Quantum Symmetries And Marginal Deformations

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$\mathcal{N}=4$ Super–Yang–Mills

- $\mathcal{N}=4$ SYM has played a fundamental role in the study of gauge and string theory
- It is the unique four–dimensional gauge theory with maximal global supersymmetry (16 supercharges)
- Extremely interesting properties:
  - Perturbative Finiteness [Ferrara,Zumino],[Grisaru et al. '80],[West,Sohnius '81] [Stelle '81],[Brink et al. '83],[Mandelstam '83]
  - Nonperturbative Finiteness [ADS '84,NSVZ '86,Seiberg$\rightarrow$ Holomorphicity]
    $\Rightarrow$ 4d Superconformal Field Theory
  - AdS/CFT Correspondence [Maldacena '97]
  - Planar Integrability [Minahan,Zarembo '02], many others
  - Planar Amplitudes [Anastasiou et al. '04], [Bern et al. '05], [Alday,Maldacena '07]
- But is it the unique theory with these features?
- Are there theories which share only *some* of these features?
- How does the knowledge accumulated for $\mathcal{N}=4$ SYM help in understanding more realistic theories?

Marginal Deformations of $\mathcal{N} = 4$ SYM

- Look for theories as close as possible to $\mathcal{N} = 4$ SYM
- Preserve conformal invariance $\Rightarrow$ Marginal Deformations
- Focus on superpotential deformations $\Rightarrow$ $\mathcal{N} = 1$ SUSY
- Write the $\mathcal{N} = 4$ SYM action in $\mathcal{N} = 1$ superspace

$$\mathcal{L} = \int d^4\theta \mathrm{Tr} e^{gV} \overline{\Phi}_i e^{-gV} \Phi^i + \left(\int d^2\theta \mathcal{W} + \int d^2\bar{\theta} \overline{\mathcal{W}} \right) + \cdots$$

- Chiral Superfields $\Phi^i = \phi^i + \theta^\alpha \psi^i_\alpha + \theta^2 F^i, \; i = 1, 2, 3$
- $\mathcal{N} = 4$ superpotential:

$$\mathcal{W} = g \mathrm{Tr} \Phi^1 [\Phi^2, \Phi^3] = \frac{g}{3} \epsilon_{ijk} \mathrm{Tr} \Phi^i \Phi^j \Phi^k$$

- Most general classically marginal deformation:

$$\delta \mathcal{W} = h_{ijk} \mathrm{Tr} \Phi^i \Phi^j \Phi^k,$$

where h_{ijk} is a symmetric tensor

Exactly Marginal Deformations

- Not all of these deformations are *exactly marginal*
- Perturbative approaches [Parkes, West '84],[Jones,Mezincescu '84]
- Leigh and Strassler ('95) provided a non–perturbative proof using the NSVZ β function
- The *Leigh–Strassler* theories are defined by:

$$\mathcal{W}_{LS} = \kappa \mathrm{Tr} \left(\Phi^1 [\Phi^2, \Phi^3]_q + \frac{h}{3} \left((\Phi^1)^3 + (\Phi^2)^3 + (\Phi^3)^3 \right) \right)$$

- q–commutator $[X, Y]_q = XY - qYX$
- Finite if $f(g, \kappa, q, h) = 0$, where f unknown in general
- 1–loop finiteness condition

$$2g^2 = \kappa\bar{\kappa} \left[\frac{2}{N^2}(1+q)(1+\bar{q}) + \left(1 - \frac{4}{N^2}\right)(1 + q\bar{q} + h\bar{h}) \right]$$

- Recover $\mathcal{N} = 4$ SYM for $q = 1$, $h = 0$
- An interesting case: $q = e^{i\beta}, h = 0$ ("Real β deformation")

$\mathcal{N} = 4$ SYM vs. Leigh–Strassler?

- How do the LS theories compare with $\mathcal{N} = 4$ SYM?

	$\mathcal{N} = 4$ SYM	Leigh–Strassler
Conformally Invariant	$\surd$	$\surd$(*)
AdS/CFT dual	$\surd$	real β [Lunin,Maldacena'05]
Planar Integrability	$\surd$	basically real β (†)
Planar Amplitudes	$\surd$	real β [Khoze'05]

(*) Recent controversy over higher–loop finiteness [Elmetti et al. '06,'07], [Rossi et al. '05,'06]

(†) Berenstein&Cherkis ('04) showed mismatch of LS deformation with integrable deformation of the spin chain for complex β. However one–loop integrability persists in a particular sector for complex β [Månsson '07]. A few other integrable choices are known.

