Topologically massive gravity and the AdS/CFT correspondence

Balt van Rees

8 September 2009

Based on work with K. Skenderis and M. Taylor:
arXiv:0906.4926

Topologically massive gravity

Three-dimensional pure Einstein gravity is locally trivial

This changes when we add a gravitational Chern-Simons term to the action:

$$S = \frac{1}{16\pi G_N}\Big(\int d^3x\sqrt{-G}(R-2\Lambda) + \frac{1}{2\mu}\int d^3x(\Gamma d\Gamma + \frac{2}{3}\Gamma\wedge\Gamma\wedge\Gamma)\Big)$$

This gives a third-order equation of motion:

$$R_{\mu\nu} - \frac{1}{2}RG_{\mu\nu} + \Lambda G_{\mu\nu} + \frac{1}{2\mu}(\epsilon_\mu{}^{\alpha\beta}\nabla_\alpha R_{\beta\nu} + \mu\leftrightarrow\nu) = 0$$

which does allow for local degrees of freedom in a three-dimensional theory of gravity

Problems with stability. For $\Lambda < 0$:

- perturbative solutions around AdS_3 have negative energy (in our conventions)
- BTZ black hole has positive energy

Deser, Jackiw, Templeton (1982)

Quantum topologically massive gravity

Topologically massive gravity recently received more attention

Li, Song, Strominger (2008)

Inspired by renewed interest in three-dimensional Einstein gravity
Search for a possible dual CFT ($\Lambda < 0$)

Witten (2007)

What would be the dual CFT for topologically massive gravity with $\Lambda < 0$?

Is it consistent, unitary? Can we learn anything about higher-dimensional theories?

- $+$ Dynamics might give a more realistic theory
- $-$ Problems with positivity of energy

Some properties of TMG

Action for $\Lambda = -1$:

$$S = \frac{1}{16\pi G_N} \int d^3x \sqrt{-G}(R+2) + \frac{1}{32\pi G_N \mu} \int d^3x \sqrt{-G} \epsilon^{\lambda\mu\nu} \left(\Gamma^{\rho}_{\lambda\sigma} \partial_\mu \Gamma^{\sigma}_{\rho\nu} + \frac{2}{3} \Gamma^{\rho}_{\lambda\sigma} \Gamma^{\sigma}_{\mu\tau} \Gamma^{\tau}_{\nu\rho} \right)$$

Equations of motion:

$$R_{\mu\nu} - \frac{1}{2} R G_{\mu\nu} - G_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0$$

$$C_{\mu\nu} = \frac{1}{2} \epsilon_{\mu}{}^{\alpha\beta} \nabla_\alpha R_{\beta\nu} + \mu \leftrightarrow \nu$$

Properties of the Cotton tensor:

$$C^{\mu}_{\mu} = 0 \qquad \nabla^\mu C_{\mu\nu} = 0$$

If $G_{\mu\nu}$ is Einstein, so $R_{\mu\nu} = -2G_{\mu\nu}$, then $C_{\mu\nu} = 0$ and $G_{\mu\nu}$ is also a solution of TMG

All solutions $G_{\mu\nu}$ of TMG have $R = -6$

Perturbative spectrum

We investigate the spectrum around an AdS$_3$ background:

$$G_{\mu\nu}dx^\mu dx^\nu = -(r^2+1)dt^2 + \frac{dr^2}{r^2+1} + r^2 d\phi^2$$

Consider a small variation of the metric:

$$G_{\mu\nu} \to G_{\mu\nu} + H_{\mu\nu}$$

The equation of motion gives a third-order linear differential equation for $H_{\mu\nu}$
The solutions can be classified by the symmetry algebra $\sim SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$
with generators L_0, L_{-1}, L_1 and $\bar{L}_0, \bar{L}_{-1}, \bar{L}_1$

We search for *primary* perturbations that are:

- annihilated by L_1 and $\bar{L}_1$
- eigenfunctions of L_0 and $\bar{L}_0$:

$$L_0 H_{\mu\nu} = h H_{\mu\nu} \qquad \bar{L}_0 H_{\mu\nu} = \bar{h} H_{\mu\nu}$$

where $L_0 = \frac{i}{2}(\partial_t + \partial_\phi)$ and $\bar{L}_0 = \frac{i}{2}(\partial_t - \partial_\phi)$

Perturbative spectrum

For generic μ, there exist three primary solutions $H^L_{\mu\nu}$, $H^R_{\mu\nu}$, $H^M_{\mu\nu}$ with:

$$L_0 H^L = 2H^L \qquad \bar{L}_0 H^L = 0$$
$$L_0 H^R = 0 \qquad \bar{L}_0 H^R = 2H^R$$
$$L_0 H^M = \frac{1}{2}(\mu+3)H^M \qquad \bar{L}_0 H^M = \frac{1}{2}(\mu-1)H^M$$

For $\mu = 1$ the modes H^L and H^M coincide

Li, Song, Strominger (2008)

However, for $\mu = 1$ a new mode $\tilde{H}^M_{\mu\nu}$ arises for which:

$$L_0 \tilde{H}^M = 2\tilde{H}^M + H^L \qquad \bar{L}_0 \tilde{H}^M = H^L$$

Hints at a *logarithmic* CFT with $\tilde{H}^M$ the logarithmic partner of H^L
This mode has different falloff conditions ($\log(r)/r^2$ vs. $1/r^2$)

Grumiller, Johansson (2008)

Questions:

- Is TMG at $\mu = 1$ dual to a logarithmic CFT?
  If so, what is the precise AdS/CFT dictionary for TMG?
- Can we allow the different falloff conditions?

Setting up an AdS/CFT dictionary

Aim: compute CFT correlators from a bulk theory with action S using

$$Z_{\text{CFT}} \sim \exp(-S_{\text{on-shell}})$$

GKP, Witten (1998)

Procedure:

- Write down equations of motion from S
- Perform an asymptotic analysis near the conformal boundary of spacetime
  - Fix the *leading* behaviour of the fields (asymptotically AdS, sources $\phi_{(0)}$)
  - Solve the equations of motion asymptotically
  - We find an asymptotic expansion of every possible bulk solution
  - In particular, the possible *subleading* behaviour of the fields is determined dynamically
- This asymptotic solution can be substituted into S and leads to divergences
- Holographically renormalize by adding a boundary counterterm action S_{ct} to S
- The renormalized action $S_{\text{ren}} = S + S_{\text{ct}}$ is finite on-shell
- Find the full solution to the equations of motion with sources $\phi_{(0)}$ (perhaps perturbatively)
- Substitute this solution into S_{ren} which gives $S_{\text{on-shell,ren}}[\phi_{(0)}]$
- Use $Z_{\text{CFT}}[\phi_{(0)}] \sim \exp(-S_{\text{on-shell,ren}}[\phi_{(0)}])$ to compute correlation functions

Skenderis (2002)

Asymptotic analysis

We will now work in Fefferman-Graham coordinates. The metric takes the form:

$$G_{\mu\nu}dx^\mu dx^\nu = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}g_{ij}(x,\rho)dx^i dx^j$$

For an asymptotically AdS spacetime, the conformal boundary is at $\rho = 0$ and:

$$g_{ij}(x,\rho) = g_{(0)ij}(x) + \dots$$

where $g_{(0)ij}$ is nondegenerate

For TMG, the equations of motion for $\mu = 1$ give the most general asymptotic solution:

$$g_{ij} = b_{(0)ij}\log(\rho) + g_{(0)ij} + b_{(2)ij}\rho\log(\rho) + \rho g_{(2)ij} + \dots$$

Following the usual AdS/CFT dictionary, we interpret the leading terms as CFT sources

$$g_{(0)ij} \leftrightarrow T_{ij} \qquad\qquad b_{(0)ij} \leftrightarrow t_{ij}$$

The subleading terms $b_{(2)ij}$ and $g_{(2)ij}$ are *partially* determined by the asymptotic analysis and these terms enter in the one-point functions

