Supergravity with Pure Spinors

Martin Cederwall

Strings, M-Theory and Quantum Gravity
Ascona, July 26, 2010

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Plan

Background

$D = 11$ supergravity

- Component fields and action
- On-shell formulation in superspace
- What is needed to go off-shell?

Pure spinors

- BRST charge
- Cohomologies and linearised fields

Supersymmetric action

- Linearised action
- Batalin–Vilkovisky formalism
- Full action
- Gauge fixing

Outlook

JHEP 01 (2010) 117 [arXiv:0912.1814], 1001.0112 (and earlier)

Background

Maximally supersymmetric models (16 supercharges without gravity, 32 with gravity) have on-shell supermultiplets. There is no finite set of auxiliary fields.

Examples:

- $D = 10$ super-Yang–Mills theory
- $N = (2,0)$ model in $D = 6$
- IIB supergravity in $D = 10$
- $D = 11$ supergravity
- BLG model in $D = 3$
- Dimensional reductions of above

How does one formulate an action principle preserving manifest supersymmetry? This is of course desirable, especially for examining quantum properties.

Pure spinors provide an answer (in the cases self-dual fields are not present).

Quantum calculations with pure spinors are traditionally performed in a first-quantised, superparticle, framework. This is a heritage from string theory. The rules for constructing superparticle amplitudes should preferrably be derived from a supersymmetric action, when one exists. Then all symmetries of amplitudes are under control.

$D = 11$ supergravity

Component fields

The fields are

Bosonic: metric g_{mn}, 3-form C_{mnp};

Fermionic: gravitino ψ_m^α.

Action:

$$S = \tfrac{1}{2\kappa^2} \int d^{11}x \sqrt{-g} \left(R - \tfrac{1}{48} H^{mnpq} H_{mnpq}\right)$$
$$+ \tfrac{1}{12\kappa^2} \int C \wedge H \wedge H + \text{terms with fermions} \; .$$

Supergravity is formulated as Cartan geometry on superspace (analogous statements true for other supersymmetric gauge theories).

Coordinates: $Z^M = (x^m, \theta^\mu)$.

Vielbein: $E^A = dZ^M E_M{}^A$.

Spin connection 1-form (Lorentz valued): $\Omega_A{}^B$.

Torsion 2-form: $T^A = DE^A = dE^A + E^B \wedge \Omega_B{}^A$.

Curvature 2-form: $R_A{}^B = d\Omega_A{}^B + \Omega_A{}^C \wedge \Omega_C{}^B$.

Bianchi identities: $DT^A = E^B \wedge R_B{}^A$, $DR_A{}^B = 0$.

($M = (m, \mu)$, $A = (a, \alpha)$.)

[Cremmer, Ferrara 1980; Brink, Howe 1980]

Too many superfields. Conventional constraints remove all independent superfield except the lowest-dimensional one, ${E_\mu}^a$.

They are used to set all of the dimension-0 torsion to zero, except

$$T_{\alpha\beta}{}^c = 2\gamma^c_{\alpha\beta} + \frac{1}{2}U^c{}_{e_1 e_2}\gamma_{\alpha\beta}^{e_1 e_2} + \frac{1}{5!}V^c{}_{e_1\ldots e_5}\gamma_{\alpha\beta}^{e_1\ldots e_5}$$

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If U and V are set to 0, the torsion Bianchi identities imply the equations of motion.

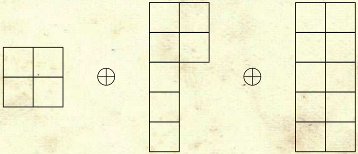
All physical fields are contained in the supergeometry. For example,

$$T_{a\beta}{}^\gamma \propto H_{ae_1e_2e_3}(\gamma^{e_1e_2e_3})_\beta{}^\gamma - \frac{1}{8}H^{e_1e_2e_3e_4}(\gamma_{ae_1e_2e_3e_4})_\beta{}^\gamma$$

[see e.g. Cederwall, Gran, Nilsson 2005]