- What makes the real β deformation so special?
- Take a closer look at the symmetries
- Work at the level of the classical lagrangian

Symmetries: $\mathcal{N} = 4$ SYM

- 4d Superconformal group: $\mathrm{PSU}(2,2|4)$
- Focus on the R–symmetry subgroup $\mathrm{SU}(4) \sim \mathrm{SO}(6)$
- In $\mathcal{N} = 1$ superspace notation, the $\mathcal{N} = 4$ theory has manifest $\mathrm{SU}(3) \times \mathrm{U}(1)_R$ symmetry

$$\mathcal{W} = g\mathrm{Tr}\Phi^1[\Phi^2, \Phi^3] = \frac{g}{3}\epsilon_{ijk}\mathrm{Tr}\Phi^i\Phi^j\Phi^k$$

- ϵ_{ijk} is the invariant tensor of $\mathrm{SU}(3)$

$$\epsilon_{ijk}U^i{}_l U^j{}_m U^k{}_n = (\mathrm{det}U)\epsilon_{lmn} = \epsilon_{lmn}$$

- Transforming $\Phi^i \to U^i_l \Phi^l$ leaves the superpotential invariant
- Note that $\mathrm{SL}(3)$ would be enough for the superpotential, $\mathrm{SU}(3)$ comes from the kinetic term $\sim \overline{\Phi}_i\Phi^i$

Symmetries: Leigh–Strassler

- The generic LS deformation breaks $\mathrm{SU}(3)$ to a discrete subgroup

$$\mathcal{W}_{LS} = \kappa \mathrm{Tr} \left(\Phi^1 [\Phi^2, \Phi^3]_q + \frac{h}{3} \left((\Phi^1)^3 + (\Phi^2)^3 + (\Phi^3)^3 \right) \right)$$

- This superpotential has the following $\mathbb{Z}_3$ symmetries:

$$\mathbb{Z}_3^A : \ \Phi^1 \to \Phi^2 \quad , \quad \Phi^2 \to \Phi^3 \quad , \quad \Phi^3 \to \Phi^1$$

$$\mathbb{Z}_3^B : \ \Phi^1 \to \omega \Phi^1 \quad , \quad \Phi^2 \to \omega^2 \Phi^2 \quad , \quad \Phi^3 \to \Phi^3 \quad (\omega^3 = 1)$$

- Together with a third $\mathbb{Z}_3$ within $\mathrm{U}(1)_R$ ($\Phi^i \to \omega \Phi^i$), they form a trihedral group known as Δ_{27} [Aharony et al. '02]
- For real β the symmetry group is enhanced to $\mathrm{U}(1)^3$
- Is this all?

More symmetry?

- Let us naively rewrite $\mathcal{W}_{LS}$ as

$$W_{LS} = \frac{1}{3} E_{ijk} \mathrm{Tr} \Phi^i \Phi^j \Phi^k$$

where

$$E_{123} = E_{231} = E_{312} = \kappa \ ,$$
$$E_{321} = E_{213} = E_{132} = -\kappa q \ ,$$
$$E_{111} = E_{222} = E_{333} = \kappa h$$

- Similarly $F^{ijk} = \overline{E}_{ijk}$ defines $\overline{\mathcal{W}} = \frac{1}{3} F^{ijk} \mathrm{Tr} \overline{\Phi}_i \overline{\Phi}_j \overline{\Phi}_k$
- Would like to find some t^i_j such that $\Phi^i \to \mathrm{t}^i_l \Phi^l$ is a symmetry, i.e.

$$E_{ijk} \mathrm{t}^i_l \mathrm{t}^j_m \mathrm{t}^k_n = E_{lmn}$$

- Clearly there exists no Lie group with this property...
$\Rightarrow$ Quantum Groups

