)T(-t) = \mathcal{A}(I+t)$, T has positive generator P and $T(t)\Omega = \Omega$ where Ω is the vacuum vactor.

Let V be a unitary on $\mathcal{H}$ commuting with T. The following are equivalent:

(*i*) $V\mathcal{A}(I_2)V^*$ commutes with $\mathcal{A}(I_1)$ for all intervals I_1, I_2 of $\mathbb{R}$ such that $I_2 > I_1$ (I_2 is contained in the future of I_1).

(*ii*) $V\mathcal{A}(a,\infty)V^* \subset \mathcal{A}(a,\infty)$ for every $a \in \mathbb{R}$.

(*iii*) $V\mathcal{A}(0,\infty)V^* \subset \mathcal{A}(0,\infty)$.

The semigroup $\mathcal{E}(\mathcal{A})$

$$\mathcal{E}(\mathcal{A}) \equiv \text{semigroup of unitaries } V \text{ as above}$$

$\mathcal{A}$ conformal net & $V \in \mathcal{E}(\mathcal{A}) \longrightarrow$ Boundary QFT $\mathcal{A}_V$

$$\mathcal{A}_V(\mathcal{O}) \equiv \mathcal{A}(I_1) \vee V\mathcal{A}(I_2)V^*$$

where I_1, I_2 are intervals of time-axis such that $I_2 > I_1$ and $\mathcal{O} = I_1 \times I_2$.

$\mathcal{A}$ with the split property, $V \in \mathcal{E}(\mathcal{A})$ then $\mathcal{A}_V$ is *locally isomorphic* to $\mathcal{A}_+ = \mathcal{A}_I$.

As an immediate consequence, if V_t is a one-parameter semigroup of unitaries in $\mathcal{E}(\mathcal{A})$, the family $\mathcal{A}_{V_t}$ gives a *deformation* of the conformal net $\mathcal{A}_+$ on M_+ with translation covariant nets on M_+ that are locally isomorphic to $\mathcal{A}_+$.

Inner functions

$\mathbb{D} \equiv \{z \in \mathbb{C} : |z| < 1\}$ unit disk, $\mathbb{H}^\infty(\mathbb{D})$ Hardy space.

$\varphi \in \mathbb{H}^\infty(\mathbb{D}) \Rightarrow \exists\, \varphi(e^{i\theta}) \equiv \lim_{r\to 1^-} \varphi(re^{i\theta})$ a.e. on $\partial\mathbb{D}$

$\varphi \in \mathbb{H}^\infty(\mathbb{D})$ is an *inner function* if $|\varphi(z)| = 1$ for almost all $z \in \partial\mathbb{D}$.

Examples:
$B_0(z) \equiv z$, or its Möbius transform:
$B_a(z) = \frac{|a|}{a} \frac{z-a}{1-\bar{a}z}$ (Blaschke factor),

$$B(z) \equiv \prod_{n=1}^{\infty} B_{a_n}(z) \quad \text{(Blaschke product)},$$

$a_n \in \mathbb{D}$, $\sum_{n=1}^{\infty}(1-|a_n|) < \infty$.

$B(z)$ has zeros exactly at $\{a_n\}$, with multiplicity.
If an inner function φ has no zeros on $\mathbb{D}$, then φ is called a *singular* inner function.
φ is an inner function iff (uniquely)

$$\varphi(z) = \alpha B(z) \exp\left(-\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \mathrm{d}\mu(e^{i\theta})\right),$$

where μ is a positive, Lebesgue singular measure on $\partial\mathbb{D}$, $B(z)$ is a Blaschke product and α is a constant with $|\alpha| = 1$. All the zeros of φ come from B so φ is singular if and only if B is the identity.
Inner functions form a (multiplicative) *semigroup*, singular inner functions a sub-semigroup.

One-param. semigroup $\{\varphi_t\}$ of inner functions:

$$\varphi_t(z) = e^{it\lambda} \exp\left(-t \int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \mathrm{d}\mu(e^{i\theta})\right)$$

Symmetric inner functions:

φ is symmetric $\bar{\varphi}(z) = \varphi(\bar{z})$.

Notions go $\mathbb{S}_\infty$ and $\mathbb{S}_\pi$: $h(z) \equiv i\frac{1+z}{1-z}$,

$$\mathbb{D} \xrightarrow{h} \mathbb{S}_\infty \xrightarrow{\log} \mathbb{S}_\pi$$

$\varphi \in \mathbb{H}^\infty(\mathbb{S}_\pi)$ inner: $|\varphi(q)| = |\varphi(q + i\pi)| = 1$
symmetric: $\varphi(q + i\pi) = \bar{\varphi}(q)$, $q \in \mathbb{R}$ a.e.

$\varphi \in \mathbb{H}^\infty(\mathbb{S}_\infty)$ inner: $|\varphi(q)| = 1$, $q > 0$
symmetric: $\varphi(-q) = \bar{\varphi}(q)$ a.e.

Scattering functions
A scattering function is a symmetric innere function f on $\mathbb{S}_\pi$ s.t. $\varphi(-p) = \varphi(p)$.