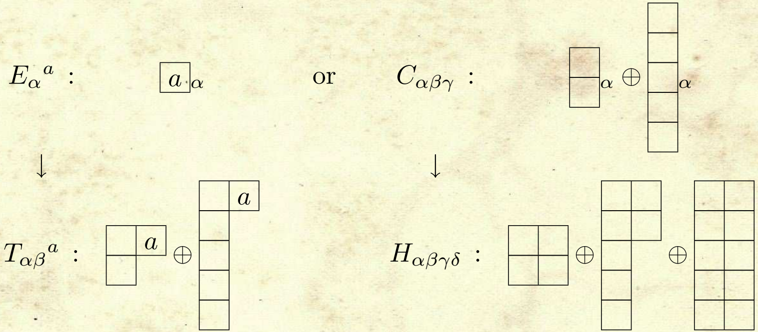
There is a closed 4-form on superspace, whose purely bosonic leg is the physical 4-form field strength.

The construction of the super-4-form relies on supergeometric data (the torsion), so this is not an independent construction. However, $C_{\alpha\beta\gamma}$ contains the entire linearised supermultiplet, and the linearised equations of motion are obtained by demanding that the irreducible modules



in $H_{\alpha\beta\gamma\delta}$ vanish (the rest are conventional constraints).

The physical fields and equations of motion reside in superfields



One needs an action containing the upper superfields, and whose equations of motion contain the lower ones.

The operation of going from fields to equations of motion looks like an exterior derivative in a fermionic direction.

Torsion in (flat) superspace, generically:

$$\{D_\alpha, D_\beta\} = -T_{\alpha\beta}{}^c D_c = -2\gamma^c_{\alpha\beta} D_c \ .$$

If a bosonic spinor λ^α is *pure*, *i.e.*, if the vector part $(\lambda\gamma^a\lambda)$ of the spinor bilinear vanishes, the operator $q = \lambda^\alpha D_\alpha$ becomes nilpotent,

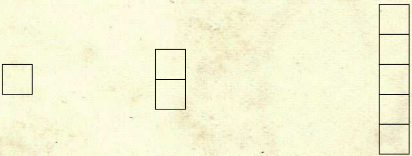
$$q^2 = 0 \ ,$$

and may be used as a BRST operator. Physical states may be defined as cohomology of q.

λ is a ghost variable.

In $D = 11$, the bilinears of a spinor λ are

$$(\lambda\gamma^{a}\lambda)\ ,\quad (\lambda\gamma^{a_1a_2}\lambda)\quad \text{and}\quad (\lambda\gamma^{a_1a_2a_3a_4a_5}\lambda)\ .$$



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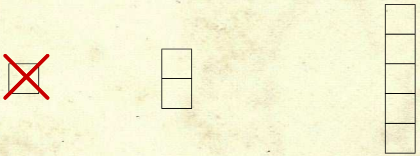

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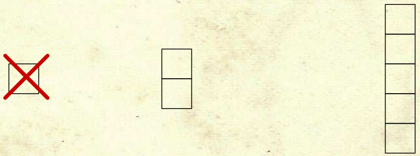
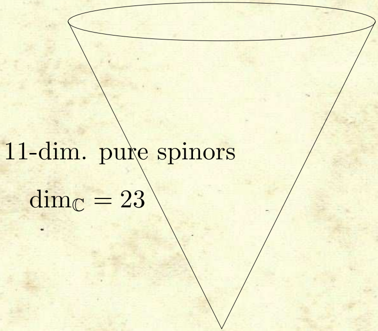


The reducibility leads to a more complicated structure of the pure spinor space than in $D = 10$.

Solution of the pure spinor constraint requires λ to be complex.