An Example: The Manin Plane

- A simpler setting that illustrates the main ideas
- $x^1, x^2 \in V$, where V is a noncommutative space

$$x^1 x^2 = \frac{1}{q} x^2 x^1$$

- Coordinates of a 2–dimensional quantum plane
- These commutation relations can be obtained from a matrix $R : V \otimes V \to V \otimes V$

$$R = q^{-\frac{1}{2}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} ,$$

via the relation ${R^i}_k{}^j{}_l x^k x^l = q^{\frac{1}{2}} x^j x^i$ (or $R_{12} x_1 x_2 = q^{\frac{1}{2}} x_2 x_1$)

- Think of R as acting on basis $\{|11\rangle, |12\rangle, |21\rangle, |22\rangle\}$ (e.g. ${R^1}_2{}^2{}_1 = q^{\frac{1}{2}} - q^{-\frac{3}{2}}$)

Quantum Plane Symmetries

- The function $f(x^1, x^2) = x^1 x^2 - q^{-1} x^2 x^1$ is invariant under

$$x^i \to \mathrm{t}^i_l x^l$$

  if the matrix t^i_j satisfies the FRT, or RTT relations:

$$R^{i\,k}_{\,a\,b} t^a_j t^b_l = t^k_b t^i_a R^{a\,b}_{\,j\,l} \qquad (R_{12} t_1 t_2 = t_2 t_1 R_{12})$$

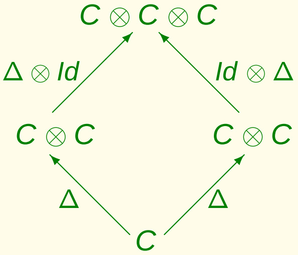
- Writing $t = \begin{pmatrix} \mathrm{t}^1_1 & \mathrm{t}^1_2 \\ \mathrm{t}^2_1 & \mathrm{t}^2_2 \end{pmatrix}$, we find

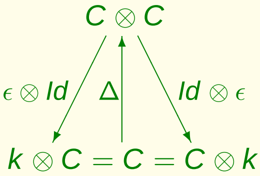
$$\mathrm{t}^1_1 \mathrm{t}^1_2 = q^{-1} \mathrm{t}^1_2 \mathrm{t}^1_1 \,,\; \mathrm{t}^1_1 \mathrm{t}^2_1 = q^{-1} \mathrm{t}^2_1 \mathrm{t}^1_1 \,,$$
$$\mathrm{t}^1_2 \mathrm{t}^2_2 = q^{-1} \mathrm{t}^2_2 \mathrm{t}^1_2 \,,\; \mathrm{t}^2_1 \mathrm{t}^2_2 = q^{-1} \mathrm{t}^2_2 \mathrm{t}^2_1 \,,$$
$$\mathrm{t}^1_2 \mathrm{t}^2_1 = \mathrm{t}^2_1 \mathrm{t}^1_2 \,,\; \mathrm{t}^1_1 \mathrm{t}^2_2 - \mathrm{t}^2_2 \mathrm{t}^1_1 = (q^{-1} - q) \mathrm{t}^1_2 \mathrm{t}^2_1 \,,$$

- The elements of t are noncommutative!
- These commutation relations define a *quantum group*

Quantum Groups

- What are quantum groups? Some definitions
- Recall an *algebra* $(\mathcal{C}, +, \cdot, \eta; k)$ is a vector space together with a product $\cdot : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ and a unit map $\eta : k \to \mathcal{C}$
- A *coalgebra* $(\mathcal{C}, +, \Delta, \epsilon; k)$ is instead equipped with a coproduct $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ and a counit $\epsilon : \mathcal{C} \to k$





- A *bialgebra* $(H, +, \cdot, \eta, \Delta, \epsilon; k)$ is both an algebra and a coalgebra in a compatible way

Hopf Algebras

- A Hopf Algebra is a bialgebra equipped with an antipode $S \ : \mathcal{C} \rightarrow \mathcal{C}$

$$\cdot\,(S \otimes \mathrm{id}) \circ \Delta = \cdot\,(\mathrm{id} \otimes S) \circ \Delta = \eta \circ \epsilon\ .$$

- We are working in the Quantum Matrix Algebra picture, where the coproduct and counit are

$$\Delta g^m_n = \sum_k g^m_k \otimes g^k_n, \qquad \epsilon g^m_n = \delta^m_n$$

  and the antipode s satisfies : $t^i_k s^k_j = \delta^i_j = s^i_k t^k_j$

- In this picture, it is the *product* whose noncommutativity is controlled by the matrix R through the RTT relations
- For $R = I$ we are left with a Lie Algebra
- Dual picture $\Rightarrow$ Universal Enveloping Algebra