Inverse scattering: construct QFT models from scattering function (cf. Lechner models)

Beurling-Lax theorem (1949-1959)

S shift operator on $H^2(\mathbb{D})$:

$$Sf(z) = zf(z)$$

A *closed S-invariant subspace K* of $H^2(\mathbb{D})$ has the form

$$K = \varphi H^2(\mathbb{D}), \quad \varphi \textit{ an inner function}$$

This implies: $f \in H^2$ (or $f \in H^p, p \geq 1$) has a factorization:

$$f(z) = \varphi(z)\psi(z)$$

φ is inner and ψ is outer $\psi(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log|f(e^{it})| \mathrm{d}t\right)$

Lax generalization to $H^2(\mathbb{S}_\infty)$, one-param. unitary translations in Fourier transform.

Standard real Hilbert subspaces

$\mathcal{H}$ complex Hilbert space and $H \subset \mathcal{H}$ a real linear subspace.
Symplectic complement:

$$H' = \{\xi \in H : \Im(\xi, \eta) = 0 \ \forall \eta \in H\}$$

$H' = (iH)^{\perp}$ (real orthogonal complement), $H_1 \subset H_2 \Leftrightarrow H_2' \subset H_1'$

A standard subspace H of $\mathcal{H}$ is a closed, real linear subspace of $\mathcal{H}$ which is both cyclic ($\overline{H + iH'} = \mathcal{H}$) and separating ($H \cap iH' = \{0\}$). H is standard iff H' is standard.

H standard subspace $\rightarrow$ anti-linear operator $S : D(S) \subset \mathcal{H} \rightarrow \mathcal{H}$,

$$S : \xi + i\eta \rightarrow \xi - i\eta, \ \xi, \eta \in H$$

$S^2 = 1|_{D(S)}$. S is closed and densely defined.

Conversely, S densely defined, closed, anti-linear involution on $\mathcal{H} \rightarrow H = \{\xi \in D(S) : S\xi = \xi\}$ is a standard subspace:

$$H \leftrightarrow S \textit{ is a bijection}$$

Modular theory for standard real Hilbert subspaces

Set $S = J\Delta^{1/2}$, polar decomposition of S.

Then J is an anti-unitary involution, $\Delta > 0$ and we have:

$$\Delta^{it}H = H, \quad JH = H'$$

(one particle Tomita-Takesaki theorem).

Borchers theorem (real subspace analog)

H standard subspace, T a one-parameter group with positive generator s.t. $T(s)H \subset H, \ s > 0$.

Then:

$$\begin{cases} \Delta^{it}T(s)\Delta^{-it} = T(e^{-2\pi t}s) \\ JT(s)J = T(-s), \ t, s \in \mathbb{R} \end{cases}$$

Consequence: If T has no non-zero fixed vector, H is unique up to multiplicity

Endomorphisms of standard subspaces

A *standard pair* of $\mathcal{H}$ is a pair (H, T) such that

- H is a standard subspace,
- T is a one-par. unitary group, with positive generator P, s.t. $T(t)H \subset H$, $t \geq 0$.

Thm. Assume (H, T) to be irreducible and let $K \subset H$ be a real subspace. The following are equivalent:

(*i*) $T(t)K \subset K, t \geq 0$,

(*ii*) $K = VH$ where V is a unitary commuting with T,

(*iii*) $K = VH$ where $V = \psi(Q)$ with $Q \equiv \log P$ and $\psi \in L^\infty(\mathbb{R}, \mathrm{d}q)$ is the boundary value of an inner function in $H^\infty(\mathbb{S}_\pi)$ such that $\psi(q + i\pi) = \bar{\psi}(q)$, for almost all $q \in \mathbb{R}$.

The semigroup $\mathcal{E}(H)$ of endomorphisms of (H, T) is isomorphic to the semigroup of symmetric inner functions on the strip $0 < \Im z < \pi$.

Constructing models

$\mathcal{A}$ free field on $\mathbb{R}$ acting on the Fock space $F(\mathcal{H})$.

H standard subspace of $\mathcal{H}$ $\rightarrow$ von Neumann algebra on $F(\mathcal{H})$

$$\mathcal{A}(H) = \{W(h) : h \in H\}''$$

Take $H = H(0, \infty)$.

$$V \in \mathcal{E}(H) \rightarrow \Gamma(V) \in \mathcal{E}(\mathcal{A})$$

therefore

symmetric inner function $\rightarrow$ $V \in \mathcal{E}(\mathcal{A})$ $\rightarrow$ Boundary QFT net $\mathcal{A}_V$ on M_+

In particular

φ scattering function $\rightarrow$ Boundary QFT

More general BQFT's

$\mathcal{A} = \mathcal{A}_N$ Buchholz-Mach-Tododorv extension of $U(1)$-current net:

symmetric inner function Hölder continuous at 0 & $V \in \mathcal{E}(\mathcal{A})$

$\downarrow$

Boundary QFT net $\mathcal{A}_V$ on M_+

Examples: $\mathcal{A}_1$ associated with level 1 $\widehat{su(2)}$-Kac-Moody algebra with $c = 1$, $\mathcal{A}_2$ Bose subnet of free complex Fermi field net, $\mathcal{A}_3$ appears in the $\mathbb{Z}_4$-parafermion current algebra analyzed by Zamolodchikov and Fateev, and in general $\mathcal{A}_N$ is a coset model $SO(4N)_1/SO(2N)_2$.