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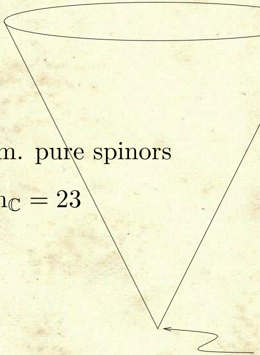

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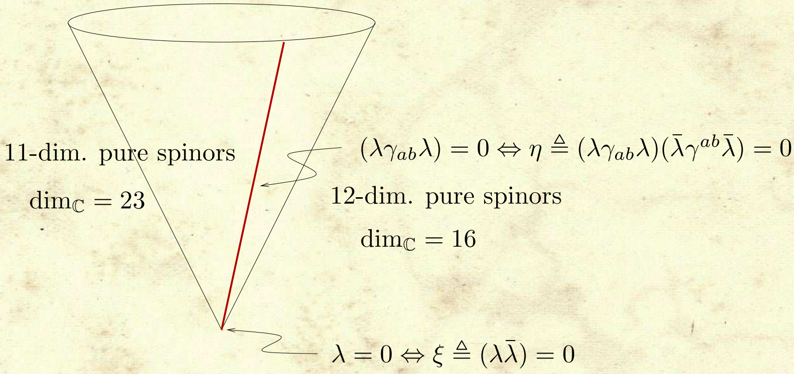
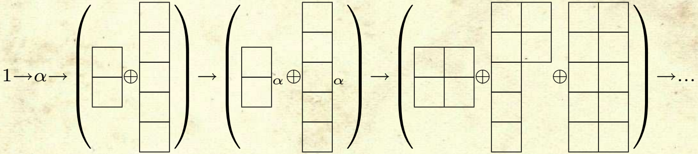


Cohomologies and linearised fields

A scalar field $\Psi(x, \theta, \lambda)$, when expanded in a power series in λ, contains



We recognise the modules of $C_{\alpha\beta\gamma}$ and of the equations of motion. The cohomology of q gives the linearised equations of motion!

An analogous statement holds for a field Φ^a and the linearised supergeometry, where Φ^a enjoys the extra gauge symmetry $\Phi^a \approx \Phi^a + (\lambda\gamma^a\rho)$.

[Cederwall, Nilsson, Tsimpis 2002]

To define integration on pure spinor superspace, extra "non-minimal" variables are needed, without changing the cohomology.

$$Q = \lambda^\alpha D_\alpha + r_\alpha \frac{\partial}{\partial \bar{\lambda}_\alpha}$$

[Berkovits 2005]

I will not go into details about integration (or regularisation).

In addition to the physical fields, pure spinor superfield cohomology contains ghosts, ghosts for ghosts,... and all Batalin–Vilkovisky antifields.

Ψ must be considered to be the "fundamental field", since Φ^a only contains the 3-form C through its field strength $H = dC$.

It is possible to relate the fields Ψ and Φ^a through an operator R^a of non-trivial cohomology as

$$\Phi^a = R^a \Psi \ .$$

where

$$R^a = \eta^{-1}(\bar{\lambda}\gamma^{ab}\bar{\lambda})\partial_b + \ldots$$

[Cederwall 2009]

Supersymmetric action

Linearised action

A linearised action is

$$S = \int [dZ] \Psi Q \Psi \ .$$

In order to introduce interaction, the concept of cohomology (which is inherently linear) must be generalised. The appropriate language is the Batalin–Vilkovisky formalism. This is already hinted at by the fact that ghosts and antifields are included in the cohomology.

The action itself is the generator of "gauge transformations", generated as $\delta X = (S, X)$, where $(\cdot, \cdot)$ is the antibracket. In a component formalism:

$$(A, B) = \int [dx] \left(A \frac{\overleftarrow{\delta}}{\delta\phi^A(x)} \frac{\overrightarrow{\delta}}{\delta\phi^\star_A(x)} B - A \frac{\overleftarrow{\delta}}{\delta\phi^\star_A(x)} \frac{\overrightarrow{\delta}}{\delta\phi^A(x)} B \right) .$$

The governing equation generalising $Q^2 = 0$ is the BV master equation $(S, S) = 0$.