Quasitriangular Hopf Algebras

- In a *quasitriangular* Hopf algebra, the matrix R controlling noncommutativity satisfies the Quantum Yang–Baxter Equation (QYBE) (but note without spectral parameter):

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

$$\left(\text{in index notation:} \quad R^{i}{}_{s}{}^{j}{}_{r}R^{s}{}_{l}{}^{k}{}_{p}R^{r}{}_{m}{}^{p}{}_{n} = R^{j}{}_{s}{}^{k}{}_{p}R^{i}{}_{r}{}^{p}{}_{n}R^{r}{}_{l}{}^{s}{}_{m}\right)$$

- Among other things, this condition guarantees that the resulting algebra is not too trivial
- We call a quasitriangular Hopf algebra a *quantum group*
- The matrix R defining the Manin plane does satisfy QYBE!
- It corresponds to the quantum group $\mathrm{SU}_q(2)$

Why $\mathrm{SU}_q(2)$ and not $\mathrm{GL}_q(2)$?

- We have constructed the quantum group $\mathrm{SU}_q(2)$ as the invariance group of the Manin plane
- The quantum determinant $\mathbb{D} := \mathrm{t}^1{}_1\mathrm{t}^2{}_2 - q^{-1}\mathrm{t}^2{}_1\mathrm{t}^1{}_2$ is *central* and we can set it equal to one $\Longrightarrow \mathrm{SL}_q(2)$
- There is an analogous construction for the coplane $u_1u_2 = qu_2u_1$ ($u_i \in V^*$) defined by

$$q^{\frac{1}{2}}u_au_b = u_ju_iR^{ji}{}_{ba}$$

- Compatibility of the plane and coplane imposes a reality condition on the matrix R:

$$\overline{R^{i\ j}{}_{k\ l}} = R^{l\ k}{}_{j\ i} \qquad (\hat{R} = PR \ \text{ hermitian})$$

- This lets us define ${\mathrm{t}^i{}_j}^* = \mathrm{s}^i{}_j \Longrightarrow \mathrm{SU}_q(2)$

Three–Dimensional Quantum Planes

- Apply these ideas to the Leigh–Strassler theories!
- Note the F–term conditions: [Berenstein et al. '00]

$$\phi^1\phi^2 = q\phi^2\phi^1 - h(\phi^3)^2$$
$$\phi^2\phi^3 = q\phi^3\phi^2 - h(\phi^1)^2$$
$$\phi^3\phi^1 = q\phi^1\phi^3 - h(\phi^2)^2$$

(non–commutative moduli space)

- The three scalars will play the role of the quantum plane coordinates: $\Phi^i \to x^i$
- Can think of the $\mathbb{C}^3$ tranverse to the $D3$–branes becoming noncommutative
- We need to examine the symmetries of three–dimensional quantum planes

Quantum deformations of $\mathrm{GL}(3)$

- Ewen & Ogievetsky ('94) classified quantum deformations of $\mathrm{GL}(3)$, and the corresponding quantum planes
- Starting point were the q–epsilon tensors E_{ijk} and their duals F^{ijk}
- Given the following conditions

$$\delta^i_j = \frac{1}{2} E_{jmn} F^{mni} \quad \text{and} \quad E_{ajm} F^{mib} E_{ebk} F^{kcj} = \delta^c_a \delta^i_e + \delta^i_a \delta^c_e$$

they show that

$$\hat{R}^{ij}{}_{kl} = \delta^i_k \delta^j_l - E_{kln} F^{nij}$$

satisfies QYBE and defines a quantum plane through

$$\hat{R}_{12} x_1 x_2 = x_1 x_2$$

LS as a quantum symmetry deformation

- Can the E&O approach be applied to LS?
- The desired q–epsilon tensors are:
  $E_{123} = \kappa$, $E_{132} = -\kappa q$, $E_{111} = \kappa h$ (+cyclic), $F^{ijk} = \overline{E}_{ijk}$
- Imposing the first E&O condition ($\delta^i_j = \frac{1}{2} E_{jmn} F^{mni}$) we find
  $\kappa\bar{\kappa} = 1/(1 + q\bar{q} + h\bar{h}) \Rightarrow$ planar 1–loop finiteness condition!
- $H^{i\,j}_{kl} = E_{kln} F^{nij}$ is the 1–loop spin chain Hamiltonian [Roiban '03]