Recent works

- QFT on the interior of the Lorentz hyperboloid (K.H. Rehren, R.L.)
- KMS states in CFT (P. Camassa, Y. Tanimoto, M. Weiner, R.L.)
- Computation of KMS states in models (P. Camassa, Y. Tanimoto, M. Weiner, R.L.)
- The semigroup in lattice models (M. Bischoff).
- Deformation of two-dimensional models (Y. Tanimoto)

Boundary CFT on the interior of Lorentz hyperboloid $\mathfrak{H}_R$

$\mathfrak{H}_R = x^2 - t^2 = R^2$ $(x > 0)$.

Boundary condition: no energy flow across the boundary $T^{0\mu}\epsilon_{\mu\nu}dx^\nu = 0$: $uT_L(u)|_{uv=1} = -vT_R(-v)|_{uv=1}$ (chiral coordinates), so

$$uT_L(u) = -\frac{1}{u}T_R\left(-\frac{1}{u}\right) \equiv T(u) \ .$$

so

$$T_{00}(u,v) = \frac{1}{u}T(u) - vT\left(-\frac{1}{v}\right), \qquad T_{01}(u,v) = \frac{1}{u}T(u) + vT\left(-\frac{1}{v}\right),$$

Local von Neumann algebras $\mathcal{A}(\mathcal{O})$ on $\mathfrak{H}_R$:

$$\mathcal{A}(\mathcal{O}) = \mathcal{A}_0(I) \vee \mathcal{A}_0(J^{-1}) \ ,$$

where $\mathcal{A}_0$ is generated by the chiral stress-energy tensor (Virasoro net).

Dilation covariance of $\mathcal{A}_0|_{\mathbb{R}^+}$ gives boost covariance of $\mathcal{A}$ and the KMS property of the vacuum state on $\mathcal{A}_0(\mathbb{R}^+)$ (Bisognano-Wichmann property) gives the KMS property of the vacuum state on $\mathcal{A}(\mathfrak{H}_R)$ w.r.t. the boosts at Hawking-Unruh inverse temperature $\beta = 2\pi$.

The semigroup $\mathcal{E}_\delta(\mathcal{A})$

$$\mathcal{E}_\delta(\mathcal{A}) \equiv \text{semigroup of unitaries } V:$$
$$V\mathcal{A}(0,\infty)V^* = \mathcal{A}(0,\infty),\;\; V\mathcal{A}(1,\infty)V^* \subset \mathcal{A}(1,\infty)$$

$\mathcal{A}$ conformal net & $V \in \mathcal{E}_\delta(\mathcal{A}) \longrightarrow$ Boundary QFT on $\mathfrak{H}_R$ $\mathcal{A}_V$

$$\mathcal{A}_V(\mathcal{O}) \equiv \mathcal{A}(I) \vee V\mathcal{A}(J^{-1})V^*$$

where I, J are intervals of time-axis such that $J > I$ and $\mathcal{O} = I \times J$.

$\mathcal{A}_V$ is in a thermal state

Constructing local nets on $\mathfrak{H}_R$

$\mathcal{A}$ free field on $\mathbb{R}$ acting on the Fock space $F(\mathcal{H})$.

H standard subspace of $\mathcal{H} \to$ von Neumann algebra on $F(\mathcal{H})$

$$\mathcal{A}(H) = \{W(h) : h \in H\}''$$

Take $H = H(0, \infty)$.

$$V \in \mathcal{E}_\delta(H) \to \Gamma(V) \in \mathcal{E}_\delta(\mathcal{A})$$

therefore

symmetric inner function $\to V \in \mathcal{E}_\delta(\mathcal{A}) \to$ Boundary QFT net $\mathcal{A}_V$ on $\mathfrak{H}_R$

In particular

φ scattering function $\to$ Boundary QFT on $\mathfrak{H}_R$ in a KMS state

Surprisingly: Symmetric inner functions appear here too and play the same role.

A classification of KMS states (Camassa, Tanimoto, Weiner, L.)

KMS state w.r.t. translation on a conformal net $\mathcal{A}$ $\to$ Boundary QFT on $\mathfrak{H}_R$ in a KMS state.

Home many KMS states do there exist?

- $\mathcal{A}$ completely rational: only one KMS state (geometrically constructed)
- $\mathcal{A}$ $U(1)$ current: classified (one-parameter family)
- Virasoro net:
  $c < 1$ only one (completely rational)
  $c = 1$ classified (one parameter family)
  $c > 1$ one parameter family (probably all)

Problems

- Is there a connection with Lechner models?
- Which BQFT's are associated with loop group models?