Current algebras and higher genus CFT partition functions

Roberto Volpato

Institute for Theoretical Physics – ETH Zurich

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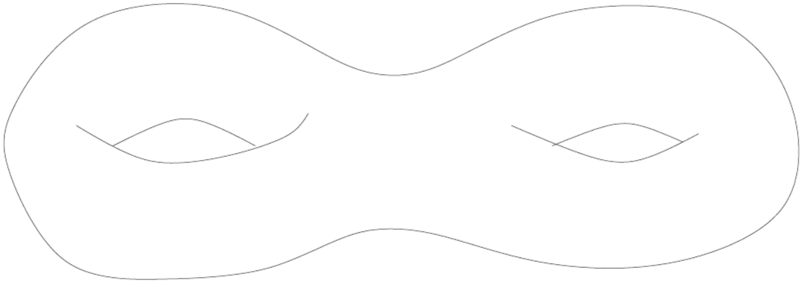
Based on: M. Gaberdiel and R.V., arXiv: 0903.4107 [hep-th]
M. Gaberdiel, C. Keller, R.V., work in progress

2-D CFT

- 2-D Conformal Field Theory on a surface of genus g



- Amplitudes depend on the choice of a complex structure

$$\left\langle \Phi_1(z_1, \bar{z}_1)\Phi_2(z_2, \bar{z}_2)\cdots\Phi_2(z_2, \bar{z}_2)\right\rangle_{\Sigma}$$

Partition functions

- Moduli space $\mathcal{M}_g$

$$\dim_{\mathbb{C}} \mathcal{M}_g = \begin{cases} 0 & \text{for } g = 0 \\ 1 & \text{for } g = 1 \\ 3g - 3 & \text{for } g > 1 \end{cases}$$

- Partition function on Riemann surface Σ

$Z_g(m_i, \bar{m}_i)$ corresponds to $\langle 1 \rangle_\Sigma$

Ex: $Z_{g=1}(\tau, \bar{\tau}) = \mathrm{Tr}_{\mathcal{H}}\left(q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{c}{24}}\right)$ $\qquad q = e^{2\pi i \tau}$

Motivations

- How much information in the partition function?
  - Genus 1 $\longrightarrow$ spectrum of the theory
  - Genus 2,3,… $\longrightarrow$?
  - Can we reconstruct a CFT from partition functions?

[Friedan, Schenker '87]

- Which functions on $\mathcal{M}_g$ are CFT partition functions?
  - Modular invariance, factorisation, … what else?

- Applications to Ads/CFT correspondence
  - 3-d pure quantum gravity, chiral gravity, ...

[Witten '07]
[Li, Song, Strominger '08]

Main results

- The affine Lie algebra of currents in a CFT is uniquely determined by its PFs (and representations are strongly constrained)