Holographic renormalization

We substitute the asymptotic expansion in the action for TMG and find divergences (e.g. a volume divergence)

We need to *holographically renormalize* by adding a boundary counterterm action S_{ct}

However, the most general asymptotic solution is:

$$g_{ij} = b_{(0)ij} \log(\rho) + g_{(0)ij} + b_{(2)ij} \rho \log(\rho) + \rho g_{(2)ij} + \dots$$

For nonzero $b_{(0)ij}$, this is no longer asymptotically AdS

- we cannot do an all-orders renormalization
- we treat $b_{(0)ij}$ as infinitesimal and renormalize perturbatively
- in the dual theory $b_{(0)ij}$ sources a (marginally) irrelevant operator and the boundary theory with finite $b_{(0)ij}$ *is* only no longer completely renormalizable

We did a *linearized* analysis at the level of the equation of motion
$\longrightarrow$ This is equivalent to a *quadratic* analysis at the level of the action
so we computed S_{ren} to second order in $b_{(0)ij}$
$\longrightarrow$ This is sufficient to compute two-point functions

Full linearized solutions

We begin with an AdS_3 background

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij} dx^i dx^j \qquad g_{ij} dx^i dx^j = dz d\bar{z}$$

and study perturbations:

$$g_{ij} \to g_{ij} + h_{ij}$$

At the linearized level we find:

$$\begin{aligned}
h_{z\bar{z}} &= \qquad\qquad h_{(0)z\bar{z}} - \frac{1}{2}\rho\log(\rho)\partial^2 b_{(0)\bar{z}\bar{z}} + \rho h_{(2)z\bar{z}}[h_{(0)}, b_{(0)}] + \dots \\
h_{\bar{z}\bar{z}} &= b_{(0)\bar{z}\bar{z}}\log(\rho) + h_{(0)\bar{z}\bar{z}} - \frac{1}{2}\rho\log(\rho)\bar{\partial}\partial b_{(0)\bar{z}\bar{z}} + \rho h_{(2)\bar{z}\bar{z}} + \dots \\
h_{zz} &= \qquad\qquad h_{(0)zz} + \frac{1}{2}\rho\log(\rho) b_{(2)\bar{z}\bar{z}} + \rho h_{(2)zz} + \dots
\end{aligned}$$

with $h_{(2)z\bar{z}}[h_{(0)}, b_{(0)}] = -\frac{1}{2}\partial^2 h_{(0)\bar{z}\bar{z}} - \frac{1}{2}\bar{\partial}^2 h_{(0)zz} + \bar{\partial}\partial h_{(0)z\bar{z}} - \frac{1}{2}\partial^2 b_{(0)\bar{z}\bar{z}}$.

We search for regular solutions as $\rho \to \infty$ which constrains the subleading terms to be:

$$\begin{aligned}
h_{(2)\bar{z}\bar{z}} &= \frac{\bar{\partial}}{\partial} h_{(2)z\bar{z}} + \frac{4\gamma - 3}{2}\bar{\partial}\partial b_{(0)\bar{z}\bar{z}} \\
b_{(2)\bar{z}\bar{z}} &= \frac{1}{2}\frac{\partial^3}{\bar{\partial}} b_{(0)\bar{z}\bar{z}} \\
h_{(2)zz} &= \left(2\gamma - 1 + \log(-\partial\bar{\partial})\right)\frac{\partial^3}{\bar{\partial}} b_{(0)\bar{z}\bar{z}} + \frac{\partial}{\bar{\partial}} h_{(2)z\bar{z}}
\end{aligned}$$

Correlation functions

After holographic renormalization we find the one-point functions from:

$$\langle T_{ij}\rangle = 4\pi \frac{\delta S_{\text{TMG, on-shell, ren}}}{\delta h^{ij}_{(0)}} \qquad \langle t_{zz}\rangle = -4\pi \frac{\delta S_{\text{TMG, on-shell, ren}}}{\delta b^{zz}_{(0)}}$$