$$H_{l,l+1} = \frac{1}{1+q\bar{q}+h\bar{h}} \begin{pmatrix} h\,\bar{h} & 0 & 0 & 0 & 0 & \bar{h} & 0 & -\bar{h}\,q & 0 \\ 0 & 1 & 0 & -q & 0 & 0 & 0 & 0 & h \\ 0 & 0 & q\,\bar{q} & 0 & -h\,\bar{q} & 0 & -\bar{q} & 0 & 0 \\ 0 & -\bar{q} & 0 & q\,\bar{q} & 0 & 0 & 0 & 0 & -h\,\bar{q} \\ 0 & 0 & -\bar{h}\,q & 0 & h\,\bar{h} & 0 & \bar{h} & 0 & 0 \\ h & 0 & 0 & 0 & 0 & 1 & 0 & -q & 0 \\ 0 & 0 & -q & 0 & h & 0 & 1 & 0 & 0 \\ -h\,\bar{q} & 0 & 0 & 0 & 0 & -\bar{q} & 0 & q\,\bar{q} & 0 \\ 0 & \bar{h} & 0 & -\bar{h}\,q & 0 & 0 & 0 & 0 & h\,\bar{h} \end{pmatrix}$$

- Hermitian, cyclic: $H^{i\,j}_{k\,l} = H^{(i+1)\,(j+1)}_{(k+1)\,(l+1)}$ etc.
- Define $\hat{R}^{ij}_{\ \ kl} = \delta^i_k \delta^j_l - H^{i\,j}_{kl} \quad \Longrightarrow \quad R^{i\ \ j}_{\ k\ \ l} = \hat{R}^{j\ \ i}_{\ k\ \ l}$

Does it work?

- Given R, the RTT relations will produce a bialgebra $\mathcal{A}(R)$
- But our R is *not* part of the E&O classification
- In particular, R does not satisfy QYBE!
  (apart from special cases, e.g. real β)
- Differences from E&O:
  - We have not imposed the second E&O condition
  - We are interested in a cyclic quantum plane structure (while E&O look at ordered planes, e.g. $x^i x^j = q x^j x^i \;, \;\; i < j$)
- We cannot have a quasitriangular Hopf Algebra, but is it still a Hopf Algebra?
- Need careful analysis of the RTT relations
- Two main new features:
  - Possibility of no (nontrivial) solutions
  - Associativity will imply higher relations

Solving the RTT relations

- When R satisfies QYBE, we are guaranteed that

$$R^{i\ k}_{\ a\ b} t^a_j t^b_l = t^k_b t^i_a R^{a\ b}_{\ j\ l}$$

has nontrivial solutions for t^i_j

- In our case we need to explicitly show that out of the 81 equations, only 36 are independent
- This turns out to be the case!
- Quadratic commutation relations of $\mathcal{A}(R)$:

(a) $$\mathrm{t}^a_c t^{a+1}_c - q t^{a+1}_c \mathrm{t}^a_c + h t^{a-1}_c t^{a-1}_c = h\left(t^a_{c+1} t^{a+1}_{c-1} - \bar{q} t^a_{c-1} t^{a+1}_{c+1} + \bar{h} t^a_c t^{a+1}_c\right)$$

(b) $$q[\mathrm{t}^{a+1}{}_{c+1}, \mathrm{t}^a{}_c] = -q^2 \mathrm{t}^{a+1}{}_c \mathrm{t}^a{}_{c+1} + hq \mathrm{t}^{a-1}{}_c \mathrm{t}^{a-1}{}_{c+1} + h \mathrm{t}^{a-1}{}_{c+1} \mathrm{t}^{a-1}{}_c + \mathrm{t}^a{}_{c+1} \mathrm{t}^{a+1}{}_c$$

(c) $$-q t^{a+1}_c t^a_{c+1} + \bar{q} t^a_{c+1} t^{a+1}_c = \bar{h} t^a_{c-1} t^{a+1}_{c-1} - h t^{a-1}_c t^{a-1}_{c+1}$$