M. Gaberdiel and R.V., JHEP 0906:048 (2009)
[arXiv:0903.4107]

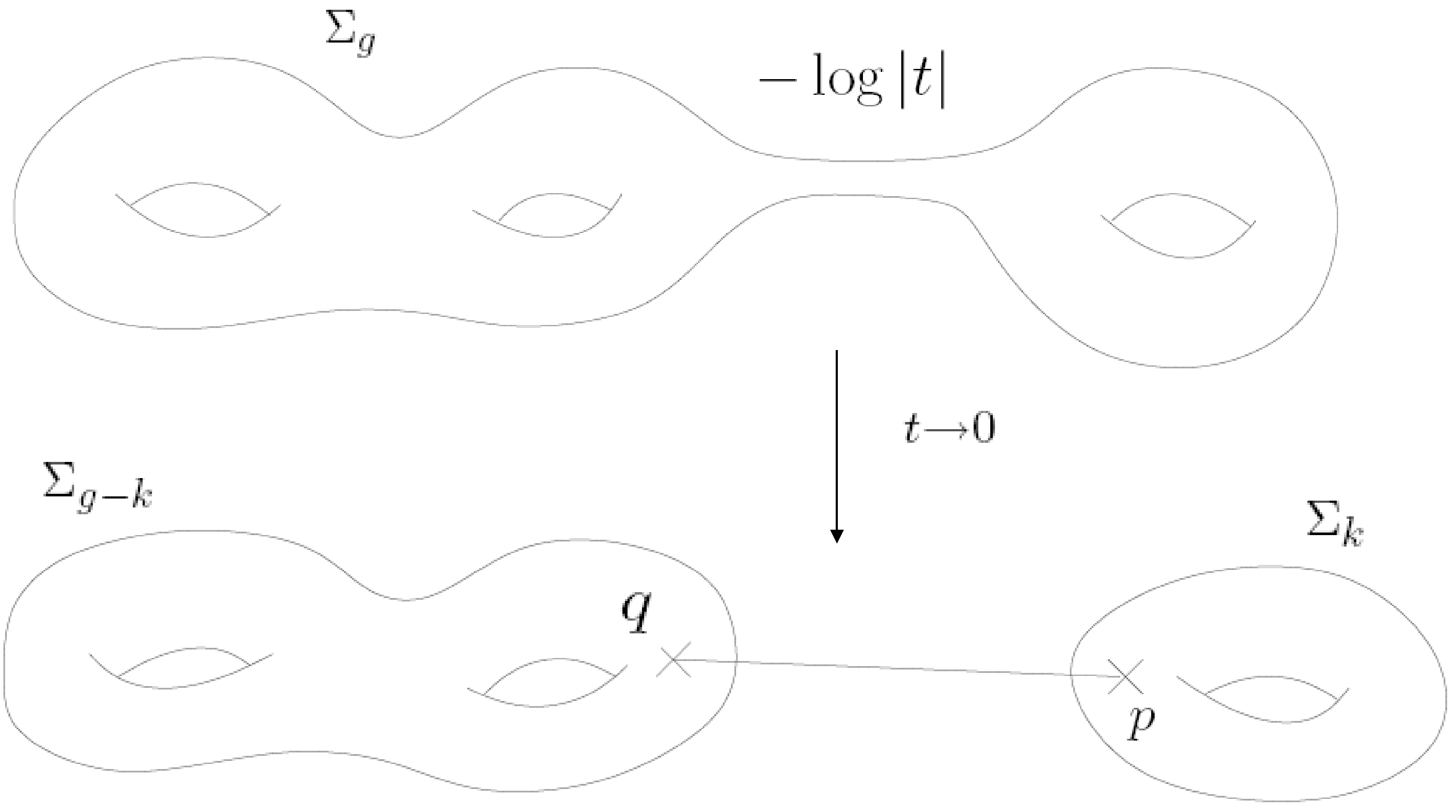
- Constraints for meromorphic unitary theories from genus 2 partition functions

M. Gaberdiel, C. Keller and R.V., work in progress

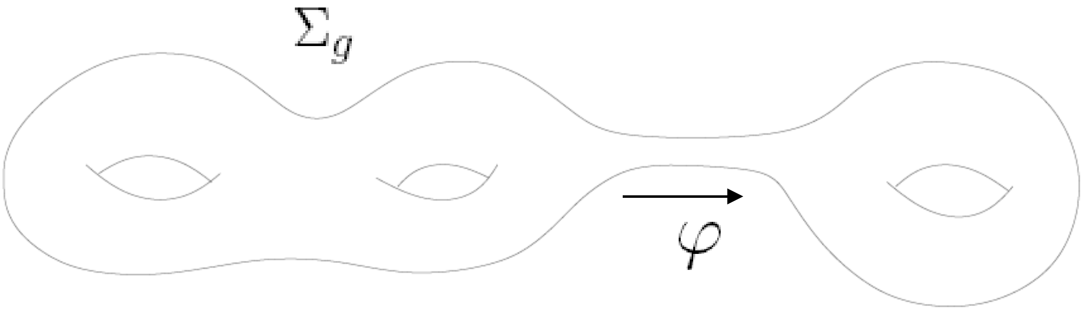
How can we obtain information on a CFT from its partition function?