We for example find:

$$\langle T_{zz}\rangle = -\frac{1}{2G_N} b_{(2)zz} + \text{local} = -\frac{1}{4G_N}\left(\frac{\partial^3}{\bar\partial} b_{(0)\bar z\bar z} + \text{local}\right)$$

which is a linear and nonlocal function of the sources

Differentiating once more with respect to the sources we obtain the two-point functions:

$$\langle t(z,\bar z)t(0)\rangle = \frac{3}{G_N}\frac{\log(m^2|z|^2)}{z^4} \qquad \langle t(z,\bar z)T(0)\rangle = \frac{-3/G_N}{2z^4}$$

$$\langle T(z,\bar z)T(0)\rangle = 0 \qquad \langle \bar T(z,\bar z)\bar T(0)\rangle = \frac{3/G_N}{2\bar z^4}$$

where $t = t_{zz}$, $T = T_{zz}$ and $\bar T = T_{\bar z\bar z}$

We read off that:

$$c_L = 0 \qquad c_R = 3/G_N$$

and we find *logarithmic* correlation functions

Logarithmic CFT

We indeed find the structure of a logarithmic CFT (Gurarie 1993) for topologically massive gravity at $\mu = 1$

Such CFT's have logarithms in correlation functions which are related to an indecomposible representation of the Virasoro algebra

$$L_0 \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} h & 0 \\ 1 & h \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} \qquad L_m \begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0 \qquad (m > 0)$$

One then finds logarithms in correlation functions:

$$\langle \phi(z)\phi(w) \rangle = 0 \qquad \langle \phi(z)\chi(w) \rangle = \frac{1}{z^{2h}} \qquad \langle \chi(z)\chi(w) \rangle = \frac{\log |z|^2}{z^{2h}}$$

A logarithmic CFT is not unitary. Maybe a restriction to the right-moving sector is consistent and results in a unitary theory? Maloney, Song, Strominger (2009)

Logarithmic CFT

It is instructive to compute the same correlation functions in the vicinity of $\mu = 1$

There are still four sources, three for T_{ij} and a fourth for a new operator X
The correlation functions become:

$$\begin{aligned}
\langle \bar{T}(z,\bar{z})\bar{T}(0)\rangle &= \frac{3}{2G_N}\left(1+\frac{1}{\mu}\right)\frac{1}{2\bar{z}^4}, \\
\langle T(z,\bar{z})T(0)\rangle &= \frac{3}{2G_N}\left(1-\frac{1}{\mu}\right)\frac{1}{2z^4}, \\
\langle X(z,\bar{z})X(0)\rangle &= \frac{-1}{8G_N}\,\frac{(\mu-1)(\mu+1)(\mu+2)}{\mu}\,\frac{1}{z^{\mu+3}\bar{z}^{\mu-1}}
\end{aligned}$$

One finds *negative norm states* for $\mu > 1$ and *negative conformal weights* for $\mu < 1$

As $\mu \to 1$ we find that a new operator appears:

$$t = \lim_{\mu\to 1} \frac{-2}{\mu-1}(T+X)$$

which is the logarithmic partner of T. This mimicks a construction in the LCFT literature (Kogan, Nichols 2002)

Charges from the dual field theory

We may define conserved charges in the CFT in the usual way, for example:

$$M = -\oint d\phi T^t_t \qquad J = -\oint d\phi T^t_\phi$$

Our asymptotic analysis was completely general
$\longrightarrow$ these are *finite* charges for all bulk solutions

They are also the correct *gravitational* charges (Papadimitriou, Skenderis 2005)

We in particular find:

$$\langle X|H|X\rangle < 0$$

which is the CFT counterpart of the negative energy found in the bulk

Summary

The AdS/CFT techniques were applied to topologically massive gravity with $\Lambda < 0$

This allows for the computation of correlation functions and finite charges

We found evidence for a logarithmic CFT at $\mu = 1$

Away from $\mu = 1$ we find negative conformal dimensions or negative norm states

Future directions:

- Three-point functions and chirality
- Condensed matter applications
- Adaptation to "new massive gravity" Bergshoeff, Hohm, Townsend (2009)
- ...