[Batalin, Vilkovisky 1981]

For the pure spinor superfield Ψ, the antibracket takes the simple form

$$(A, B) = \int A \frac{\overleftarrow{\delta}}{\delta\Psi(Z)} [dZ] \frac{\overrightarrow{\delta}}{\delta\Psi(Z)} B .$$

[Cederwall 2009]

Full action

The full BV action for $D = 10$ super-Yang–Mills (and its dimensional reductions) is the Chern–Simons-like action

$$S = \int [dZ] \mathrm{Tr} \left(\tfrac{1}{2} \Psi Q \Psi + \tfrac{1}{3} \Psi^3 \right) \ .$$

implicit in [Berkovits 2001,2005; Cederwall, Nilsson, Tsimpis 2001]

Note that there is only a 3-point coupling; the quartic interaction arises on elimination of "auxiliary fields", notably the lowest component in the superfield $A_\alpha(x, \theta)$.

An analogous formulation exists for the Bagger–Lambert–Gustavsson and Aharony–Bergman–Jafferis–Maldacena models in $D = 3$.

The simplification there is even more radical: The component actions contain 6-point couplings, but the pure spinor superfield actions only have minimal coupling (*i.e.*, 3-point interactions).

[Cederwall, 2008]

But I would like to turn to supergravity.

The algebraic properties of the operator R^a ensure that an interaction term

$$S_3 \propto \int [dZ](\lambda\gamma_{ab}\lambda)\Psi R^a \Psi R^b \Psi$$

is a nontrivial deformation respecting the master equation.

The factor $(\lambda\gamma_{ab}\lambda)$ ensures that dimension and gh# are correct, guarantees the invariance under $\Phi^a \approx \Phi^a + (\lambda\gamma^a\rho)$, and makes possible a contraction of Ψ^a's.

Some terms have been checked explicitly (CS term, coupling of diffeomorphism ghosts), so it is clear that this gives the 3-point couplings of $D = 11$ supergravity.

One may expect that an expansion around flat space would be non-polynomial. This is however not the case. Checking the master equation to higher order in the field involves commutators of R^a's. The R^a's don't commute, but "almost".

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$$\frac{1}{2}(\lambda\gamma_{ab}\lambda)[R^a, R^b] = \frac{3}{2}\{Q, T\}$$

where $T = 8\eta^{-3}(\bar{\lambda}\gamma^{ab}\bar{\lambda})(\bar{\lambda}r)(rr)(\lambda\gamma_{ab}w)$.

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The master equation is *exactly* satified by

$$S = \int [dZ] \left[\tfrac{1}{2}\Psi Q \Psi + \tfrac{1}{6}(\lambda\gamma_{ab}\lambda)(1 - \tfrac{3}{2}T\Psi)\Psi R^a \Psi R^b \Psi \right] \ .$$

Note the similarity of the 3-point coupling ($\propto \Psi\Phi\Phi$) to the Chern–Simons term (which it indeed contains).

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After a field redefinition $\Psi = (1 + \frac{1}{2}T\tilde{\Psi})\tilde{\Psi}$:

$$S = \int [dZ] \left[\tfrac{1}{2}(1 + T\tilde{\Psi})\tilde{\Psi} Q \tilde{\Psi} + \tfrac{1}{6}(\lambda\gamma_{ab}\lambda)\tilde{\Psi} R^a \tilde{\Psi} R^b \tilde{\Psi} \right] \ .$$

Covariant gauge fixing amounts to demanding

$$b\Psi = 0 \ ,$$

where b is the composite b-ghost, satisfying $[Q, b] = \Box$. The propagator then becomes $b\Box^{-1}$.

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The $D = 11$ b-ghost has been constructed,

$$b = \tfrac{1}{2}\eta^{-1}(\bar{\lambda}\gamma_{ab}\bar{\lambda})(\lambda\gamma^{ab}\gamma^{i}D)\partial_i + \ldots$$

[Aisaka, Berkovits, Cederwall, work in progress]

Outlook

The framework described resolves the issue of supersymmetric actions for maximally supersymmetric theories.

The interaction terms are generically much simpler and of lower order than in a component language; for supergravity to the extent that the action becomes polynomial.

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[Berkovits, Nekrasov 2006; Aisaka, Berkovits 2009]

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Lots of other issues. How is U-duality realised? Models connected to generalised geometry, with enlarged structure groups, may possibly provide generalised models of gravity.

Geometry? Background invariance? The polynomial property should be better understood.