Factorisation properties under degeneration limits

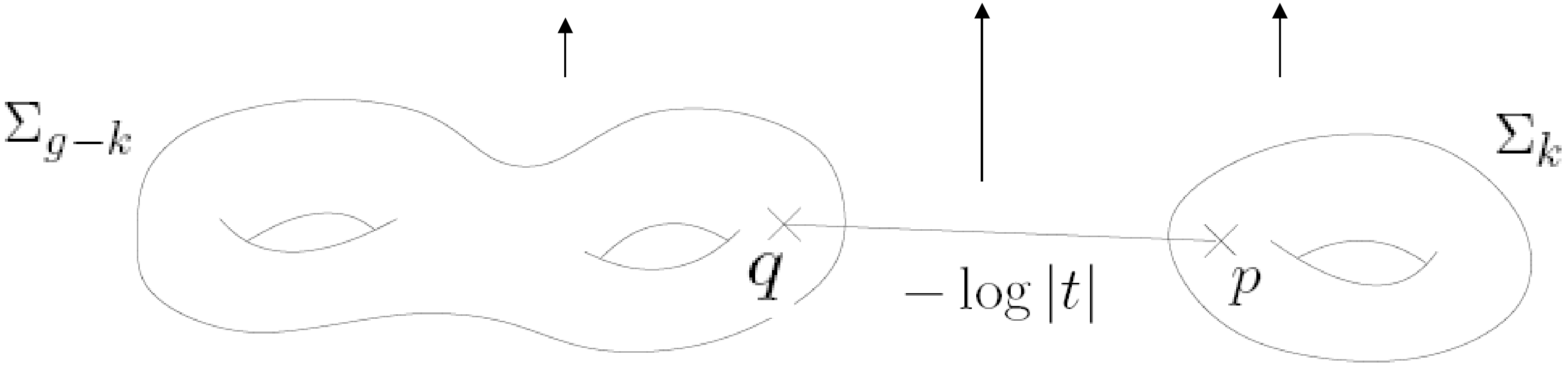
Degeneration limit



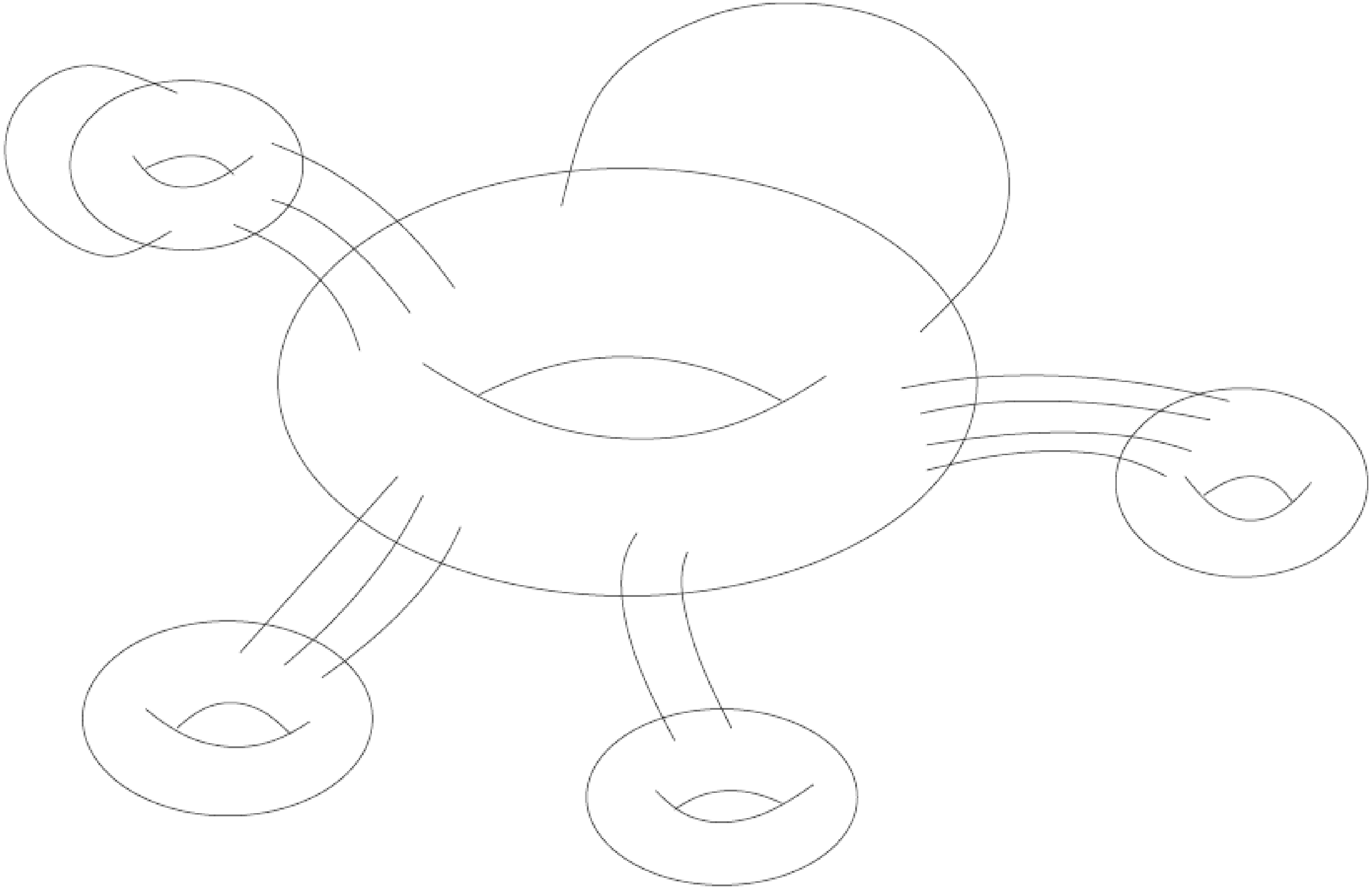
Factorization



$$Z_g \xrightarrow{t\to 0} \sum_{\varphi} \langle \bar{\varphi}(q) \rangle_{\Sigma_{g-k}} \; t^{h_\varphi} \bar{t}^{\tilde{\bar{h}}_{\bar{\varphi}}} \quad \langle \varphi(p) \rangle_{\Sigma_k}$$

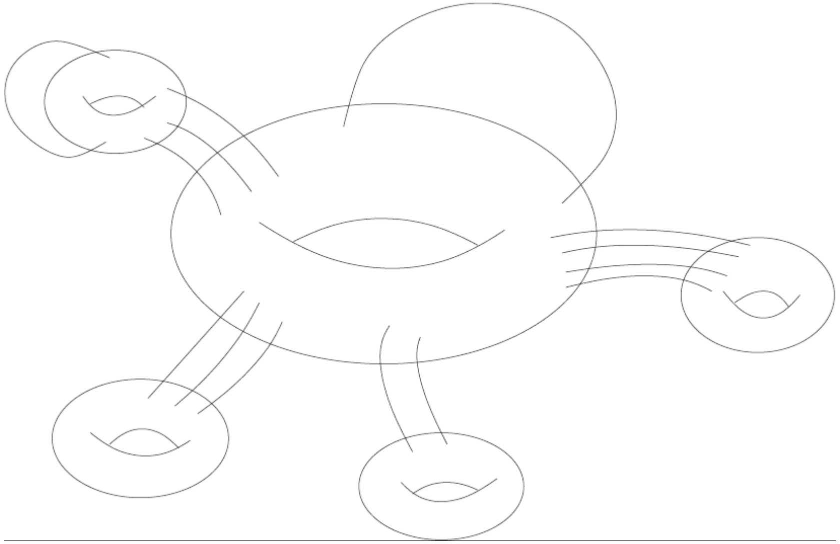


Multiple degenerations…



…2n-point amplitudes

- n parameters
- 2n-point correlators



$$Z_g \longrightarrow$$

$$\sum_{\varphi_1,\dots,\varphi_n} \langle \varphi_1(p_1)\bar{\varphi}_1(q_1)\cdots\varphi_n(p_n)\bar{\varphi}_n(q_n)\rangle\, t_1^{h_1}\bar{t}_1^{\tilde{h}_1}\cdots t_n^{h_n}\bar{t}_n^{\tilde{h}_n}$$

- Can we obtain directly all correlators?

NO

- Can we reconstruct the whole CFT?

Open problem

[Friedan, Schenker ’87]

- Can we reconstruct the algebra of currents?

YES

[M. Gaberdiel and R.V. ’09]

Kac-Moody affine algebras

- Currents (conformal weight 1) $J^a(z) \quad a = 1, \ldots, N$
- Mode expansion (sphere) $J^a(z) = \sum_n z^{-n-1} J^a_n$
- Kac-Moody affine algebra

$$[J^a_m, J^b_n] = m\, \hat{k} \delta^{ab} \delta_{n+m,0} + i\, f^{ab}{}_c J^c_{n+m}$$