(d) $$h(t^a_{c+1} t^a_{c-1} - \bar{q} t^a_{c-1} t^a_{c+1}) = \bar{h}(t^{a+1}_c t^{a-1}_c - q t^{a-1}_c t^{a+1}_c)$$

Associativity

- The QYBE also guarantees no new relations arise at higher levels
- In our case, associativity leads to new cubic relations

a) $R_{12}R_{13}R_{23}\mathbf{t}_1\mathbf{t}_2\mathbf{t}_3 = R_{12}R_{13}\mathbf{t}_1\mathbf{t}_3\mathbf{t}_2R_{23} = R_{12}\mathbf{t}_3\mathbf{t}_1\mathbf{t}_2R_{13}R_{23} = \mathbf{t}_3\mathbf{t}_2\mathbf{t}_1R_{12}R_{13}R_{23}$

b) $R_{23}R_{13}R_{12}\mathbf{t}_1\mathbf{t}_2\mathbf{t}_3 = R_{23}R_{13}\mathbf{t}_2\mathbf{t}_1\mathbf{t}_3R_{12} = R_{23}\mathbf{t}_2\mathbf{t}_3\mathbf{t}_1R_{13}R_{12} = \mathbf{t}_3\mathbf{t}_2\mathbf{t}_1R_{23}R_{13}R_{12}$

- These relations would be the same if QYBE were satisfied, but now they have to be imposed to guarantee associativity
- Danger is that they will trivialise the quantum determinant

$$\begin{aligned}\mathbb{D} =& \frac{1}{6}E_{ijk}\mathrm{t}^i{}_l\mathrm{t}^j{}_m\mathrm{t}^k{}_mF^{lmn}\\ =& \mathrm{t}^1{}_1\mathrm{t}^2{}_2\mathrm{t}^3{}_3 - q\mathrm{t}^2{}_1\mathrm{t}^1{}_2\mathrm{t}^3{}_3 + h\mathrm{t}^3{}_1\mathrm{t}^3{}_2\mathrm{t}^3{}_3 + \mathrm{t}^3{}_1\mathrm{t}^1{}_2\mathrm{t}^2{}_3 - q\mathrm{t}^1{}_1\mathrm{t}^3{}_2\mathrm{t}^2{}_3 + h\mathrm{t}^2{}_1\mathrm{t}^2{}_2\mathrm{t}^2{}_3\\ &+ \mathrm{t}^2{}_1\mathrm{t}^3{}_2\mathrm{t}^1{}_3 - q\mathrm{t}^3{}_1\mathrm{t}^2{}_2\mathrm{t}^1{}_3 + h\mathrm{t}^1{}_1\mathrm{t}^1{}_2\mathrm{t}^1{}_3\end{aligned}$$

- We have checked that this is not the case
- $\mathbb{D}$ is nontrivial and central $\Rightarrow$ Can set $\mathbb{D} = 1$

The Quantum Symmetry Algebra

- We have also shown that there exists an antipode

$$s^{1+i}_{\;1+k} = t^{2+k}_{\;2+i}t^{3+k}_{\;3+i} - \bar{q}t^{2+k}_{\;3+i}t^{3+k}_{\;2+i} + \bar{h}t^{2+k}_{\;1+i}t^{3+k}_{\;1+i} = t^{2+k}_{\;2+i}t^{3+k}_{\;3+i} - qt^{3+k}_{\;2+i}t^{2+k}_{\;3+i} + ht^{1+k}_{\;2+i}t^{1+k}_{\;3+i} \,.$$

- The bialgebra $\mathcal{A}(R)$ is thus a Hopf algebra
- We have found a Hopf algebra underlying the general Leigh–Strassler deformation
  - Transform $\Phi^i \to \mathrm{t}^i_j \Phi^j$, $t \in \mathcal{A}(R)$
  - $\mathbb{D} = 1$ guarantees invariance of the superpotential:

$$\mathcal{W} = \frac{1}{3} E_{ijk} \mathrm{Tr} \Phi^i \Phi^j \Phi^k \Longrightarrow E_{ijk} \mathrm{t}^i_l \mathrm{t}^j_m \mathrm{t}^k_{\;n} = \mathbb{D} E_{lmn} \,.$$

