Pure Spinor Helicity Methods

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R.B., arXiv:0908.0738 [hep-th]

Why you should pay attention, an experiment

The greatest common denominator of the audience?

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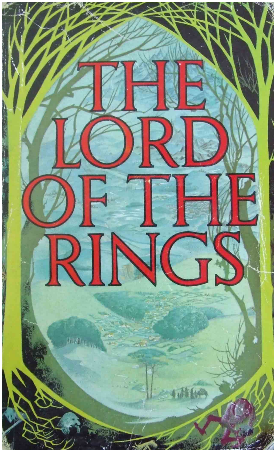

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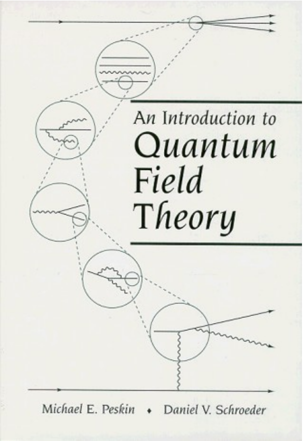

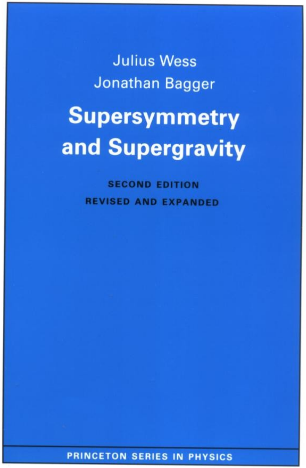

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precise control over Poincaré and Susy quantum numbers for all legs simultaneously

Outline

1. Motivation

2. Covariant representation theory of
   - Poincaré algebra
   - Spin algebra
   - Susy algebra

3. Outlook

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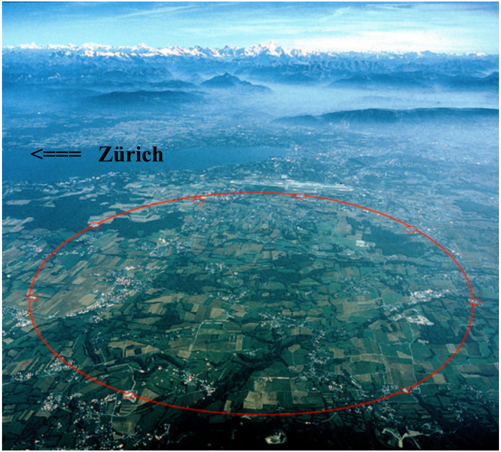

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- pure spinor spaces are higher dimensional twistor spaces

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- broken gauge theory in $D = 4 \rightarrow$ [R.B., Christian Schwinn, to appear]

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polarization vectors (in q lightcone gauge)

$$e^\mu(0, \ldots, \pm_j, \ldots) = \frac{1}{\sqrt{2}}\left(\tilde{n}^\mu_{2j-1} \pm \mathrm{i}\tilde{n}^\mu_{2j}\right)(k)$$

- requires choice of complex structure

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- 'helicity equal' in $D = 4$
- 'one helicity unequal' does not vanish in $D > 4$
- class of Einstein gravity amplitudes through KLT
- six gluon open string amplitude [Oprisa, Stieberger, 2005]

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- phases, schmases

Spinor helicity

- there is a natural spinor inner product
  $\overline{\psi\left(h_0, \vec{h}\right)}\psi'\left(h'_0, \vec{h}'\right) \sim \delta\left(h_0 + h'_0\right)\delta\left(\vec{h} - \vec{h}'\right)$ where $\overline{\psi'}\psi \equiv \psi'^{\dagger}\gamma_0\psi$

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$$k_\mu\gamma^\mu = \sum_{\vec{h}} \psi_{\vec{h}}\overline{\psi_{\vec{h}}} \qquad q_\mu\gamma^\mu = \sum_{\vec{h}} \xi_{\vec{h}}\overline{\xi_{\vec{h}}}$$

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for any $\vec{h}$. Similar formulae for polarization vectors.

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- representation is redundant $(D = 4, 1)$, $(D = 6, 2)$, $(D = 10, 4)$

Calculability

- choose highest weight $\xi(\frac{1}{2},\ldots\frac{1}{2}) \quad (q\xi = 0)$
- define (phases of) other states using (ordered) lowering operators, e.g.

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numerics & lightcone analysis...

- pick a frame in which $q = (1, 0, \ldots, 0, 1)$, $\tilde{n}_i = (0, \ldots, 1_i, 0, \ldots)$.
- pick a gamma matrix representation.
- find eigenvectors of $q_\mu \gamma^\mu \xi = 0$
- other quantum numbers are then easy $\leftarrow$ momentum independent
- all solutions to the massless Dirac equation by applying $\gamma_\mu k^\mu$

Some remarks

Checks

Extensions

Intriguing further structure

To do

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- $4D$ spinor helicity
- proposal for 6D [Cheung, O' Donnal, 09]
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To do

- study supersymmetry
- find more amplitudes (on-shell recursion!)

Outline

1 Motivation

2 Covariant representation theory of
- Poincaré algebra
- Spin algebra
- Susy algebra

3 Outlook

Susy Ward identities

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  1. Lagrangian [Grisaru, Pendleton, Van Nieuwenhuizen, 76]
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In any susy S-matrix theory

$$0 = \langle 0| S\, \Phi_{\mathrm{in}}\, Q\, |0\rangle = \langle 0| S\, [Q, \Phi_{\mathrm{in}}] |0\rangle = \langle 0| S\, Q\, |\mathrm{in}\rangle$$

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- need: covariant supersymmetry representation theory

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in terms of Lorentz invariant generators

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→ action of any Q known covariantly

Example: $\mathcal{N} = 1, D = 10$

- bosonic states are e.g. $\vec{h} = (\pm 1, 0, 0, 0)$ $(\# = 8)$
- fermionic ones are $\vec{h} = (\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, a)$, last 'bit' fixed by chirality
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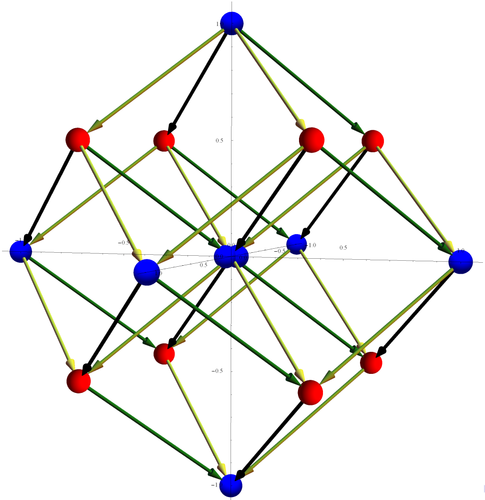

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- generic central charges ('technicality')

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- complete spinor helicity construction from covariant representation theory
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- join the fun!
  - non-zero amplitudes? $\rightarrow$ on-shell recursion (work in progress)
  - less nuts and bolts?
  - connection to Berkovits' pure spinors?
  - spontaneous symmetry breaking?

Example: $\mathcal{N} = 1, D = 11$