Level

Structure constants

Assumptions

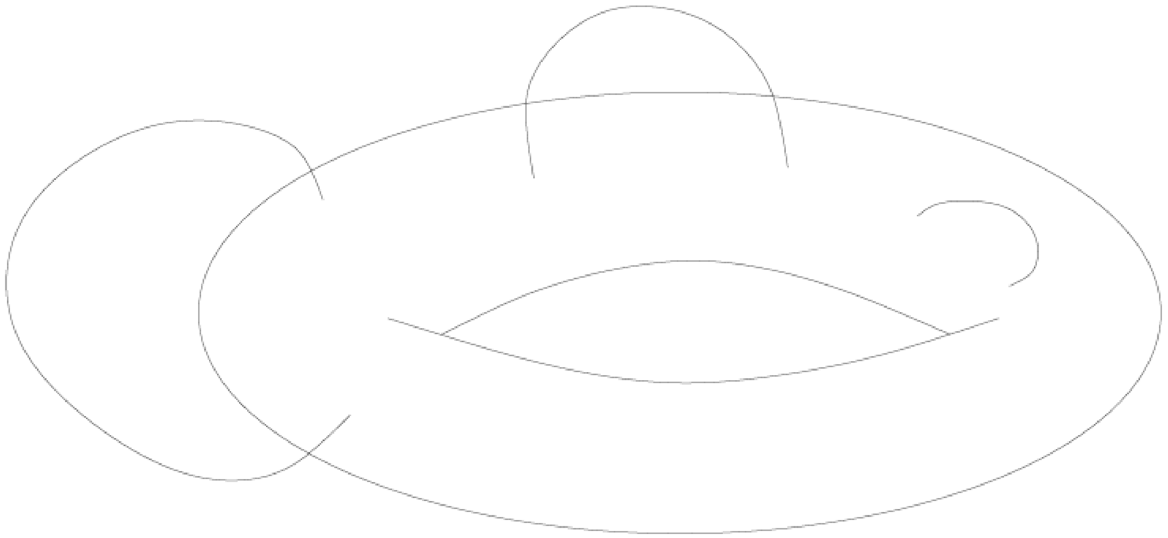
- We only consider unitary bosonic meromorphic self-dual CFTs (but results hold more generally)

$$Z_g(m_i, \bar{m}_i) = Z_g(m_i) \qquad Z_1(-1/\tau) = Z_1(\tau)$$

- Lattice theories: CFT of free chiral bosons on even unimodular lattice
  - Example: 16 chiral bosons in heterotic strings ($E_8 \times E_8$ and $Spin(32)/\mathbb{Z}_2$)

General procedure

- Consider a genus g partition function Z_g
- Take the degeneration limit to a torus



General procedure

- Consider a genus g partition function Z_g
- Take the degeneration limit to a torus
- Consider the $t_1 \cdots t_n$ term in the power expansion of Z_g $(n \leq g-1)$

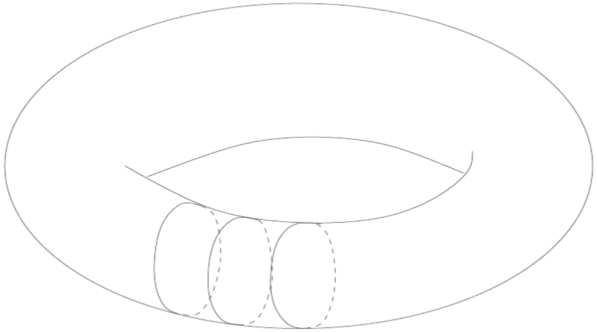
$$q = e^{2\pi i \tau}$$

$$Z_g \longrightarrow \mathrm{Tr}\left(q^{L_0}\right) + \ldots$$

$$+ \ t_1 \cdots t_n \ \mathrm{Tr}\left(q^{L_0} J^{a_1}(p_1) J^{a_1}(q_1) \cdots J^{a_n}(p_n) J^{a_n}(q_n)\right)$$

General procedure

- Consider a genus g partition function Z_g
- Take the degeneration limit to a torus
- Consider the $t_1 \cdots t_n$ term in the power expansion of Z_g $(n \leq g-1)$
- Integrate the coefficient over non-trivial cycle



$$\mathrm{Tr}\left(q^{L_0}\, J_0^{a_1} J_0^{a_1} \cdots J_0^{a_n} J_0^{a_n}\right)$$

General procedure

- Consider a genus g partition function Z_g
- Take the degeneration limit to a torus
- Consider the $t_1 \cdots t_n$ term in the power expansion of Z_g $(n \leq g-1)$
- Integrate the coefficient over non-trivial cycle
- Expand in powers of q

$$\sum_h q^h \,\mathrm{Tr}_{\mathcal{H}_h}(J_0^{a_1} J_0^{a_1} \cdots J_0^{a_n} J_0^{a_n})$$