  - The antipode guarantees invariance of the kinetic terms

$$\overline{\Phi}_i \Phi^i \to \overline{\Phi}_j t^j_{\;i}{}^* \mathrm{t}^i_k \Phi^k = \overline{\Phi}_j \delta^j_{\;k} \Phi^k$$

- The full Leigh–Strassler lagrangian is invariant under $\mathcal{A}(R)$
- The $\mathbb{Z}_3$'s appear as automorphisms of $\mathcal{A}(R)$

Integrable Cases

- The Hopf algebra $\mathcal{A}(R)$ becomes quasitriangular for special choices of (q, h)
- All known integrable deformations of $\mathcal{N} = 4$ can be obtained in this way
- Can show that they arise as Hopf algebra twists of the real β case

$$R_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(this is also a twist of the $\mathcal{N} = 4$ case [Beisert, Roiban '05])

- E.g. $q = 0, \bar{h} = 1/h$ can be obtained by

$$R' = \mathcal{F}_{21} R_\beta \mathcal{F}^{-1} , \quad \mathcal{F} = U \otimes U^2$$

where

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Relation to Noncommutativity

- It is known that the real β deformation can be described by a star product [Berenstein et al. '00], [Kulaxizi, Zoubos '04], [Lunin, Maldacena '05]
- Non(–anti)commutativity on open string side leads to NS (and RR) fields on closed string side [Schomerus '99], [Seiberg, Witten '99]
- Can try to apply these ideas to construct the dual gravity background to the Leigh–Strassler theories [Kulaxizi '06]
- Works fine for real β, but problems with associativity for general case. Still, a solution was found to third order
- The resulting noncommutativity relations

$$[z^i, z^j]_* = i\beta\Theta^{ij}{}_{kl} z^k z^l, \quad [z^i, \bar{z}^{\bar{j}}]_* = i\beta\Theta^{i\bar{j}}{}_{k\bar{l}} z^k z^{\bar{l}}, \quad [\bar{z}^{\bar{i}}, \bar{z}^{\bar{j}}]_* = i\beta\Theta^{\bar{i}\bar{j}}{}_{\bar{k}\bar{l}} z^{\bar{k}} z^{\bar{l}}$$

  can be mapped to our (extended) quantum plane relations

$$R^i{}_k{}^j{}_l x^k x^l = x^j x^i \ , \ \ u_k u_l R^k{}_i{}^l{}_j = u_j u_i \ , \ \ u_l R^j{}_k{}^l{}_i x^k = x^j u_i \ , \ \ x^k \widetilde{R}^i{}_k{}^l{}_j u_l = u_j x^i$$

  by expanding $R = I + \rho r + O(\rho^2)$, ($\rho = \beta, h$)

- r is the classical r–matrix

Summary

- We have exhibited a Hopf algebra structure underlying the general Leigh–Strassler deformation
- The $\mathrm{SU}(3) \times \mathrm{U}(1)$ R–symmetry of $\mathcal{N} = 4$ is not broken, it is q–deformed to $\mathcal{A}(R) \times \mathrm{U}(1)$
- This algebra appears to be a new deformation of $\mathrm{SU}(3)$
- This quantum symmetry appears at the level of the classical Lagrangian
- It is also a symmetry of the 1–loop spin chain Hamiltonian

$$R_{12}t_1t_2 = t_2t_1R_{12} \Rightarrow \hat{R}_{12}t_1t_2 = t_1t_2\hat{R}_{12} \Rightarrow (t_2)^{-1}(t_1)^{-1}H_{12}t_1t_2 = H_{12}$$

- It reduces to known structures: Quasitriangular Hopf for integrable cases, star products at first order

Still Lots To Do

- Mathematical side
  - Better understanding of the algebra $\mathcal{A}(R)$ (e.g. higher order relations)
  - Classification of such algebras?
  - Could $\mathcal{A}(R)$ be reformulated as a (non-associative) quasi-Hopf algebra? [Drinfel'd '89], [Mack, Schomerus '92]
  - Add spectral parameter dependence?
  - Are there other integrable deformations?

- Physics side
  - What happens at the quantum level?
  - Regularisation at higher loops
  - Construction of dual backgrounds
  - Is there a relation between perturbative finiteness and quantum symmetry?