Space of conf. weight h

General procedure

- Consider a genus g partition function Z_g
- Take the degeneration limit to a torus
- Consider the $t_1 \cdots t_n$ term in the power expansion of Z_g $(n \leq g-1)$
- Integrate the coefficient over non-trivial cycle
- Expand in powers of q
- We obtain Lie algebra invariants (Casimirs)
- The degree of Casimir depends on g

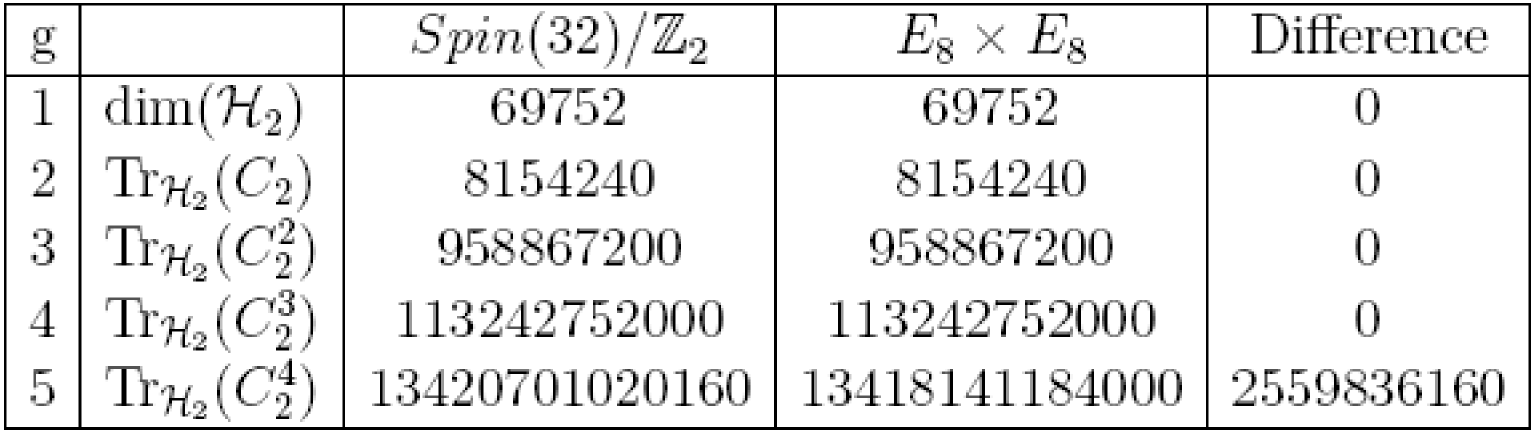
Example: $E_8\times E_8$ and $Spin(32)/\mathbb{Z}_2$

This can be used to prove that two partition functions are different

- Same PFs at $g=1,2,3,4$ but not 5

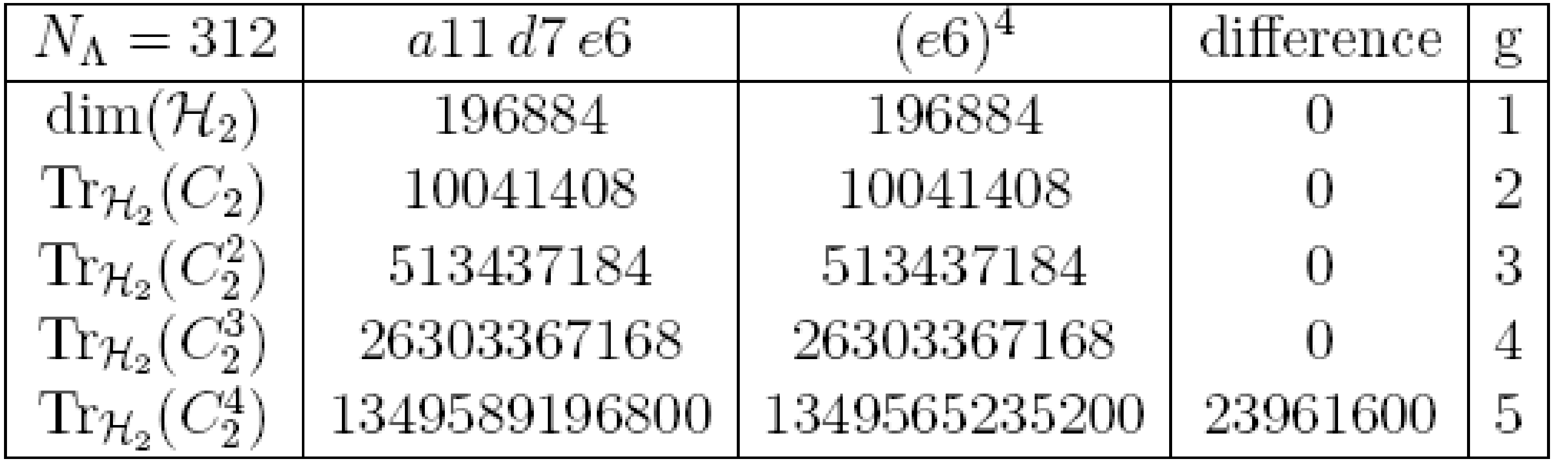
[Grushevsky, Salvati Manni '08]

- Consider $\mathrm{Tr}_{\mathcal{H}_2}(C_2^n)=\mathrm{Tr}_{\mathcal{H}_2}\left(\left(\sum_a J_0^a J_0^a\right)^n\right)$

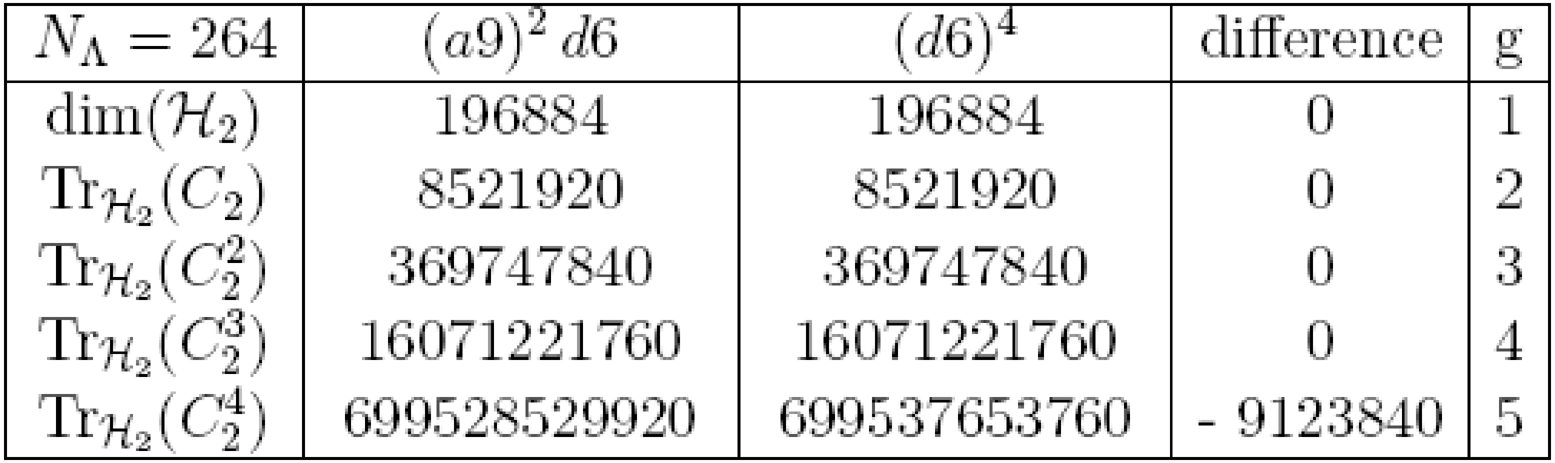


g		$Spin(32)/\mathbb{Z}_2$	$E_8\times E_8$	Difference
1	$\dim(\mathcal{H}_2)$	69752	69752	0
2	$\mathrm{Tr}_{\mathcal{H}_2}(C_2)$	8154240	8154240	0
3	$\mathrm{Tr}_{\mathcal{H}_2}(C_2^2)$	958867200	958867200	0
4	$\mathrm{Tr}_{\mathcal{H}_2}(C_2^3)$	113242752000	113242752000	0
5	$\mathrm{Tr}_{\mathcal{H}_2}(C_2^4)$	13420701020160	13418141184000	2559836160

More examples: c=24



$N_\Lambda = 312$	$a11\, d7\, e6$	$(e6)^4$	difference	g
$\dim(\mathcal{H}_2)$	196884	196884	0	1
$\mathrm{Tr}_{\mathcal{H}_2}(C_2)$	10041408	10041408	0	2
$\mathrm{Tr}_{\mathcal{H}_2}(C_2^2)$	513437184	513437184	0	3
$\mathrm{Tr}_{\mathcal{H}_2}(C_2^3)$	26303367168	26303367168	0	4
$\mathrm{Tr}_{\mathcal{H}_2}(C_2^4)$	1349589196800	1349565235200	23961600	5



$N_\Lambda = 264$	$(a9)^2\, d6$	$(d6)^4$	difference	g
$\dim(\mathcal{H}_2)$	196884	196884	0	1
$\mathrm{Tr}_{\mathcal{H}_2}(C_2)$	8521920	8521920	0	2
$\mathrm{Tr}_{\mathcal{H}_2}(C_2^2)$	369747840	369747840	0	3
$\mathrm{Tr}_{\mathcal{H}_2}(C_2^3)$	16071221760	16071221760	0	4
$\mathrm{Tr}_{\mathcal{H}_2}(C_2^4)$	699528529920	699537653760	- 9123840	5

Systematic procedure

- Different factorizations → different Casimirs

$$\mathrm{Tr}_{\mathcal{H}_1}(J_0^{a_1} J_0^{a_2})\,\mathrm{Tr}_{\mathcal{H}_1}(J_0^{b_1} J_0^{b_2} J_0^{b_3} J_0^{b_4})\,\mathrm{Tr}_{\mathcal{H}_h}(J_0^{a_1} J_0^{a_2} J_0^{b_1} J_0^{b_2} J_0^{b_3} J_0^{b_4})$$

- In particular, all independent Casimirs for adjoint representation $\mathcal{H}_1$ can be obtained

↓ Lie algebra machinery…

The affine symmetry is uniquely determined by the partition functions

Distinguishing reps?

- Example: overall spin flip in $Spin(32)/\mathbb{Z}_2$ cannot be detected by PFs

- We cannot generate the whole algebra of Casimir invariants from PFs

- Evidence that representation content can be distinguished by PFs
  (up to Lie algebra outer automorphisms)

PARTITION FUNCTIONS AND MODULAR FORMS

PFs and modular forms

Riemann surface of generus g

Riemann period matrix Ω

$$\Omega_{ij} = \Omega_{ji}$$

$$\mathrm{Im}\,\Omega > 0$$

Genus g partition function for MCFT

Modular form $F_g(\Omega)$ of weight c/2

Example: $Z_{g=1}(\tau) = F_1(\tau)/\Delta^{c/24}(\tau)$

PFs and modular forms

Genus g PF for MCFT (centr. charge c)

- Modular properties $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(2g, \mathbb{Z})$

$$F_g\left((A\Omega+B)(C\Omega+D)^{-1}\right) = \det\left(C\Omega + D\right)^{c/2} F_g(\Omega)$$

- Factorization properties

$$F_g\begin{pmatrix} \Omega_k & 0 \\ 0 & \Omega_{g-k} \end{pmatrix} = F_k(\Omega_k)F_{g-k}(\Omega_{g-k})$$

PFs and modular forms

$F_g(\Omega)$ must satisfy some basic constraints (factorisation, modular properties)

↓

Finite number of parameters determine $F_g(\Omega)$

- Ex.: for $g = 1, 2, 3, 4$
  - $c \leq 16$: no free parameters
  - $c = 24$: 1 parameter (number of currents N)

PFs and modular forms

- Consequence: all invariants from Z_g depend on these parameters
- Example: $c = 24$ with N currents

$$\mathrm{Tr}_{\mathcal{H}_2}(J_0^a J_0^a) = -2N^2 + 32808N$$

$$\mathrm{Tr}_{\mathcal{H}_2}((J_0^a J_0^a)^2) = -\frac{23N^3}{36} + \frac{16421N^2}{3} + 40N$$

$$\mathrm{Tr}_{\mathcal{H}_2}(W_0^i W_0^i) = 230N^2 - 4160N + 909665088$$

Conclusions and to do

- From partition functions we can reconstruct the affine symmetry of a CFT
- Do PF’s determine representations?

- Partition functions of meromorphic CFT depend on finite number of parameters
- New consistency conditions on PFs?